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Bifurcation of the travelling wave solutions in a perturbed (1 + 1)-dimensional dispersive long wave equation via a geometric approach

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Choosing κ (horizontal ordinate of the saddle point associated to the homoclinic orbit) as bifurcation parameter, bifurcations of the travelling wave solutions is studied in a perturbed $(1+1)$-dimensional dispersive long wave equation. The solitary wave solution exists at a suitable wave speed c for the bifurcation parameter $\kappa \in \left(0, 1-\frac{\sqrt{3}}{3}\right) \cup \left(1+\frac{\sqrt{3}}{3}, 2\right)$, while the kink and anti-kink wave solutions exist at a unique wave speed $c^* = \sqrt{15}/3$ for $\kappa = 0$ or $\kappa = 2$. The methods are based on the geometric singular perturbation (GSP, for short) approach, Melnikov method and invariant manifolds theory. Interestingly, not only the explicit analytical expression of the complicated homoclinic Melnikov integral is directly obtained for the perturbed long wave equation, but also the explicit analytical expression of the limit wave speed is directly given. Numerical simulations are utilized to verify our mathematical results.

Keywords: wave equation; solitary wave solutions; geometric singular perturbation

2020 *Mathematics Subject Classification:* 35Q35; 35L05; 74J30; 34D15

1. Introduction

1.1. Model formulation

The water wave equations (e.g., KdV equation, BBM equation and CH or rotation-CH equation) have attracted a lot of scholars' attentions. These nonlinear wave equations have been extensively used to describe dynamical behaviour of nonlinear waves in shallow water. Since the interaction of nonlinear and dispersion factors, long wave in shallow water would admit many special characteristics. One of the important property of long waves, is that, they retain their shapes and forms after mutual interactions and collisions. In 1996, Wu and Zhang [**37**] derived

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an equation describing nonlinear dispersive long gravity waves travelling in two horizontal directions on shallow waters of uniform depth format formulated as

$$\begin{cases} \theta_t + \theta\theta_x + u\theta_y + v_x = 0, \\ u_t + \theta u_x + uu_y + v_y = 0, \\ v_t + (\theta v)_x + (uv)_y + \frac{1}{3}(\theta_{xxx} + \theta_{xyy} + u_{xxy} + u_{yyy}) = 0, \end{cases} \tag{1.1}$$

where θ (resp., u) is the surface velocity of water along the x (resp., y) direction and v is the elevation of the water wave. For dispersive long wave equation (1.1), Chen, Tang and Lou [**15**] obtained a special type of multisoliton solution by the Weiss-Tabor-Carnvale Painlevé truncation expansion. Equation (1.1) reduces to the following $(1+1)$-dimensional dispersive long wave equation (for short, DLWE) by symmetry reduction and scale transition:

$$\begin{cases} u_t = -uu_x - v_x, \\ v_t = -vu_x - uv_x - \frac{1}{3}u_{xxx}. \end{cases} \tag{1.2}$$

Due to it models the nonlinear water wave availably, $(1+1)$-dimensional DLWE (1.2) is often applied to coastal design and harbour construction. So far, a lot of articles have been concerned with the exact solutions of $(1+1)$-dimensional DLWE (1.2) by using various methods, such as a modified Conte's invariant Painlevé expansion (Chen and Lou [**4**]), a new Jacobi elliptic function rational expansion method (Wang, Chen and Zhang [**32**]), a new general algebraic method with symbolic computation (Chen and Wang [**5**]), a generalized extended rational expansion method (Zeng and Wang [**41**]) and extended tanh-function method (Fan [**15**]).

However, in the real world application, particularly for nonlinear wave, there are many factors of uncertainty and unpredictable perturbations. Thus, small perturbation terms are usually added to describe such unpredictable perturbations when modelling the nonlinear waves. For shallow water wave equation, weak backward diffusion u_{xx} and dissipation u_{xxxx} are called Kuramoto–Sivashinsky (KS, for short) perturbation. However, to the best of our knowledge, there are no papers considering the perturbed $(1+1)$-dimensional DLWE. In this paper, we study a perturbed $(1+1)$-dimensional DLWE described by

$$\begin{cases} u_t = -uu_x - v_x, \\ v_t = -vu_x - uv_x - \frac{1}{3}u_{xxx} + \varepsilon(u_{xx} + u_{xxxx}), \end{cases} \tag{1.3}$$

where $0 < \varepsilon \ll 1$ (a sufficiently small parameter), u_{xx} is backward diffusion and u_{xxxx} represents a dissipation term.

1.2. Methods, motivation and contributions

Note that the perturbed $(1+1)$-dimensional DLWE (1.3) is singular perturbation system due to the introduction of εu_{xxxx}. Considering the geometric singular perturbations in the evolution equation is a effective method to describe the real situation for the nonlinear waves. One of important themes of shallow water waves is to study the existence of travelling wave solutions. In this paper, we will apply the geometric singular perturbation theory to study the travelling wave solutions of the singular perturbed DLWE (1.3).

In fact, the theory of the geometric singular perturbation have been well developed by (Fenichel [**16**], Jones [**21**], Szmolyan [**31**], Guckenheimer and Holmes [**18**], Wiggins [**35**], Zhang [**42**] and Li *et al.* [**23**–**25**]). And it is a powerful tool to solve the applications arising in nonlinear waves, biological systems and other dynamic systems. Up till now, there are an extensive literature applying GSP theory to study such systems, including delayed CH equation (Du *et al.* [**11**]), CH Kuramoto–Sivashinsky equation (Du *et al.* [**10**]), generalized CH equation (Qiu *et al.* [**28**]), perturbed BBM equation (Chen *et al.* [**2**]), generalized BBM equation (Sun *et al.* [**30**], Zhu *et al.* [**45**]), the delayed DP equation (Cheng and Li [**8**]), KdV equation (Derks and Gils [**9**], Ogama [**26**]), generalized KdV equation (Chen *et al.* [**7**], Yan *et al.* [**39**], Zhang *et al.* [**44**]), biological model (Chen and Zhang [**6**], Wang and Zhang [**33**]), reaction-diffusion equation (Yang and Ni [**40**], Wu and Ni [**36**]), Belousov–Zhabotinskii system (Du and Qiao [**14**]), MEMS model (Iuorio *et al.* [**20**]), piecewise-smooth dynamical systems (Buzzi *et al.* [**1**]), perturbed Gardner equation (Wen [**34**]), delayed Schrödinger equation (Xu *et al.* [**38**]), perturbed mKdV and mK$(3,1)$ equation (Zhang *et al.* [**43**, **44**]), generalized Keller–Segel system (Du *et al.* [**13**], Qiao and Zhang [**27**]), and so on.

To apply the geometric singular perturbation approach to track invariant manifolds of corresponding ordinary differential equations (ODEs), usually, it is connected with the zeroes of the Melnikov functions associated to the perturbed ODEs. However, it is not easy to determine the zeroes of the Melnikov functions because of the complexity of the expression. To analyse the Melnikov functions, usually, some auxiliary tools are needed. For examples, one of the effective method is to detect the monotonicity of the ratio of Abelian integral (for short, MRAI) associated to the Melnikov function. For instance, Derks and Gils [**9**], Ogawa [**26**] computed the MRAI of perturbed KdV equation. Chen *et al.* [**2**, **3**, **7**] detected the MRAI of perturbed BBM equation, perturbed defocusing mKdV equation and perturbed generalized KdV equation by the similar method. Du *et al.* [**12**] considered MRAI of a generalized Nizhnik–Novikov–Veselov equation with diffusion term. Then they all proved the existence of travelling wave solutions. Different from them, Sun *et al.* [**29**] employed the Chebyshev criterion to detect the MRAI of a shallow water fluid to analyse the coexistence of the solitary and periodic wave solutions. Sun and Yu [**30**] also illustrated the existence and uniqueness of periodic waves of a generalized BBM equation based on the same technique.

Different from applying the method of MRAI ([**2**, **3**, **7**, **9**, **12**, **26**, **29**, **30**]) and the method of Fredholm orthogonality (Du and Qiao [**14**]), in this paper, we directly compute the explicit expression of Melnikov function to determine the zeroes of the Melnikov function. The main purpose of this paper is to investigate the existence of solitary and kink (anti-kink) wave solutions for $(1+1)$-dimensional DLWE under a small singular perturbation. Firstly, we introduce a new parameter κ (horizontal ordinate of the saddle point associated to the homoclinic orbit) to obtain the solitary wave solutions and kink (anti-kink) wave solutions of the unperturbed $(1+1)$-dimensional DLWE. Then, the geometric singular perturbation approach is used to construct homoclinic or heteroclinic orbits by tracking invariant manifolds of corresponding ODEs. Finally, by Melnikov method, we analyse conditions such as

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the wave speed and parameter for the existence of solitary and kink (anti-kink) wave solutions. The contributions and novelty of this paper is summarized as follows:

(1) By GSP approach and the bifurcation analysis, we prove that the solitary wave solution exists at a suitable wave speed c for the bifurcation parameter $\kappa \in \left(0, 1-\frac{\sqrt{3}}{3}\right) \cup \left(1+\frac{\sqrt{3}}{3}, 2\right)$, while the kink and anti-kink wave solutions exist at a unique wave speed $c^* = \sqrt{15}/3$ for $\kappa = 0$ or $\kappa = 2$. This indicates that the solitary and kink (anti-kink) wave solutions can't coexist at the same bifurcation parameter value in $(1+1)$-dimensional DLWE.

(2) To prove the existence of solitary and kink (anti-kink) wave solutions, we use the analytical Melnikov method and bifurcation theory. Not only the analytical expression of Melnikov integral is directly obtained by hand (not by mathematical software) for a perturbed PDE, but also the analytical expression of the limit wave speed c is directly obtained by hand [see (5.1), (5.2), (5.17)–(5.19) in §5]. Indeed, the calculations and expressions are verified by the mathematical softwares (e.g., Maple).

(3) To obtained the analytical expressions, we use the following two mathematical skills.
   (i) We apply a factorization technique together with the handbook of integral [**17**] to the calculate $I(\kappa)$ and $J(\kappa)$, which lead us to obtain analytical Melnikov integral in §5. We perform very detailed computations and skilful mathematical analysis to obtain the expressions of $I(\kappa)$ and $J(\kappa)$ [see (5.9)–(5.16) in §5].

   (ii) $I(\kappa)$ and $J(\kappa)$ are integrated over a closed curve. Usually, it is difficult to be calculated. We transfer them to an definite integral by translating the time variable ζ into the state variable z on homoclinic orbit. Then the up and lower limits of the definite integral depend on the coordinates of the intersections between the homoclinic orbit and z axis. For example,

$$I(\kappa) = \oint_{\Gamma(\kappa)} y^2 \, \mathrm{d}\zeta = \oint_{\Gamma(\kappa)} y \, \mathrm{d}z = 2\int_{z_+}^{\kappa} f(z) \, \mathrm{d}z,$$

   where $f(z)$ is the expression related to the homoclinic orbit [see (5.9) in §5].

(4) We introduce a new bifurcation parameter κ (horizontal ordinate of the saddle point associated to the homoclinic orbit) instead of the integral constant G. This benefits to the factorization of the integrand because there is a factor $(z-\kappa)^2$. For example, when we compute $I(\kappa)$, it will lead us to

$$I(\kappa) = \cdots = \int_{z_+}^{\kappa} \sqrt{\frac{3}{4}(z-\kappa)^2(z-z_+)(z-z_-)} \, \mathrm{d}z = \cdots$$

[see (5.9) in §5].

(5) Numerical simulations are utilized to verify the theoretical results.

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1.3. Outline of the paper

The outline of the paper is as follows. Some preliminaries including geometric singular perturbation theory are introduced in §2. Reduction of the model by geometric singular perturbation theory and dynamical system method is presented in §3. By introducing a bifurcation parameter κ, the solitary and kink (anti-kink) wave solutions are determined in §4. In §5, combining the GSP approach and Melnikov method, the existence of solitary and kink (anti-kink) wave solutions for a perturbed $(1+1)$-dimensional DLWE is investigated. In §6, numerical simulations are carried out to show the effectiveness of previous theoretical results.

2. Preliminaries

In this section, we introduce some known results on the theory of geometric singular perturbation (see. e.g., [**21**]). Consider the system

$$\begin{cases} x_1' = f(x_1, x_2, \epsilon), \\ x_2' = \epsilon g(x_1, x_2, \epsilon), \end{cases} \tag{2.1}$$

where $' = \frac{\mathrm{d}}{\mathrm{d}t}$, $x_1 \in \mathbb{R}^n$, $x_2 \in \mathbb{R}^l$ and ϵ is a positive real parameter, $U \subseteq \mathbb{R}^{n+l}$ is open subset, and I is an open subset of $\mathbb{R}$, containing 0. f and g are C^∞ on a set $U \times I$. Moreover, the x_1 (resp., x_2) variables are called fast (resp., slow) variables. Letting $\tau = \epsilon t$ which gives the following equivalent system

$$\begin{cases} \epsilon \dot{x_1} = f(x_1, x_2, \epsilon), \\ \dot{x_2} = g(x_1, x_2, \epsilon), \end{cases} \tag{2.2}$$

where $\cdot = \frac{\mathrm{d}}{\mathrm{d}\tau}$. We refer to t (resp., τ) as the fast time scale or fast time (resp., slow time scale or slow time). Each of the scalings is naturally associated with a limit as ϵ tend to zero. These limits are respectively given by

$$\begin{cases} x_1' = f(x_1, x_2, 0), \\ x_2' = 0, \end{cases} \tag{2.3}$$

and

$$\begin{cases} 0 = f(x_1, x_2, 0), \\ x_2' = g(x_1, x_2, 0). \end{cases} \tag{2.4}$$

System (2.3) is called the layer problem and system (2.4) is reduced system.

DEFINITION 2.1 see [**8**, **11**, **21**]. *A manifold M_0 on which $f(x_1, x_2, 0) = 0$ is called a critical manifold or slow manifold. A critical manifold M_0 is said to be normally hyperbolic if the linearization of system (2.1) at each point in M_0 has exactly l eigenvalues on the imaginary axis $Re(\lambda) = 0$.*

DEFINITION 2.2 see [**8**, **11**, **21**]. *A set M is locally invariant under the flow of system (2.1) if it has neighbourhood V so that no trajectory can leave M without also leaving V. In other words, it is locally invariant if for all $x_1 \in M$, $x_1 \cdot [0, t] \subseteq V$ implies*

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that $x_1 \in M$, $x_1 \cdot [0,t] \subseteq M$, *similarly with* $[0,t]$ *replaced by* $[t,0]$ *when* $t < 0$, *where* $x_1 \cdot [0,t]$ *denotes the application of a flow after time* t *to the initial condition* x_1.

Lemma 2.3 see [8, 11, 16]. *Let* M_0 *be a compact, normally hyperbolic critical manifold given as a graph* $\{(x_1, x_2) : x_1 = h^0(x_2)\}$. *Then for sufficiently small positive* ϵ *and any* $0 < r < +\infty$,

- *there exists a manifold* M_ϵ, *which is locally invariant under the flow of system* (2.1) *and* C^r *in* x_1, x_2, ϵ. *Moreover,* M_ϵ *is given as graph:*

$$M_\epsilon = \{(x_1, x_2) : x_1 = h^\epsilon(x_2)\}$$

  for some C^r *function* $h^\epsilon(x_2)$;
- M_ϵ *possesses locally invariant stable and unstable manifold* $W^s(M_\epsilon)$ *and* $W^u(M_\epsilon)$ *lying within* $O(\epsilon)$ *and being* C^r *diffeomorphic to the stable and unstable manifold* $W^s(M_0)$ *and* $W^u(M_0)$ *of the critical manifold* M_0;
- $W^s(M_\epsilon)$ *is partitioned by moving invariant submanifolds* $\mathcal{F}^s(p_\epsilon)$, *which are* $O(\epsilon)$ *close and diffeomorphic to* $\mathcal{F}^s(p_0)$, *with base point* p_ϵ *belonging to* M_ϵ. *Moreover, they are* C^r *with respect to* p *and* ϵ. *Moving invariance means the submanifold* $\mathcal{F}^s(p_\epsilon)$ *is mapped under the time* t *flow to another submanifold* $\mathcal{F}^s(p_\epsilon \cdot t)$ *whose base point is the time* t *evolution image of the taken base point* p_ϵ;
- *the dynamics on* M_ϵ *is a regular perturbation of that generated by system* (2.4).

3. Reduction of the model by geometric singular perturbation theory and dynamical system method

In this section, to consider the exact solutions of the unperturbed $(1+1)$-dimensional DLWE (1.3), we firstly reduce the model by GSP theory and dynamical system method. From the view of physical meanings, we only consider the case of $c > 0$ in this paper.

By introducing the following transformations:

$$u(x,t) = \phi(\xi), \quad v(x,t) = \psi(\xi), \quad \xi = x - ct, \tag{3.1}$$

we obtain

$$\begin{aligned} &\frac{\partial u(x,t)}{\partial t} = -c\phi_\xi, \quad \frac{\partial v(x,t)}{\partial t} = -c\psi_\xi, \\ &\frac{\partial^2 u(x,t)}{\partial x^2} = \phi_{\xi\xi}, \quad \frac{\partial^3 u(x,t)}{\partial x^3} = \phi_{\xi\xi\xi}, \quad \frac{\partial^4 u(x,t)}{\partial x^4} = \phi_{\xi\xi\xi\xi}, \end{aligned} \tag{3.2}$$

where ϕ_ξ and ψ_ξ are the first order derivative with respect to ξ. And $\phi_{\xi\xi}$, $\phi_{\xi\xi\xi}$ and $\phi_{\xi\xi\xi\xi}$ means the second, third and fourth order derivative with respect to ξ, respectively.

Substituting (3.1) and (3.2) into (1.3), then system (1.3) is given by

$$\begin{cases} c\phi_\xi = \phi\phi_\xi + \psi_\xi, \\ c\psi_\xi = \psi\phi_\xi + \phi\psi_\xi + \frac{1}{3}\phi_{\xi\xi\xi} - \varepsilon(\phi_{\xi\xi} + \phi_{\xi\xi\xi\xi}). \end{cases} \tag{3.3}$$

Integrating both sides of the first equation of system (3.3) once with respect to ξ and letting the integration constant be zero yields

$$\psi(\xi) = c\phi(\xi) - \frac{1}{2}\phi^2(\xi). \tag{3.4}$$

We substitute (3.4) into the second equation of (3.3), then the coupled system (3.3) becomes the following ODE:

$$\frac{1}{3}\phi_{\xi\xi\xi} - \frac{3}{2}\phi^2\phi_\xi + 3c\phi\phi_\xi - c^2\phi_\xi - \varepsilon(\phi_{\xi\xi} + \phi_{\xi\xi\xi\xi}) = 0. \tag{3.5}$$

Integrating both sides on (3.5) once with respect to ξ and rescaling $\epsilon = 3\varepsilon$, it follows that

$$\phi_{\xi\xi} - \frac{3}{2}\phi^3 + \frac{9}{2}c\phi^2 - 3c^2\phi - \epsilon(\phi_\xi + \phi_{\xi\xi\xi}) = g, \tag{3.6}$$

where g is an integration constant ($g \in \mathbb{R}$).

Introducing new variables $\zeta = c\xi$, $z = \frac{\phi}{c}$ and $G = \frac{g}{c^3}$, equation (3.6) is equivalent to

$$-3z + \frac{9}{2}z^2 - \frac{3}{2}z^3 + \frac{\mathrm{d}^2 z}{\mathrm{d}\zeta^2} - \epsilon\left(\frac{1}{c}\frac{\mathrm{d}z}{\mathrm{d}\zeta} + c\frac{\mathrm{d}^3 z}{\mathrm{d}\zeta^3}\right) = G. \tag{3.7}$$

Obviously, equation (3.7) reduces to a three-dimensional system:

$$\begin{cases} \dfrac{\mathrm{d}z}{\mathrm{d}\zeta} = y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}\zeta} = w, \\ \epsilon c\dfrac{\mathrm{d}w}{\mathrm{d}\zeta} = -3z + \frac{9}{2}z^2 - \dfrac{3}{2}z^3 + w - \dfrac{\epsilon}{c}y - G, \end{cases} \tag{3.8}$$

where ϵ is a sufficiently small parameter such that $0 < \epsilon \ll 1$. Therefore, the travelling wave solutions of equation (1.3) can be obtained by studying the corresponding orbits of system (3.8).

Obviously, system (3.8) is a singularly perturbed system described as 'slow system.' Rescaling $\zeta = \epsilon\eta$, we have the following equivalent 'fast system':

$$\begin{cases} \dfrac{\mathrm{d}z}{\mathrm{d}\eta} = \epsilon y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}\eta} = \epsilon w, \\ c\dfrac{\mathrm{d}w}{\mathrm{d}\eta} = -3z + \dfrac{9}{2}z^2 - \dfrac{3}{2}z^3 + w - \dfrac{\epsilon}{c}y - G. \end{cases} \tag{3.9}$$

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Let $\epsilon = 0$ in system (3.8) and (3.9). Then, the corresponding reduced system is given by

$$\begin{cases} \dfrac{\mathrm{d}z}{\mathrm{d}\zeta} = y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}\zeta} = w, \\ 0 = -3z + \dfrac{9}{2}z^2 - \dfrac{3}{2}z^3 + w - G, \end{cases} \tag{3.10}$$

and the layer system is

$$\begin{cases} \dfrac{\mathrm{d}z}{\mathrm{d}\eta} = 0, \\ \dfrac{\mathrm{d}y}{\mathrm{d}\eta} = 0, \\ c\dfrac{\mathrm{d}w}{\mathrm{d}\eta} = -3z + \frac{9}{2}z^2 - \dfrac{3}{2}z^3 + w - G, \end{cases} \tag{3.11}$$

which admits a two-dimensional critical manifold as follows

$$M_0 = \left\{ (z, y, w) \in \mathbb{R}^3 | w = 3z - \frac{9}{2}z^2 + \frac{3}{2}z^3 + G \right\}. \tag{3.12}$$

Suppose A to be the linearized matrix of system (3.11), then it is given by

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \dfrac{-3 + 9z - \dfrac{9}{2}z^2}{c} & 0 & \dfrac{1}{c} \end{pmatrix}, \tag{3.13}$$

that the eigenvalues of A are 0, 0 and $\frac{1}{c}$. Thus, M_0 is a normally hyperbolic invariant manifold (see definitions 2.1 and 2.2). There exists a two-dimensional slow submanifold M_ϵ which is C^1 $O(\epsilon)$ close to M_0 (see lemma 2.3). The invariant submanifold M_ϵ is represented by

$$\begin{aligned} M_\epsilon = \Bigg\{ (z, y, w) \in \mathbb{R}^3 | w = 3z - \frac{9}{2}z^2 + \frac{3}{2}z^3 + G \\ + \epsilon \left[cy \left(\frac{9}{2}z^2 - 9z + 3 + \frac{1}{c^2} \right) \right] + O(\epsilon^2) \Bigg\}. \end{aligned} \tag{3.14}$$

The dynamical behaviour of slow system (3.8) or fast system (3.9) is governed by

$$\begin{cases} \dfrac{\mathrm{d}z}{\mathrm{d}\zeta} = y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}\zeta} = 3z - \dfrac{9}{2}z^2 + \frac{3}{2}z^3 + G + \epsilon[cy(\dfrac{9}{2}z^2 - 9z + 3 + \dfrac{1}{c^2})] + O(\epsilon^2). \end{cases} \tag{3.15}$$

Obviously, the unperturbed system $(3.15)|_{\epsilon=0}$ admits homoclinic, heteroclinic and periodic orbits with the parameter G taking different values. By the bifurcation

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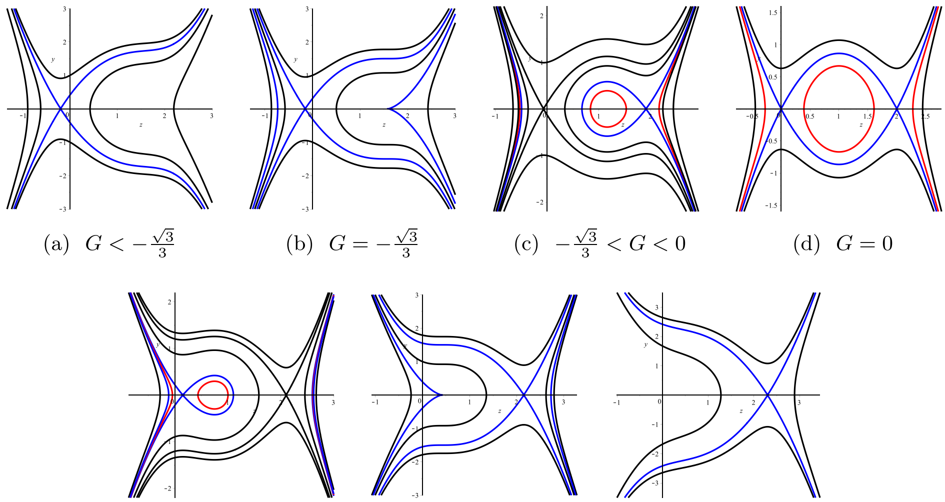

(a) $G < -\frac{\sqrt{3}}{3}$ (b) $G = -\frac{\sqrt{3}}{3}$ (c) $-\frac{\sqrt{3}}{3} < G < 0$ (d) $G = 0$

(e) $0 < G < \frac{\sqrt{3}}{3}$ (f) $G = \frac{\sqrt{3}}{3}$ (g) $G > \frac{\sqrt{3}}{3}$

Figure 1. The bifurcation and phase portraits of unperturbed system $(3.15)|_{\epsilon=0}$.

theory of planar dynamical systems (see [**22**]), we have the following proposition of unperturbed system $(3.15)|_{\epsilon=0}$:

PROPOSITION 3.1.

(i) *If* $\mid G \mid > \frac{\sqrt{3}}{3}$*, there exists a single saddle point [see figures 1(a) or (g)].*

(ii) *If* $\mid G \mid = \frac{\sqrt{3}}{3}$*, there exist a saddle point and a cusp point [see figures 1(b) or (f)].*

(iii) *If* $G = 0$*, there exist two saddle points* $(0, 0)$ *and* $(2, 0)$*, and a centre point* $(1, 0)$*. It has two heteroclinic orbits surrounding the centre point* $(1, 0)$ *to two saddle points* $(0, 0)$ *and* $(2, 0)$ *[see figure 1(d)].*

(iv) *If* $0 <\mid G \mid < \frac{\sqrt{3}}{3}$*, there exist a saddle point and a centre point. It has a homoclinic orbit surrounding the centre point to one saddle point [see figures 1(c) or (e)].*

Assume that $z(\zeta)$ is a solution of equation (3.7) satisfying $\lim_{\zeta \longrightarrow \infty} z(\zeta) = \kappa$, thus $u(x, t) = cz(\zeta) = cz[c(x - ct)]$ is a solitary wave solution or a kink wave solution of equation (3.7).

For $-\frac{\sqrt{3}}{3} < G \leqslant 0$ and $0 \leqslant G < \frac{\sqrt{3}}{3}$, the unperturbed system $(3.15)|_{\epsilon=0}$ admits a saddle point defined by $S_0(\kappa, 0)$. We know that $G = -\left(3\kappa - \frac{9}{2}\kappa^2 + \frac{3}{2}\kappa^3\right)$, then we obtain $1 + \frac{\sqrt{3}}{3} < \kappa \leqslant 2$ for $G \in \left(-\frac{\sqrt{3}}{3}, 0\right]$ and $0 \leqslant \kappa < 1 - \frac{\sqrt{3}}{3}$ for $G \in \left[0, \frac{\sqrt{3}}{3}\right)$, respectively. We can regard κ as a bifurcation parameter to find the homoclinic

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orbits of system (3.15). Therefore, we rewrite system (3.15), that is:

$$\begin{cases} \dfrac{\mathrm{d}z}{\mathrm{d}\zeta} = y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}\zeta} = 3z - \dfrac{9}{2}z^2 + \dfrac{3}{2}z^3 - \left(3\kappa - \dfrac{9}{2}\kappa^2 + \dfrac{3}{2}\kappa^3\right) \\ \quad + \epsilon\left[cy\left(\dfrac{9}{2}z^2 - 9z + 3 + \dfrac{1}{c^2}\right)\right] + O(\epsilon^2). \end{cases} \tag{3.16}$$

4. Travelling wave solutions of equation (1.3)$|_{\varepsilon=0}$

In this section, we study the solitary and kink (anti-kink) wave solutions of equation (1.3)$|_{\varepsilon=0}$. By dynamical method, we consider the travelling wave solution of system (3.16)$|_{\epsilon=0}$ with $\kappa \in \left[0, 1-\frac{\sqrt{3}}{3}\right) \cup \left(1+\frac{\sqrt{3}}{3}, 2\right]$. The unperturbed system (3.16) is of the form:

$$\begin{cases} \dfrac{\mathrm{d}z}{\mathrm{d}\zeta} = y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}\zeta} = 3z - \dfrac{9}{2}z^2 + \dfrac{3}{2}z^3 - \left(3\kappa - \dfrac{9}{2}\kappa^2 + \dfrac{3}{2}\kappa^3\right). \end{cases} \tag{4.1}$$

It is easy to obtain the first integral of system (4.1), it is given by

$$H(z, y) = \frac{1}{2}y^2 - \frac{3}{8}z^4 + \frac{3}{2}z^3 - \frac{3}{2}z^2 + \left(\frac{3}{2}\kappa^3 - \frac{9}{2}\kappa^2 + 3\kappa\right)z. \tag{4.2}$$

The homoclinic orbit $\Gamma(\kappa)$ to the saddle $(\kappa, 0)$ is defined by

$$\begin{aligned} H(z, y) &= \frac{1}{2}y^2 - \frac{3}{8}z^4 + \frac{3}{2}z^3 - \frac{3}{2}z^2 + \left(\frac{3}{2}\kappa^3 - \frac{9}{2}\kappa^2 + 3\kappa\right)z \\ &= \frac{9}{8}\kappa^4 - 3\kappa^3 + \frac{3}{2}\kappa^2. \end{aligned} \tag{4.3}$$

And the heteroclinic orbits $\Upsilon(\kappa)_\pm$ (where $\pm$ represents upper and lower branch curves) to the two saddle points $(0, 0)$ and $(2, 0)$, namely, $\kappa = 0$ or $\kappa = 2$, the heteroclinic curve is determined by

$$H(z, y) = \frac{1}{2}y^2 - \frac{3}{8}z^4 + \frac{3}{2}z^3 - \frac{3}{2}z^2 = 0. \tag{4.4}$$

4.1. Solitary wave solutions of equation (1.3)$|_{\varepsilon=0}$

By (4.3) and proposition 3.1(iv), we can obtain the expression of y as follows:

$$\begin{aligned} y &= \pm\sqrt{\frac{3}{4}z^4 - 3z^3 + 3z^2 - (3\kappa^3 - 9\kappa^2 + 6\kappa)z + \frac{9}{4}\kappa^4 - 6\kappa^3 + 3\kappa^2} \\ &= \pm\sqrt{\frac{3}{4}(z-\kappa)^2(z-z_+)(z-z_-)}, \end{aligned} \tag{4.5}$$

where $z_\pm = -\kappa + 2 \pm \sqrt{-2\kappa^2 + 4\kappa}$. It can be seen that $z_- < z_+ < \kappa$ for $1 + \frac{\sqrt{3}}{3} < \kappa < 2$ [see figure 1(c)] and $\kappa < z_- < z_+$ for $0 < \kappa < 1 - \frac{\sqrt{3}}{3}$ [see figure 1(e)].

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Since $\frac{\mathrm{d}z}{\mathrm{d}\zeta} = y$, it implies

$$\frac{\mathrm{d}z}{\mathrm{d}\zeta} = \pm\sqrt{\frac{3}{4}(z-\kappa)^2(z-z_+)(z-z_-)}, \tag{4.6}$$

which yields

$$\zeta = \pm\int_{z_+}^{z} \frac{1}{\sqrt{\frac{3}{4}(s-\kappa)^2(s-z_+)(s-z_-)}}\,\mathrm{d}s, \quad \left(1+\frac{\sqrt{3}}{3} < \kappa < 2\right), \tag{4.7}$$

and

$$\zeta = \pm\int_{z_-}^{z} \frac{1}{\sqrt{\frac{3}{4}(s-\kappa)^2(z_+-s)(z_--s)}}\,\mathrm{d}s, \quad \left(0 < \kappa < 1-\frac{\sqrt{3}}{3}\right). \tag{4.8}$$

Thus, for $1+\frac{\sqrt{3}}{3} < \kappa < 2$, the parametric representation of the dark solitary wave of system $(3.16)|_{\epsilon=0}$ can be obtained by [see figure 2(a)]

$$z(\zeta) = -\frac{2(6\kappa^2-12\kappa+4)}{2\cosh\left(\frac{1}{2}\sqrt{18\kappa^2-36\kappa+12}\zeta\right)\sqrt{-2\kappa^2+4\kappa}+4\kappa-4}. \tag{4.9}$$

Then, it corresponds to a dark solitary wave solution [see figure 2(b)] of equation $(1.3)|_{\varepsilon=0}$ is given by:

$$u(x,t) = -\frac{2(6\kappa^2-12\kappa+4)c}{2\cosh\left[\frac{1}{2}\sqrt{18\kappa^2-36\kappa+12}c(x-ct)\right]\sqrt{-2\kappa^2+4\kappa}+4\kappa-4}. \tag{4.10}$$

For $0 < \kappa < 1-\frac{\sqrt{3}}{3}$, the expression of the bright solitary wave of system $(3.16)|_{\epsilon=0}$ can be obtained by [see figure 3(a)]

$$z(\zeta) = -\frac{2(6\kappa^2-12\kappa+4)}{4\kappa-4-2\cosh\left(\frac{1}{2}\sqrt{18\kappa^2-36\kappa+12}\zeta\right)\sqrt{-2\kappa^2+4\kappa}}. \tag{4.11}$$

The bright solitary wave solution of equation $(1.3)|_{\varepsilon=0}$ [see figure 3(b)] is of the form

$$u(x,t) = -\frac{2(6\kappa^2-12\kappa+4)c}{4\kappa-4-2\cosh\left[\frac{1}{2}\sqrt{18\kappa^2-36\kappa+12}c(x-ct)\right]\sqrt{-2\kappa^2+4\kappa}}. \tag{4.12}$$

In summary, we give the theorem as follows.

THEOREM 4.1. *For any wave speed* $c > 0$ *and* $\kappa \in \left(0, 1-\frac{\sqrt{3}}{3}\right) \cup \left(1+\frac{\sqrt{3}}{3}, 2\right)$, *equation* $(1.3)|_{\varepsilon=0}$ *has solitary wave solutions given by* (4.10) *or* (4.12).

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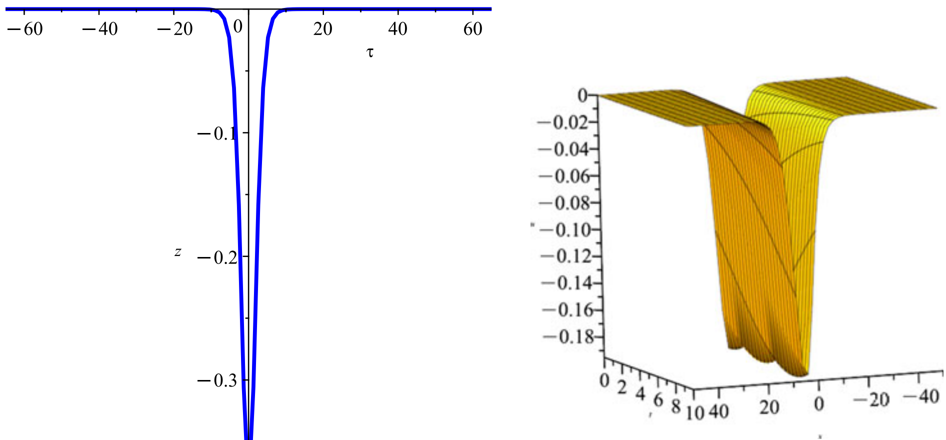
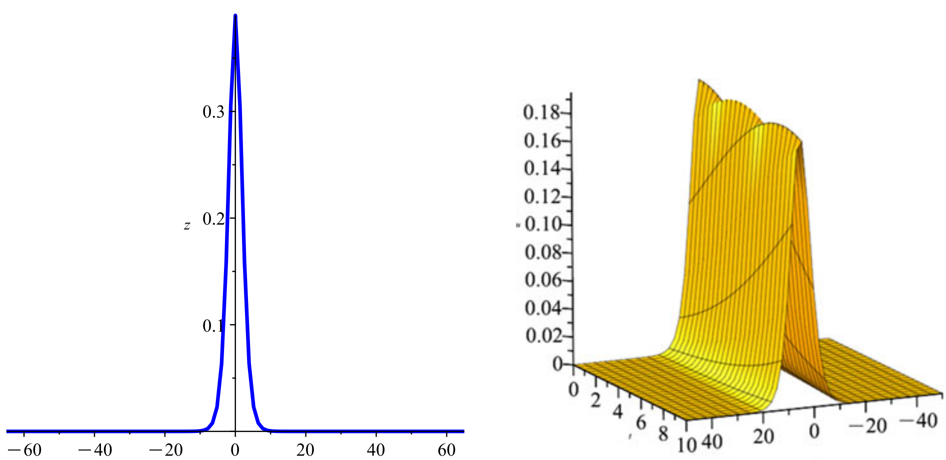


(a) (b)

Figure 2. Dark solitary wave of system ([3.16](#))$|_{\epsilon=0}$ and exact solution of equation ([1.3](#))$|_{\varepsilon=0}$ for $c = 0.5$ and $\kappa = 1.7$. (a) Dark solitary wave. (b) Exact solution.



(a) (b)

Figure 3. Bright solitary wave of system ([3.16](#))$|_{\epsilon=0}$ and exact solution of equation ([1.3](#))$|_{\varepsilon=0}$ for $c = 0.5$ and $\kappa = 0.3$. (a) Bright solitary wave (b) Exact solution.

4.2. Kink and anti-kink wave solutions of equation (1.3)$|_{\varepsilon=0}$

According (4.4) and proposition 3.1(iii), we can obtain the expression of y as follows:

$$y = \pm\sqrt{\frac{3}{4}z^4 - 3z^3 + 3z^2} = \pm\sqrt{\frac{3}{4}(z-0)^2(2-z)^2} = \pm\frac{\sqrt{3}}{2}z(z-2). \tag{4.13}$$

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Similarly, due to $\frac{\mathrm{d}z}{\mathrm{d}\zeta}=y$, we have

$$\zeta=\pm\int_{1}^{z}\frac{1}{\frac{\sqrt{3}}{2}s(s-2)}\,\mathrm{d}s. \tag{4.14}$$

For $\kappa=0$ or $\kappa=2$, the expression of the kink or anti-kink wave of system (3.16)$|_{\epsilon=0}$ can be obtained by [see figures 4(a) and (c)]

$$z(\zeta)=\frac{2c}{1+\mathrm{e}^{\pm\sqrt{3}c\zeta}}. \tag{4.15}$$

Therefore, we obtain the following kink and anti-kink wave solutions (see figures 4(b) and (b)]:

$$u(x,t)=\frac{2c}{1+\mathrm{e}^{\pm\sqrt{3}c(x-ct)}}. \tag{4.16}$$

Based on above analysis, we have the following theorems:

THEOREM 4.2. *For any wave speed $c>0$ and $\kappa=0$ or $\kappa=2$, equation* (1.3)$|_{\varepsilon=0}$ *has kink or anti-kink solutions given by* (4.16)*.*

5. Bifurcations of the travelling wave solutions for the perturbed equation (1.3)

In this section, before stating our main results on the existence of the solitary and kink wave solutions for the perturbed equation (1.3), denote

$$\begin{aligned}
c_1(\kappa)\Big|_{\kappa\in\left(0,1-\frac{\sqrt{3}}{3}\right)} = \frac{\sqrt{15}}{3}\Big[\Big(&6\ln2\kappa^3+6\kappa^3\ln\left(\sqrt{6\kappa^2-12\kappa+4}+2\kappa-2\right)-18\ln2\kappa^2\\
&-18\kappa^2\ln\left(\sqrt{6\kappa^2-12\kappa+4}+2\kappa-2\right)+12\ln2\kappa\\
&+12\kappa\ln\left(\sqrt{6\kappa^2-12\kappa+4}+2\kappa-2\right)+2\sqrt{6\kappa^2-12\kappa+4}\Big]\\
&-\sqrt{3}\kappa\Big[2\kappa^2\ln\left(-\sqrt{2\kappa-\kappa^2}\right)\\
&+3\ln2\kappa^2-6\kappa\ln\left(-\sqrt{2\kappa-\kappa^2}\right)-9\ln2\kappa\\
&+4\ln\left(-\sqrt{2\kappa-\kappa^2}\right)+6\ln2\Big)\Big/\Big(27\kappa^4\sqrt{6\kappa^2-12\kappa+4}\\
&+30\kappa^3\ln\left(-\sqrt{2\kappa-\kappa^2}\right)+60\kappa\ln\left(-\sqrt{2\kappa-\kappa^2}\right)+15\ln2\kappa^3\\
&-30\kappa^3\ln\left(\sqrt{6\kappa^2-12\kappa+4}+2\kappa-2\right)-90\kappa^2\ln\left(-\sqrt{2\kappa-\kappa^2}\right)\\
&+90\kappa^2\ln\left(\sqrt{6\kappa^2-12\kappa+4}+2\kappa-2\right)\\
&+30\sqrt{2}\kappa\sqrt{(3\kappa^2-6\kappa+2)^3}
\end{aligned}$$

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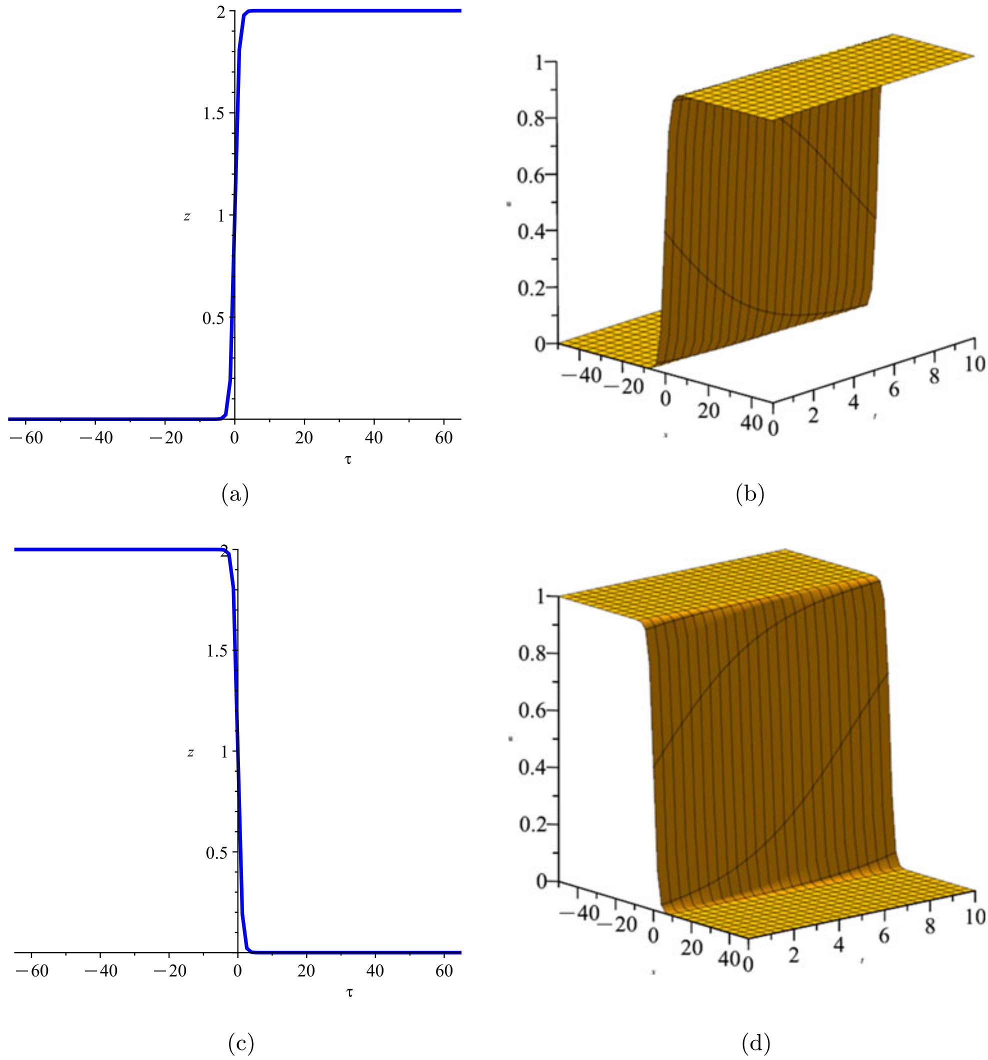

Figure 4. Kink and anti-kink wave of system (3.16)$|_{\epsilon=0}$. Kink and anti-kink wave solutions of equation (1.3)$|_{\varepsilon=0}$ for $c = 0.5$. (a) Kink wave (b) Kink wave solution (c) Anti-kink wave (d) Anti-kink wave solution.

$$
\begin{aligned}
&+ 294\kappa^2\sqrt{6\kappa^2 - 12\kappa + 4} - 45\ln 2\kappa^2 - 198\kappa^3\sqrt{6\kappa^2 - 12\kappa + 4} \\
&- 60\kappa\ln\left(\sqrt{6\kappa^2 - 12\kappa + 4} + 2\kappa - 2\right) + 10\sqrt{2}\sqrt{(3\kappa^2 - 6\kappa + 2)^3} \\
&-72\kappa\sqrt{6\kappa^2 - 12\kappa + 4} + 30\ln 2\kappa - 18\sqrt{6\kappa^2 - 12\kappa + 4}\ \Big)\Big]^{\frac{1}{2}},
\end{aligned} \tag{5.1}
$$

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and

$$
\begin{aligned}
c_2(\kappa)\Big|_{\kappa\in\left(1+\frac{\sqrt{3}}{3},2\right)} = \frac{\sqrt{15}}{3}\Big[&\Big(6\ln 2\kappa^3+6\kappa^3\ln\left(\sqrt{6\kappa^2-12\kappa+4}+2\kappa-2\right)-18\ln 2\kappa^2\\
&-18\kappa^2\ln\left(\sqrt{6\kappa^2-12\kappa+4}+2\kappa-2\right)+12\ln 2\kappa\\
&+12\kappa\ln\left(\sqrt{6\kappa^2-12\kappa+4}+2\kappa-2\right)+2\sqrt{6\kappa^2-12\kappa+4}\Big]\\
&-\sqrt{3}\kappa\left[\kappa^2\ln(2\kappa-\kappa^2)+3\ln 2\kappa^2-3\kappa\ln(2\kappa-\kappa^2)-9\ln 2\kappa\right.\\
&+2\ln(2\kappa-\kappa^2)+6\ln 2\Big)\Big/\Big(27\kappa^4\sqrt{6\kappa^2-12\kappa+4}\\
&+15\kappa^3\ln(2\kappa-\kappa^2)-198\kappa^3\sqrt{6\kappa^2-12\kappa+4}\\
&-30\kappa^3\ln\left(\sqrt{6\kappa^2-12\kappa+4}+2\kappa-2\right)\\
&+15\ln 2\kappa^3-45\kappa^2\ln(2\kappa-\kappa^2)\\
&+90\kappa^2\ln\left(\sqrt{6\kappa^2-12\kappa+4}+2\kappa-2\right)\\
&+30\sqrt{2}\kappa\sqrt{(3\kappa^2-6\kappa+2)^3}\\
&+294\kappa^2\sqrt{6\kappa^2-12\kappa+4}-45\ln 2\kappa^2+30\kappa\ln(2\kappa-\kappa^2)\\
&-60\kappa\ln\left(\sqrt{6\kappa^2-12\kappa+4}+2\kappa-2\right)+10\sqrt{2}\sqrt{(3\kappa^2-6\kappa+2)^3}\\
&-72\kappa\sqrt{6\kappa^2-12\kappa+4}+30\ln 2\kappa-18\sqrt{6\kappa^2-12\kappa+4}\Big)\Big]^{\frac{1}{2}}.
\end{aligned}
\tag{5.2}
$$

Now we are in a position to state our main results on the bifurcations of the travelling waves for the perturbed equation (1.3).

THEOREM 5.1.

(i) *If $\kappa \in \left(0, 1-\frac{\sqrt{3}}{3}\right)$, then for sufficiently small ϵ $(0<\epsilon\ll 1)$, there exists a wave speed $\tilde{c}_1(\kappa,\epsilon) = c_1(\kappa)+O(\epsilon)$ such that system (3.16) possesses a homoclinic orbit. Thus, equation (1.3) has a solitary wave solution $u = u(x,t,\kappa,\epsilon)$ with a wave speed $c = \tilde{c}_1$.*

(ii) *If $\kappa \in \left(1+\frac{\sqrt{3}}{3}, 2\right)$, then for sufficiently small ϵ $(0<\epsilon\ll 1)$, there exists a wave speed $\tilde{c}_2(\kappa,\epsilon) = c_2(\kappa)+O(\epsilon)$ such that system (3.16) possesses a homoclinic orbit. Thus, equation (1.3) has a solitary wave solution $u = u(x,t,\kappa,\epsilon)$ with a wave speed $c = \tilde{c}_2$.*

(iii) *If $\kappa = 0$ or $\kappa = 2$, then for sufficiently small ϵ $(0<\epsilon\ll 1)$, there exists a unique wave speed $\tilde{c}_3(\epsilon) = \sqrt{15}/3 + O(\epsilon)$ such that system (3.16) has a heteroclinic orbit. Therefore, equation (1.3) has a pair of kink and anti-kink wave solutions $u = u(x,t,\kappa,\epsilon)$ with a wave speed $c = \tilde{c}_3$.*

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Remark 5.2. Different kinds of travelling wave solutions exist in the different bifurcation parameter regions. For DWLE, solitary wave and kink wave can not coexist at a same wave speed.

Remark 5.3. The exact limit wave speed is given for each bifurcation parameter regions.

In order to prove theorem 5.1, we need to introduce some lemmas.

Poincaré map d_i $(i = 1, 2)$ is a powerful tool to study the bifurcation of homoclinic or heteroclinic orbits (e.g., see [**19**, **34**, **35**, **45**]).

Case a. It is defined by [see figure 5(a)]

$$d_1 : A(h) \longrightarrow P(h, c, \varepsilon),$$

and

$$d_1(h, c, \varepsilon) = \int_{A(h)}^{P(h,c,\varepsilon)} \mathrm{d}H = \int_{-\infty}^{+\infty} \left(\frac{\partial H}{\partial x} \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial H}{\partial y} \frac{\mathrm{d}y}{\mathrm{d}t} \right) |_{L_{(h,c,\varepsilon)}} \, \mathrm{d}t,$$

where $A(h)$ is the initial point and $P(h, c, \varepsilon)$ is the mapping point. $L(h, c, \varepsilon)$ and $L(h, c)$ are the perturbed and unperturbed orbits, respectively. We have $\lim_{\varepsilon \to 0} L(h, c, \varepsilon) = L(h, c)$. And $d_1(h, c, \varepsilon)$ can be Taylor expanded in ε, one has

$$d_1(h, c, \varepsilon) = \varepsilon M_{\mathrm{hom}}(h, c) + O(\varepsilon^2).$$

Case b. A heteroclinic orbit $L(h, c)$ connecting two saddle points S_1 and S_2. Then, $L_s^+(h, c, \varepsilon)$ and $L_u^-(h, c, \varepsilon)$ represent the stable and unstable manifolds of perturbed heteroclinic orbit $L(h, c, \varepsilon)$. Taking a point $P(h, c) \in L(h, c)$, we let L^* be a segment normal of $L(h, c)$ at point $P(h, c)$. For $0 < \varepsilon \ll 1$, we assume that $L_s^+(h, c, \varepsilon)$ and $L_u^-(h, c, \varepsilon)$ intersect the normal line L^* transversally at points $P_s^+(h, c, \varepsilon)$ and $P_u^-(h, c, \varepsilon)$ [see figure 5(b)].

Let

$$d_2(h, c, \varepsilon) = -\overrightarrow{n} \cdot \overrightarrow{P_s^+ P_u^-},$$

where $\overrightarrow{n} = \frac{(H_z(P(h,c)), H_y(P(h,c)))}{|(H_y(P(h,c)), -H_z(P(h,c)))|}$. And $d_2(h, c, \varepsilon)$ can be Taylor expanded in ε, we have

$$d_2(h, c, \varepsilon) = \varepsilon \cdot A \cdot M_{\mathrm{het}}(h, c) + O(\varepsilon^2),$$

where A is a constant.

Usually, $d_i(h, c, \varepsilon)$ $(i = 1, 2)$ is used to measure the distance between perturbed stable and unstable manifolds. Then, we say that $M_{\mathrm{hom}}(h, c)$ or $M_{\mathrm{het}}(h, c)$ is the so called 'first-order' Melnikov integral.

In this paper, 'first-order' Melnikov integral is employed to detect the persistence of the homoclinic and heteroclinic orbits under small perturbation, respectively.

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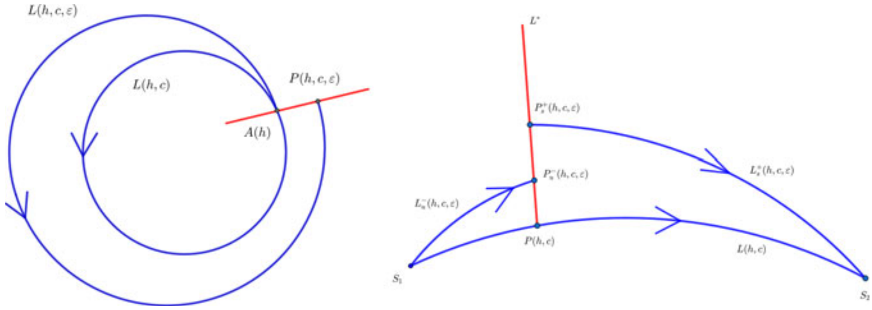

(a) (b)

Figure 5. Poincaré map. (a) The type of periodic or homoclinic orbits (b) The type of heteroclinic orbits.

Consequently, the existence of solitary and kink wave solutions are proved under small perturbation.

Firstly, the homoclinic Melnikov function of system (3.16) is defined by

$$M_{\rm hom}(c,\kappa) = \oint_{\Gamma(\kappa)} \left(\frac{9}{2}z^2 - 9z + 3 + \frac{1}{c^2}\right) y^2\, {\rm d}\zeta = \frac{1}{c^2} I(\kappa) + J(\kappa), \tag{5.3}$$

where $I(\kappa) = \oint_{\Gamma(\kappa)} y^2\, {\rm d}\zeta$ and $J(\kappa) = \oint_{\Gamma(\kappa)} \left(\frac{9}{2}z^2 - 9z + 3\right) y^2\, {\rm d}\zeta$.

LEMMA 5.4. *For any* $\kappa \in \left(0, 1 - \frac{\sqrt{3}}{3}\right) \cup \left(1 + \frac{\sqrt{3}}{3}, 2\right)$, *there exists a positive root* $c = c(\kappa)$ *of* $M_{hom}(c,\kappa) = 0$. *Moreover,* $\frac{\partial M_{hom}}{\partial c} \mid_{c=c(\kappa)} \neq 0$.

Proof. Obviously, we know that $I(\kappa) = \oint_{\Gamma(\kappa)} y^2\, {\rm d}\zeta > 0$. Because $J(\kappa)$ integrates along the homoclinic orbit $\Gamma(\kappa)$, then

$$y'' = \left(\frac{9}{2}z^2 - 9z + 3\right) z' = \left(\frac{9}{2}z^2 - 9z + 3\right) y, \tag{5.4}$$

which yields

$${\rm d}y' = \left(\frac{9}{2}z^2 - 9z + 3\right) z'\, {\rm d}\zeta = \left(\frac{9}{2}z^2 - 9z + 3\right) y\, {\rm d}\zeta. \tag{5.5}$$

By integration of parts, it is not difficult to have

$$\begin{aligned} J(\kappa) &= \oint_{\Gamma(\kappa)} (\frac{9}{2}z^2 - 9z + 3) y^2\, {\rm d}\zeta = \oint_{\Gamma(\kappa)} y\, {\rm d}y' \\ &= -\int_{\mathbb{R}} (y')^2\, {\rm d}\zeta < 0. \end{aligned} \tag{5.6}$$

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Hence, $-\frac{I(\kappa)}{J(\kappa)} > 0$, there exists a single positive root $c = c(\kappa)$ of $M_{\text{hom}}(c,\kappa) = 0$ that

$$c(\kappa) = \sqrt{-\frac{I(\kappa)}{J(\kappa)}}. \tag{5.7}$$

On the other hand, it follows from (5.3) that

$$\frac{\partial M_{\text{hom}}(c,\kappa)}{\partial c}\ |_{c=c(\kappa)} = -2\frac{1}{c^3}I(\kappa) = -2J(\kappa)\sqrt{-\frac{J(\kappa)}{I(\kappa)}} > 0. \tag{5.8}$$

The proof is completed. □

In fact, the analytical expressions of $I(\kappa)$ and $J(\kappa)$ can be computed by dividing them into two cases.

Case (I): $1 + \frac{\sqrt{3}}{3} < \kappa < 2$.

Firstly, since $I(\kappa)$ and $J(\kappa)$ are integrated over a closed curve, and the time variable ζ can be represented by the state variable z on homoclinic orbit. Thus, by (4.5), we have

$$\begin{aligned} I(\kappa) &= \oint_{\Gamma(\kappa)} y^2\,\mathrm{d}\zeta = \oint_{\Gamma(\kappa)} y\,\mathrm{d}z \\ &= 2\int_{z_+}^{\kappa} \sqrt{\frac{3}{4}z^4 - 3z^3 + 3z^2 - (3\kappa^3 - 9\kappa^2 + 6\kappa)z + \frac{9}{4}\kappa^4 - 6\kappa^3 + 3\kappa^2}\,\mathrm{d}z \\ &= 2\int_{z_+}^{\kappa} \sqrt{\frac{3}{4}(z-\kappa)^2(z-z_+)(z-z_-)}\,\mathrm{d}z \\ &= \sqrt{3}\left[\int_{z_+}^{\kappa} \kappa\sqrt{(z-z_+)(z-z_-)}\,\mathrm{d}z - \int_{z_+}^{\kappa} z\sqrt{(z-z_+)(z-z_-)}\,\mathrm{d}z\right] \\ &= \sqrt{3}\left[\kappa I_1(\kappa) - I_2(\kappa)\right], \end{aligned} \tag{5.9}$$

where $I_1(\kappa) = \int_{z_+}^{\kappa} \sqrt{(z-z_+)(z-z_-)}\,\mathrm{d}z$ and $I_2(\kappa) = \int_{z_+}^{\kappa} z\sqrt{(z-z_+)(z-z_-)}\,\mathrm{d}z$.

Note that the formulas 2.261, 2.262 (1) and 2.262 (2) in handbook [17] read:

$$\begin{aligned} \int \frac{\mathrm{d}x}{\sqrt{R}} &= \frac{1}{\sqrt{c_1}}\ln(2\sqrt{c_1 R} + 2c_1 x + b), \quad (c_1 > 0,\ 2c_1 x + b > \sqrt{-\Delta},\ \Delta < 0), \\ \int \sqrt{R}\,\mathrm{d}x &= \frac{(2c_1 x + b)\sqrt{R}}{4c_1} + \frac{\Delta}{8c_1}\int \frac{\mathrm{d}x}{\sqrt{R}}, \\ \int x\sqrt{R}\,\mathrm{d}x &= \frac{\sqrt{R^3}}{3c_1} - \frac{(2c_1 x + b)b\sqrt{R}}{8c_1^2} - \frac{b\Delta}{16c_1^2}\int \frac{\mathrm{d}x}{\sqrt{R}}, \end{aligned} \tag{5.10}$$

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where $R = a + bx + c_1x^2$ and $\Delta = 4ac_1 - b^2$. Combining (5.9) and (5.10), it follows that

$$\begin{aligned} I(\kappa) = \frac{\sqrt{3}}{3}\Big[& 6\ln 2\kappa^3 + 6\kappa^3\ln\left(\sqrt{6\kappa^2 - 12\kappa + 4} + 2\kappa - 2\right) - 18\ln 2\kappa^2 \\ & - 18\kappa^2\ln\left(\sqrt{6\kappa^2 - 12\kappa + 4} + 2\kappa - 2\right) + 12\ln 2\kappa \\ & + 12\kappa\ln\left(\sqrt{6\kappa^2 - 12\kappa + 4} + 2\kappa - 2\right) + 2\sqrt{6\kappa^2 - 12\kappa + 4}\Big] \\ & - \sqrt{3}\kappa\left[\kappa^2\ln(2\kappa - \kappa^2) + 3\ln 2\kappa^2 - 3\kappa\ln(2\kappa - \kappa^2) - 9\ln 2\kappa \right. \\ & \left. + 2\ln(2\kappa - \kappa^2) + 6\ln 2\right]. \end{aligned} \tag{5.11}$$

Secondly,

$$\begin{aligned} J(\kappa) &= \oint_{\Gamma(\kappa)} \left(\frac{9}{2}z^2 - 9z + 3\right) y^2 \,\mathrm{d}\zeta = \oint_{\Gamma(\kappa)} \left(\frac{9}{2}z^2 - 9z + 3\right) y \,\mathrm{d}z \\ &= 2\int_{z_+}^{\kappa} \left(\frac{9}{2}z^2 - 9z + 3\right) \\ &\quad \sqrt{\frac{3}{4}z^4 - 3z^3 + 3z^2 - (3\kappa^3 - 9\kappa^2 + 6\kappa)z + \frac{9}{4}\kappa^4 - 6\kappa^3 + 3\kappa^2}\,\mathrm{d}z \\ &= \sqrt{3}\int_{z_+}^{\kappa} \left(\frac{9}{2}z^2 - 9z + 3\right) \sqrt{(\kappa - z)^2(z - z_+)(z - z_-)}\,\mathrm{d}z \\ &= \sqrt{3}\int_{z_+}^{\kappa} \left[-\frac{9}{2}z^3 + \left(\frac{9}{2}\kappa + 9\right)z^2 - (9\kappa + 3)z + 3\kappa\right] \sqrt{(z - z_+)(z - z_-)}\,\mathrm{d}z \\ &= \sqrt{3}\left[-\frac{9}{2}J_1(\kappa) + \left(\frac{9}{2}\kappa + 9\right)J_2(\kappa) - (9\kappa + 3)I_2(\kappa) + 3\kappa I_1(\kappa)\right], \end{aligned} \tag{5.12}$$

where $J_1(\kappa) = \int_{z_+}^{\kappa} z^3\sqrt{(z - z_+)(z - z_-)}\,\mathrm{d}z,\ J_2(\kappa) = \int_{z_+}^{\kappa} z^2\sqrt{(z - z_+)(z - z_-)}\,\mathrm{d}z$.

Similarly, by the formulas 2.262 (3) and 2.262 (4) in [17], they are expressed by:

$$\begin{aligned} \int x^2\sqrt{R}\,\mathrm{d}x &= \left(\frac{x}{4c_1} - \frac{5b}{24c_1^2}\right)\sqrt{R^3} + \left(\frac{5b^2}{16c_1^2} - \frac{a}{4c_1}\right)\frac{(2c_1x + b)\sqrt{R}}{4c_1} \\ &\quad + \left(\frac{5b^2}{16c_1^2} - \frac{a}{4c_1}\right)\frac{\Delta}{8c_1}\int\frac{\mathrm{d}x}{\sqrt{R}}, \\ \int x^3\sqrt{R}\,\mathrm{d}x &= \left(\frac{x^2}{5c_1} - \frac{7bx}{40c_1^2} + \frac{7b^2}{48c_1^3} - \frac{2a}{15c_1^2}\right)\sqrt{R^3} - \left(\frac{7b^3}{32c_1^3} - \frac{3ab}{8c_1^2}\right)\frac{(2c_1x + b)\sqrt{R}}{4c_1} \\ &\quad - \left(\frac{7b^3}{32c_1^3} - \frac{3ab}{8c_1^2}\right)\frac{\Delta}{8c_1}\int\frac{\mathrm{d}x}{\sqrt{R}}. \end{aligned} \tag{5.13}$$

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Using (5.10), (5.12) and (5.13), we have

$$J(\kappa) = -\frac{\sqrt{3}}{5}\Big[27\kappa^4\sqrt{6\kappa^2-12\kappa+4}+15\kappa^3\ln(2\kappa-\kappa^2)-198\kappa^3\sqrt{6\kappa^2-12\kappa+4}$$
$$-30\kappa^3\ln\left(\sqrt{6\kappa^2-12\kappa+4}+2\kappa-2\right)+15\ln 2\kappa^3-45\kappa^2\ln(2\kappa-\kappa^2)$$
$$+90\kappa^2\ln\left(\sqrt{6\kappa^2-12\kappa+4}+2\kappa-2\right)+30\sqrt{2}\kappa\sqrt{(3\kappa^2-6\kappa+2)^3}$$
$$+294\kappa^2\sqrt{6\kappa^2-12\kappa+4}-45\ln 2\kappa^2+30\kappa\ln(2\kappa-\kappa^2)$$
$$-60\kappa\ln\left(\sqrt{6\kappa^2-12\kappa+4}+2\kappa-2\right)+10\sqrt{2}\sqrt{(3\kappa^2-6\kappa+2)^3}$$
$$-72\kappa\sqrt{6\kappa^2-12\kappa+4}+30\ln 2\kappa-18\sqrt{6\kappa^2-12\kappa+4}\Big]. \tag{5.14}$$

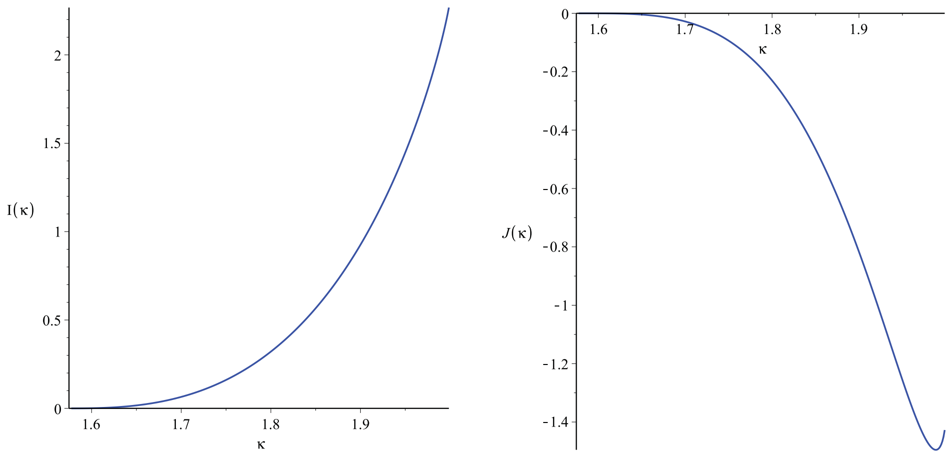
Case (II): $0<\kappa<1-\frac{\sqrt{3}}{3}$.

Same calculation method as case (I), we obtain

$$I(\kappa)=\oint_{\Gamma(\kappa)} y^2\,\mathrm{d}\zeta=\oint_{\Gamma(\kappa)} y\,\mathrm{d}z=2\int_{z_-}^{\kappa}\cdots\,\mathrm{d}z$$
$$=\frac{\sqrt{3}}{3}\Big[6\ln 2\kappa^3+6\kappa^3\ln\left(\sqrt{6\kappa^2-12\kappa+4}+2\kappa-2\right)-18\ln 2\kappa^2$$
$$-18\kappa^2\ln\left(\sqrt{6\kappa^2-12\kappa+4}+2\kappa-2\right)+12\ln 2\kappa$$
$$+12\kappa\ln\left(\sqrt{6\kappa^2-12\kappa+4}+2\kappa-2\right)+2\sqrt{6\kappa^2-12\kappa+4}\Big]$$
$$-\sqrt{3}\kappa\Big[2\kappa^2\ln\left(-\sqrt{2\kappa-\kappa^2}\right)+3\ln 2\kappa^2-6\kappa\ln\left(-\sqrt{2\kappa-\kappa^2}\right)-9\ln 2\kappa$$
$$+4\ln\left(-\sqrt{2\kappa-\kappa^2}\right)+6\ln 2\Big], \tag{5.15}$$

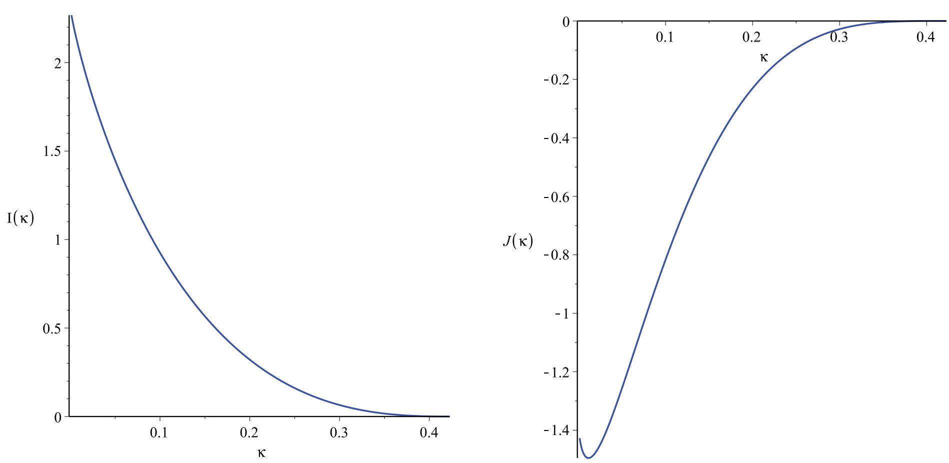
and

$$J(\kappa)=\oint_{\Gamma(\kappa)}\left(\frac{9}{2}z^2-9z+3\right)y^2\,\mathrm{d}\zeta=\oint_{\Gamma(\kappa)}\left(\frac{9}{2}z^2-9z+3\right)y\,\mathrm{d}z=2\int_{z_-}^{\kappa}\cdots\mathrm{d}z$$
$$=-\frac{\sqrt{3}}{5}\Big[27\kappa^4\sqrt{6\kappa^2-12\kappa+4}+30\kappa^3\ln\left(-\sqrt{2\kappa-\kappa^2}\right)+60\kappa\ln\left(-\sqrt{2\kappa-\kappa^2}\right)$$
$$-30\kappa^3\ln\left(\sqrt{6\kappa^2-12\kappa+4}+2\kappa-2\right)+15\ln 2\kappa^3-90\kappa^2\ln\left(-\sqrt{2\kappa-\kappa^2}\right)$$
$$+90\kappa^2\ln\left(\sqrt{6\kappa^2-12\kappa+4}+2\kappa-2\right)+30\sqrt{2}\kappa\sqrt{(3\kappa^2-6\kappa+2)^3}$$
$$+294\kappa^2\sqrt{6\kappa^2-12\kappa+4}-45\ln 2\kappa^2-198\kappa^3\sqrt{6\kappa^2-12\kappa+4}$$
$$-60\kappa\ln\left(\sqrt{6\kappa^2-12\kappa+4}+2\kappa-2\right)+10\sqrt{2}\sqrt{(3\kappa^2-6\kappa+2)^3}$$
$$-72\kappa\sqrt{6\kappa^2-12\kappa+4}+30\ln 2\kappa-18\sqrt{6\kappa^2-12\kappa+4}\Big]. \tag{5.16}$$



(a) (b)

Figure 6. The algebraic curves of $I(\kappa)$ and $J(\kappa)$ with respect to κ for $1+\frac{\sqrt{3}}{3}<\kappa<2$. (a) $(\kappa,\, I(\kappa))$ (b) $(\kappa,\, J(\kappa))$.



(a) (b)

Figure 7. The algebraic curves of $I(\kappa)$ and $J(\kappa)$ with respect to κ for $0<\kappa<1-\frac{\sqrt{3}}{3}$. (a) $(\kappa,\, I(\kappa))$ (b) $(\kappa,\, J(\kappa))$.

We plot the algebraic curve of $I(\kappa)$ with respect to κ and $J(\kappa)$ with respect to κ, respectively. (see figures 6 and 7)

We can see that $I(\kappa)>0$ and $J(\kappa)<0$ for $1+\frac{\sqrt{3}}{3}<\kappa<2$ in figure 6, $I(\kappa)>0$ and $J(\kappa)<0$ for $0<\kappa<1-\frac{\sqrt{3}}{3}$ in figure 7. It is shown that a good agreement with the proof of lemma 5.4.

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Hence, we substitute (5.11) and (5.14) into (5.3), and the expression of the homoclinic Melnikov function of system (3.16) for $1+\frac{\sqrt{3}}{3}<\kappa<2$ can be rewritten as

$$\begin{aligned}
&M_{\text{hom}}(c,\kappa)\Big|_{\kappa\in\left(1+\frac{\sqrt{3}}{3},2\right)}\\
&=\frac{1}{c^2}\frac{\sqrt{3}}{3}\left[6\ln 2\kappa^3+6\kappa^3\ln\left(\sqrt{6\kappa^2-12\kappa+4}+2\kappa-2\right)-18\ln 2\kappa^2\right.\\
&\quad-18\kappa^2\ln\left(\sqrt{6\kappa^2-12\kappa+4}+2\kappa-2\right)+12\ln 2\kappa\\
&\quad\left.+12\kappa\ln\left(\sqrt{6\kappa^2-12\kappa+4}+2\kappa-2\right)+2\sqrt{6\kappa^2-12\kappa+4}\right]\\
&\quad-\frac{1}{c^2}\sqrt{3}\kappa\left[\kappa^2\ln(2\kappa-\kappa^2)+3\ln 2\kappa^2-3\kappa\ln(2\kappa-\kappa^2)-9\ln 2\kappa\right.\\
&\quad\left.+2\ln(2\kappa-\kappa^2)+6\ln 2\right]-\frac{\sqrt{3}}{5}\left[27\kappa^4\sqrt{6\kappa^2-12\kappa+4}\right.\\
&\quad+15\kappa^3\ln(2\kappa-\kappa^2)-198\kappa^3\sqrt{6\kappa^2-12\kappa+4}\\
&\quad-30\kappa^3\ln\left(\sqrt{6\kappa^2-12\kappa+4}+2\kappa-2\right)+15\ln 2\kappa^3-45\kappa^2\ln(2\kappa-\kappa^2)\\
&\quad+90\kappa^2\ln\left(\sqrt{6\kappa^2-12\kappa+4}+2\kappa-2\right)+30\sqrt{2}\kappa\sqrt{(3\kappa^2-6\kappa+2)^3}\\
&\quad+294\kappa^2\sqrt{6\kappa^2-12\kappa+4}-45\ln 2\kappa^2+30\kappa\ln(2\kappa-\kappa^2)\\
&\quad-60\kappa\ln\left(\sqrt{6\kappa^2-12\kappa+4}+2\kappa-2\right)+10\sqrt{2}\sqrt{(3\kappa^2-6\kappa+2)^3}\\
&\quad\left.-72\kappa\sqrt{6\kappa^2-12\kappa+4}+30\ln 2\kappa-18\sqrt{6\kappa^2-12\kappa+4}\right].
\end{aligned}\tag{5.17}$$

It is easy to verify that for $\kappa\in\left(1+\frac{\sqrt{3}}{3},2\right)$, there is $c=c_1(\kappa)$ such that $M_{\text{hom}}(c,\kappa)\Big|_{\kappa\in\left(1+\frac{\sqrt{3}}{3},2\right)}=0$.

Similarly, the expression of the homoclinic Melnikov function of system (3.16) for $0<\kappa<1-\frac{\sqrt{3}}{3}$ is

$$\begin{aligned}
&M_{\text{hom}}(c,\kappa)\Big|_{\kappa\in\left(0,1-\frac{\sqrt{3}}{3}\right)}\\
&=\frac{1}{c^2}\frac{\sqrt{3}}{3}\left[6\ln 2\kappa^3+6\kappa^3\ln\left(\sqrt{6\kappa^2-12\kappa+4}+2\kappa-2\right)-18\ln 2\kappa^2\right.\\
&\quad-18\kappa^2\ln\left(\sqrt{6\kappa^2-12\kappa+4}+2\kappa-2\right)+12\ln 2\kappa\\
&\quad\left.+12\kappa\ln\left(\sqrt{6\kappa^2-12\kappa+4}+2\kappa-2\right)+2\sqrt{6\kappa^2-12\kappa+4}\right]\\
&\quad-\frac{1}{c^2}\sqrt{3}\kappa\left[2\kappa^2\ln\left(-\sqrt{2\kappa-\kappa^2}\right)+3\ln 2\kappa^2-6\kappa\ln\left(-\sqrt{2\kappa-\kappa^2}\right)-9\ln 2\kappa\right.
\end{aligned}$$

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$$+4\ln\left(-\sqrt{2\kappa-\kappa^2}\right)+6\ln 2\Big]-\frac{\sqrt{3}}{5}\Big[27\kappa^4\sqrt{6\kappa^2-12\kappa+4}$$
$$+30\kappa^3\ln\left(-\sqrt{2\kappa-\kappa^2}\right)+60\kappa\ln\left(-\sqrt{2\kappa-\kappa^2}\right)+15\ln 2\kappa^3$$
$$-30\kappa^3\ln\left(\sqrt{6\kappa^2-12\kappa+4}+2\kappa-2\right)-90\kappa^2\ln\left(-\sqrt{2\kappa-\kappa^2}\right)$$
$$+90\kappa^2\ln\left(\sqrt{6\kappa^2-12\kappa+4}+2\kappa-2\right)+30\sqrt{2}\kappa\sqrt{(3\kappa^2-6\kappa+2)^3}$$
$$+294\kappa^2\sqrt{6\kappa^2-12\kappa+4}-45\ln 2\kappa^2-198\kappa^3\sqrt{6\kappa^2-12\kappa+4}$$
$$-60\kappa\ln\left(\sqrt{6\kappa^2-12\kappa+4}+2\kappa-2\right)+10\sqrt{2}\sqrt{(3\kappa^2-6\kappa+2)^3}$$
$$-72\kappa\sqrt{6\kappa^2-12\kappa+4}+30\ln 2\kappa-18\sqrt{6\kappa^2-12\kappa+4}\,\Big]. \tag{5.18}$$

It is easy to verify that for $\kappa\in\left(0,1-\frac{\sqrt{3}}{3}\right)$, there is $c=c_2(\kappa)$ such that $M_{\text{hom}}(c,\kappa)\Big|_{\kappa\in\left(0,1-\frac{\sqrt{3}}{3}\right)}=0$.

Secondly, the Melnikov method is also employed to detect the existence of kink (anti-kink) wave solutions under small perturbation. The heteroclinic Melnikov function of system (3.16) is defined as follows:

$$\begin{aligned}
M_{\text{het}}(c,\kappa)&=\int_{\Upsilon(\kappa)}\left(\frac{9}{2}z^2-9z+3+\frac{1}{c^2}\right)y^2\,\mathrm{d}\zeta\\
&=\int_{-\infty}^{+\infty}\left(\frac{9}{2}z^2-9z+3+\frac{1}{c^2}\right)y^2\,\mathrm{d}\zeta\\
&=\int_0^2\left(\frac{9}{2}z^2-9z+3+\frac{1}{c^2}\right)y\,\mathrm{d}z\\
&=\frac{\sqrt{3}}{2}\int_0^2\left(\frac{9}{2}z^2-9z+3+\frac{1}{c^2}\right)z(2-z)\,\mathrm{d}z=-\frac{2\sqrt{3}(3c^2-5)}{15c^2}.
\end{aligned}\tag{5.19}$$

LEMMA 5.5. *For $\kappa=0$ or $\kappa=2$, $M_{het}(c,\kappa)=0$ has a positive root $c^*=\frac{\sqrt{15}}{3}$. Moreover, $\frac{\partial M_{het}}{\partial c}\,|_{c=c^*}\neq 0$.*

Proof. Clearly, it directly follows form (5.19) that $M_{\text{het}}(c,\kappa)=0$ has a positive root $c^*=\frac{\sqrt{15}}{3}$. Meanwhile,

$$\frac{\partial M_{\text{het}}}{\partial c}=-\frac{4\sqrt{3}}{5c}+\frac{4\sqrt{3}(3c^2-5)}{15c^3},$$

and

$$\frac{\partial M_{\text{het}}}{\partial c}\,|_{c=c^*}=-\frac{12\sqrt{5}}{25}\neq 0.$$

□

Proof of theorem 5.1. By lemmas 5.4 and 5.5, theorem 5.1 follows immediately. □

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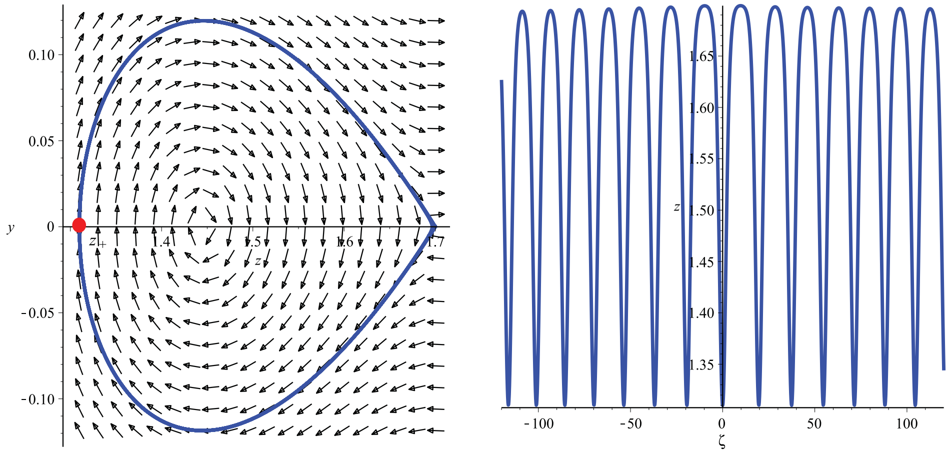

(a) (b)

Figure 8. (a) Phase portraits with $c = c(1.7)$, (b) Time history curves of (ζ, z) for system (3.16) for $\epsilon = 0.01$, $\kappa = 1.7$ and initial value $(z(0), y(0)) = (z_+, 0) = (1.309950494, 0)$. (a) $c = c(1.7)$ (b) (ζ, z).

6. Numerical analysis

Numerical simulations are employed to confirm the theoretical results derived in previous sections. Here, maple software 18.0 is used.

Firstly, we simulate the existence of solitary wave solution of perturbed $(1 + 1)$-dimensional DLWE (1.3). Let $\kappa = 1.7$ and $\epsilon = 0.01$. By (5.11) and (5.14), we obtain $I(1.7) \approx 0.0648882124$ and $J(1.7) \approx -0.02790800120$ such that $c(1.7) = \sqrt{-\frac{I(1.7)}{J(1.7)}} \approx 1.524819859$. Then, according to theorem 5.1, we take $c = c(1.7)$ and let the initial value be $(z(0), y(0)) = (z_+, 0) = (1.309950494, 0)$ (red point) which the homoclinic orbit would pass through. The phase portraits (z, y) and time history curves (ζ, z) of system (3.16) are plotted in figure 8.

One can see that the homoclinic orbit and dark solitary wave solution of system (3.16) still persist after a small singular perturbation.

Secondly, set $\kappa = 0.3$ and $\epsilon = 0.01$. By (5.15) and (5.16), we have $I(0.3) \approx 0.0648882229$ and $J(0.3) \approx -0.02790784012$ such that $c(0.3) = \sqrt{-\frac{I(0.3)}{J(0.3)}} \approx 1.524824377$. Taking $c = c(0.3)$ and initial value to be $(z(0), y(0)) = (z_-, 0) = (0.690049506, 0)$ (red point) which the homoclinic orbit would pass through. The phase portraits (z, y) and time history curves (ζ, z) of system (3.16) are plotted in figure 9.

We see that the homoclinic orbit and bright solitary wave solution of system (3.16) still exist after a small singular perturbation which verifies the theorem 5.1(i) and (ii).

Next, let us to see the existence of kink and anti-kink wave solution of system (3.16). Giving $\kappa = 0$ or $\kappa = 2$, $\epsilon = 0.01$ and initial value $(z(0), y(0)) = \left(1, \pm\frac{\sqrt{3}}{2}\right)$

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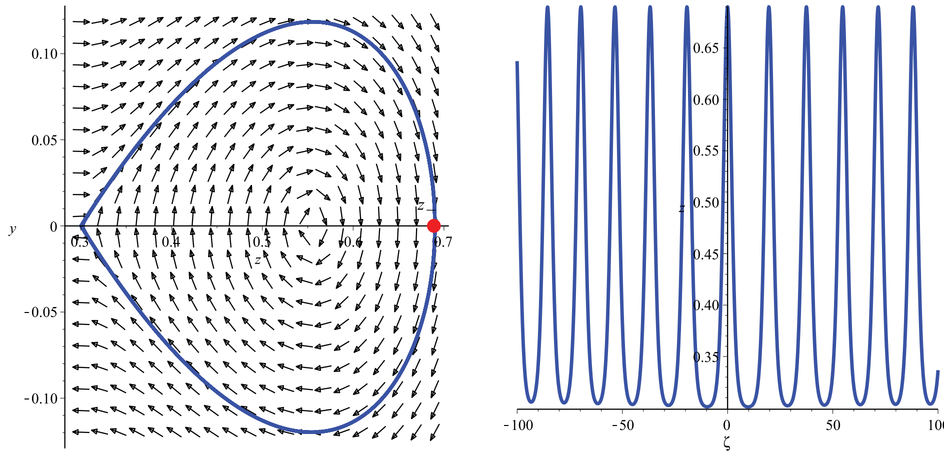
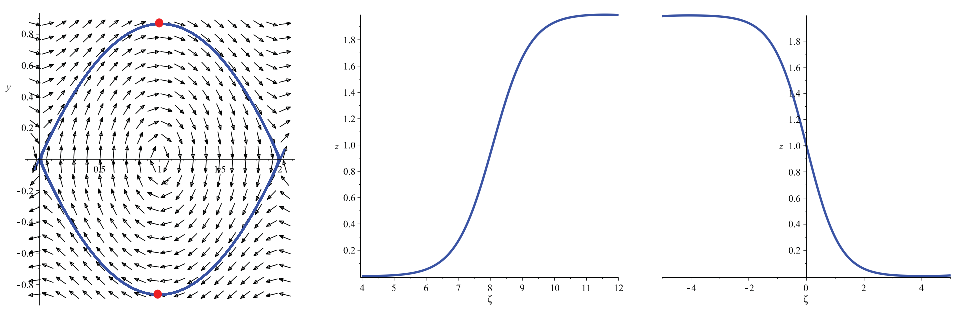


(a) (b)

Figure 9. (a) Phase portraits with $c = c(0.3)$, (b) Time history curves of (ζ, z) for system (3.16) for $\epsilon = 0.01$, $\kappa = 0.3$ and initial value $(z(0), y(0)) = (z_{-}, 0) = (0.690049506, 0)$. (a) $c = c(0.3)$ (b) (ζ, z).



(a) (b) (c)

Figure 10. Phase portraits with $c = \frac{\sqrt{15}}{3}$ for $\epsilon = 0.01$, initial value $(z(0), y(0)) = \left(1, \pm\frac{\sqrt{3}}{2}\right)$. (a) $c = \frac{\sqrt{15}}{3}$ (b) Kink wave (c) Anti-kink wave.

(red point). We draw the phase portraits of system (3.16) after taking $c = \frac{\sqrt{15}}{3}$ in figure 10. Form the figure 10(a), it is shown that the heteroclinic orbits still persist. Correspondingly, a pair of kink and anti-kink wave solutions exists [see figures 10(b) and (c)], which is consistent with theorem 5.1(iii).

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Conflict of Interest

The authors declare that they have no conflict of interest.

Data

No data was used for the research in this article.

Author contributions

All the authors have same contributions to the paper.

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