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LOCAL ENERGY DECAYS FOR WAVE EQUATIONS WITH TIME-DEPENDENT COEFFICIENTS

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§ 0. Introduction

We consider the decay of the local energy for the following equation in three dimension:

$$(0.1)\qquad \begin{gathered} u_{tt} + bu_t - \Delta u = 0 \\ u(0, x) = f(x) \quad \text{and} \quad u_t(0, x) = g(x) \,. \end{gathered}$$

Here we make the following assumption on $b = b(t, x)$:

ASSUMPTION (A). (i) $b(t, x)$ is a bounded smooth function. (ii) $b(t, x)$ is non-negative. (iii) For each $t \geq 0$,

$$(0.2)\qquad \begin{gathered} \text{the support of } b(t, x) \text{ is contained in } \{x \,|\, |x| \leq (t + \gamma)^\alpha\}, \\ 0 \leq \alpha < 1, \gamma > 1. \end{gathered}$$

(Throughout this paper the constants α and γ are used with the meaning ascribed here.)

The condition $0 \leq \alpha < 1$ means that the support of $b(t, x)$ expands at a speed strictly less than the wave speed. Therefore, it is expected that the local energy for solutions of problem (0.1) with initial data of compact support decays rapidly as $t \to \infty$. The purpose of this paper is to give a partial answer to this problem.

The problem of the decay of the local energy for wave equations with time-dependent coefficients or with moving obstacles has been studied in Bloom and Kazarinoff [1], Cooper [2] and Cooper and Strauss [3], etc. In their works it has been assumed that coefficients are asymptotically stationary or that obstacles remain in a fixed bounded region for $t \geq 0$.

Now we shall state the main theorem.

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MAIN THEOREM. *Suppose that Assumption* (A) *is satisfied and that* $0 \leq \alpha < \frac{1}{2}$. *Let* u *be a smooth solution of problem* (0.1) *with the initial data* f *and* g $(\in C_0^\infty(R^3))$ *such that the support of* f *and* g *is contained in* $|x| < \gamma^\alpha$. *Then, there exist constants* θ *and* β, $0 < \beta \leq 1$, *such that the local energy for the solution* u *decays at the rate of* $\exp(-\theta t^\beta)$ *as* $t \to \infty$.

The explicit expression of the constant β will be given in the proof of this theorem (§2).

Remark. The above result is valid for a weak solution with $f \in H^1(R^3)$ and $g \in L^2(R^3)$.

Next we consider the exterior problem with Dirichlet boundary conditions. Let $\mathscr{E}$ be a domain exterior to a star-shaped bounded domain with smooth boundary and let u be a solution of the following equation:

$$(0.3) \qquad u_{tt} + bu_t - \Delta u = 0 \quad \text{in } (0, \infty) \times \mathscr{E}$$

$$(0.4) \quad u(t, x) = 0 \quad \text{on} \quad (0, \infty) \times \partial\mathscr{E}, \ \partial\mathscr{E} \text{ being the boundary of } \mathscr{E}.$$

$$(0.5) \qquad u(0, x) = f(x), \qquad u_t(0, x) = g(x).$$

Here $b(t, x)$ satisfies Assumption (A). Then the same result as Main Theorem holds. Since the proof for the exterior problem is done with a slight modification of the proof for the whole space problem, we consider only the whole space problem in this paper. The method presented here will be useful for the problem with expanding obstacles with time and details will be discussed in the next papere.

The proof of Main Theorem is done by a generalization of the method used in Morawetz [5]. In §1 we show the uniform decay of order $t^{-\mu}$, $\mu > 0$, and in §2 we prove Main Theorem. In §3 we show that our method can be applied to wave equations with potentials of a special form.

Finally we note the following facts: (a) The symbols $C, C_1, C_2, \cdots$ are used to denote (unessential) positive constants which are not necessarily the same. (b) Integration with no domain attached is taken over the whole space. (c) We use the summation convention. (d) All the functions considered here are real-valued.

§1. Uniform decay

Let $s \geq 0$ be fixed and let $v(t;s)$ be a smooth solution of the equation

$$v_{tt} + b(t;s)v_t - v_{jj} = 0 \tag{1.1}$$

with the initial data $v(0;s)$ and $v_t(0;s)$ of compact support, where $v_{jj} = \Delta v$, $b(t;s) = b(t+s,x)$, $b(t,x)$ being the function in equation (0.1), and by (0.2)

(1.2) the support of $b(t;s)$ is contained in $\{x \mid |x| < (t+s+\gamma)^\alpha\}$ for each $t \geq 0$.

It is convenient to introduce the following notation:

$$E(v;h,T,s) = \int_{|x|<h} (|v_t(T;s)|^2 + |\nabla v(T;s)|^2)dx$$

for $0 < h \leq \infty$.

LEMMA 1.1. *Let $v(t;s)$ be a solution of problem* (1.1). *Then,*

$$E(v;\infty,T,s) \leq E(v;\infty,0,s) \tag{1.3}$$

for each $T \geq 0$, *and*

$$\int_0^\infty \int b(t;s)|v_t(t;s)|^2 \, dxdt \leq \tfrac{1}{2}E(v;\infty,0,s) \,. \tag{1.4}$$

Proof. We multiply the equation (1.1) by v_t. Then we have

$$\tfrac{1}{2}(v_t^2)_t + b(t;s)v_t^2 - (v_jv_t)_j + \tfrac{1}{2}(v_j^2)_t = 0 \,.$$

Integrating this identity over $R^3 \times (0,T)$, we easily obtain the conclusion.

We use the next identities.

LEMMA 1.2 (cf. Strauss [6], Lemma 1). *Let $\zeta(r)$ be a smooth function of $r = |x|$ and let $\chi_i = \zeta(r)\dfrac{x_i}{r}$. Then the equation*

$$(u_{tt} + bu_t - u_{jj})(2\chi_iu_i + \chi_{ii}u) = X_t(u;t) + \nabla\cdot Y(u) + Z(u)^{1)} \tag{1.5}$$

holds, where

1) $X_t(u;t) = \dfrac{\partial}{\partial t}X(u;t)$ and $Y(u) = (Y_1(u), Y_2(u), Y_3(u))$. The same notation will be used in what follows.

$$X(u;t) = u_t(2\chi_i u_i + \chi_{ii}u)$$
$$Y_j(u) = -u_j(2\chi_i u_i + \chi_{ii}u) + \chi_j(|\nabla u|^2 - u_t^2) + \tfrac{1}{2}\chi_{iij}u^2$$
$$Z(u) = 2\chi_{ij}u_i u_j - \tfrac{1}{2}\chi_{iijj}u^2 + 2bu_t\chi_i u_i + bu_t\chi_{ii}u .$$

LEMMA 1.3 (cf. Lax and Phillips [4], Appendix 3). *The equation*

$$(1.6)\qquad \begin{aligned}&(u_{tt} + bu_t - u_{jj})((r^2 + t^2)u_t + 2tru_r + 2tu)\\ &\quad = F_t(u;t) + \nabla\cdot G(u) + H(u)\end{aligned}$$

holds, where

$$\begin{aligned}F(u;t) &= \tfrac{1}{2}(r^2 + t^2)(|\nabla u|^2 + u_t^2) + 2tru_r u_t + 2tu_t u\\ &\quad + r^{-2}(r^2 + t^2)((\nabla u\cdot x)u + \tfrac{1}{2}u^2)\\ G_j(u) &= -u_j((r^2 + t^2)u_t + 2tru_r + 2tu) + x_j t(|\nabla u|^2 - u_t^2)\\ &\quad - \tfrac{1}{2}r^{-2}((r^2 + t^2)u^2)_t x_j\\ H(u) &= (r^2 + t^2)bu_t^2 + 2trbu_t u_r + 2tbu_t u ,\end{aligned}$$

and $x = (x_1, x_2, x_3)$ is a position vector.

LEMMA 1.4. *Let $0 < \delta \leq 1$ and let $v(t;s)$ be a solution of problem* (1.1). *Then, there exists a constant C independent of T and s such that for $T \geq 1$,*

$$\int_0^T\!\!\int (1 + r)^{-1-\delta}|v_r(t;s)|^2\,dxdt + \int_0^T\!\!\int (1 + r)^{-3-\delta}|v(t;s)|^2\,dxdt \leq C(T + s)^{\alpha(1+\delta)}E(v;\infty,0,s) ,$$

where the constant C depends on δ and the bound of $b(t,x)$.

Proof. We use Lemma 1.2 with $\zeta(r) = 1 - (1 + r)^{-\delta}$. Then we note the following facts:

$$(1.7)\qquad \zeta(r) \geq 0 \quad\text{and}\quad \zeta(r) \leq \delta r \qquad \text{for } r \leq 1 .$$

$$(1.8)\qquad \chi_{ij}v_i v_j = \frac{\zeta(r)}{r}(|\nabla v|^2 - v_r^2) + \zeta_r(r)v_r^2 \geq \delta(1 + r)^{-1-\delta}v_r^2 .$$

$$(1.9)\qquad \chi_{iij} = \left(\zeta_{rr}(r) + \frac{2}{r}\zeta_r(r) - \frac{2}{r^2}\zeta(r)\right)\frac{x_j}{r} .$$

$$(1.10)\qquad \chi_{iijj} = \zeta_{rrr}(r) + \frac{4}{r}\zeta_{rr}(r) \leq -\delta(1 + \delta)(1 + r)^{-3-\delta} .$$

We integrate the identity (1.5) with $u = v(t;s)$ and $b = b(t;s)$ over

$\{x \mid |x| \geq \varepsilon\} \times (0, T)$, ε, $\varepsilon > 0$, being arbitrary (small enough), we have

$$\int_{|x|>\varepsilon} (X(v\,;T) - X(v\,;0))dx + \int_0^T \int_{|x|=\varepsilon} (Y_j(v)\cdot n_j)dSdt + \int_0^T \int_{|x|>\varepsilon} Z(v)dxdt = 0\,,$$

where $n = (n_1, n_2, n_3)$ denotes the unit exterior normal to the domain $\{x \mid |x| > \varepsilon\}$. By virtue of (1.7) and (1.9), the second term tends to zero as $\varepsilon \to 0$, and taking account of (1.8) and (1.10), we obtain

$$\begin{aligned}\int_0^T 2\delta(1+r)^{-1-\delta}v_r^2dxdt &+ \frac{1}{2}\int_0^T\int \delta(1+\delta)(1+r)^{-3-\delta}v^2dxdt \\ &\leq \int |X(v\,;T)|\,dx + \int |X(v\,;0)|\,dx - 2\int_0^T\int b(t\,;s)\zeta(r)v_tv_rdxdt \\ &\quad - \int_0^T\int b(t\,;s)\Big(\zeta_r(r) + \frac{2}{r}\zeta(r)\Big)v_tvdxdt\,,\end{aligned}$$

since $\chi_iv_i = \zeta(r)v_r$ and $\chi_{ii} = \zeta_r(r) + \dfrac{2}{r}\zeta(r)$. Furthermore, since $|\chi_{ii}| \leq C_1(1+r)^{-1}$ for some C_1 by (1.7), we make use of Lemma 1.1 and Poincaré's inequality to obtain

$$\int |X(v\,;T)|\,dx \leq C_2E(v\,;\infty, 0, s)$$

with C_2 independent of T, s and v. And the last two terms are dealt with by the Schwarz inequality, so that

$$\begin{aligned}\int_0^T\int (1+r)^{-1-\delta}v_r^2dxdt &+ \int_0^T\int (1+r)^{-3-\delta}v^2dxdt \\ &\leq C_3E(v, \infty, 0, s) + C_4\int_0^T\int (1+r)^{1+\delta}b(t\,;s)v_t^2dxdt\,.\end{aligned} \tag{1.11}$$

By (1.2),

$$(1+r)^{1+\delta} \leq C_5(T+s)^{\alpha(1+\delta)}\,, \qquad T \geq 1\,,$$

on the support of $b(t, s)$, $0 \leq t \leq T$. Hence, combining (1.11) with Lemma 1.1, we conclude the proof.

The next lemma gives the uniform decay of the local energy.

LEMMA 1.5. *Let $v(t\,;s)$ be a solution of problem* (1.1). *Then, there exists a constant C independent of T and s such that for $T \geq 1$,*

$$E(v\,;\tfrac{1}{2}T, T, s) \leq C(T^{-2}d(v(0\,;s))^2 + T^{-1}(T+s)^{\alpha(2+\delta)})E(v\,;\infty, 0, s)\,,$$

where $d(u)$ denotes the radius of the ball with center at the origin containing the support of u.

Proof. We make use of Lemma 1.3 with $u = v(t\,;s)$ and $b = b(t\,;s)$. Integrating the identity (1.6) over $R^3 \times (0, T)$, we have

$$\int F(v\,;T)\,dx + \int_0^T\!\!\int H(v)dxdt = \int F(v\,;0)dx\,.$$

Since $b(t\,;s)(r^2+t^2)v_t^2 \geq 0$, and since

$$\int F(v\,;0)dx \leq C_1 d(v(0\,;s))^2 E(v\,;\infty, 0, s)$$

by Poincaré's inequality, it follows that

$$\int F(v\,;T)\,dx \leq C_1 d(v(0\,;s))^2 E(v\,;\infty, 0, s)$$
$$-2\int_0^T\!\!\int tb(t\,;s)(rv_t v_r + v_t v)dxdt\,.$$

By use of the fact that $(1+r) \leq C_2(T+s)^\alpha$, $T \geq 1$, on the support of $b(t\,;s)$, $0 \leq t \leq T$, the last term is majorized by

$$C_3 T(T+s)^\alpha \int_0^T\!\!\int ((1+r)^{1+\delta}b(t\,;s)v_t^2 + (1+r)^{-1-\delta}v_r^2 + (1+r)^{-3-\delta}v^2)dxdt\,.$$

Hence, in view of Lemma 1.4, we have

$$\int F(v\,;T)dx \leq C_4(d(v(0\,;s))^2 + T(T+s)^{\alpha(2+\delta)})E(v\,;\infty, 0, s)\,.$$

On the other hand, we obtain that $F(v\,;T)$ is non-negative and that

$$F(v\,;T) \geq \tfrac{1}{8}T^2(v_t^2 + |\nabla v|^2 + (r^{-2}v^2x_j)_j)$$

for $|x| \leq \dfrac{T}{2}$ (see pp. 264, [4]), so that

$$\int F(v\,;T)dx \geq \int_{|x|\leq T/2} F(v\,;T)dx \geq \frac{1}{8}T^2E\Big(v\,;\frac{1}{2}T, T, s\Big)\,.$$

Thus, we conclude the proof.

§ 2. Proof of Main Theorem

Let $0 \leq \alpha < \frac{1}{2}$ and δ, $0 < \delta \leq 1$, be so small that $\alpha(2+\delta) < 1$. Let

(2.1) $$p \geq \alpha(2+\delta)(1-\alpha(2+\delta))^{-1},$$

so that $p \geq \alpha(2+\delta)(p+1)$.

Let $\{T_k\}_{k=0}^{\infty}$ be the sequence defined by

$$T_k = k^p T,$$

T being large enough (determined below, Lemma 2.2), and let

$$S_k = \sum_{m=0}^{k} T_m .$$

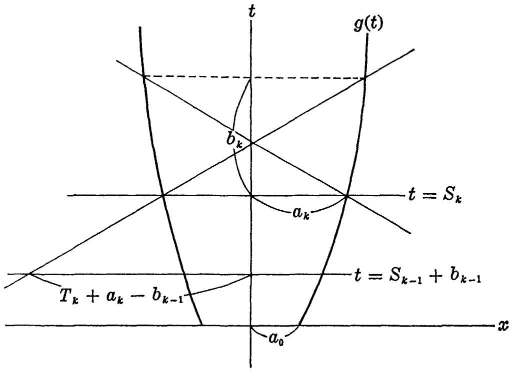

Fig. 1

Obviously,

(2.2) $$S_k \leq C_p k^{p+1} T, \qquad k \geq 0,$$

for C_p independent of k. We put $g(t) = (t+\gamma)^{\alpha}$, $\gamma > 1$, and define $\{a_k\}_{k=0}^{\infty}$, $a_k > 1$, by

(2.3) $$a_k = g(S_k), \qquad a_0 = \gamma^{\alpha}.$$

Furthermore we define the sequence $\{b_k\}_{k=0}^{\infty}$, $b_k > 0$, as follows:

(2.4) b_k is a (unique) root of the equation $t - a_k = g(t + S_k)$.

LEMMA 2.1. *There exists a constant M independent of k such that for $k > 0$*

$$a_k \leq b_k \leq M a_k .$$

Proof. The conclusion readily follows from Fig. 1.

LEMMA 2.2. *Let T_k, S_k, a_k and b_k be as above. Then, there exists a constant T (large enough) independent of $k \geq 1$ such that*

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(2.5) $$a_k + 2b_k < \tfrac{1}{2}(T_k - b_{k-1})$$

(2.6) $$a_{k-1} + b_{k-1} < T_k + a_k - b_{k-1}$$

(2.7) $$\tfrac{1}{2}k^pT < T_k - b_{k-1}\,.$$

Proof. For the proof of (2.5), in virtue of Lemma 2.1 and the monotone increasingness of a_k, it suffices to show that $T_k > (5M + 2)a_k$. By (2.2) and (2.3), $a_k \leq C_1k^{\alpha(p+1)}T^\alpha$ for C_1 independent of $k \geq 1$. Since $\alpha(p + 1) < p$ by (2.1), we can choose T independently of $k \geq 1$ so that $k^pT > C_1(5M + 2)k^{\alpha(p+1)}T^\alpha$. This implies (2.5). (2.6) and (2.7) are proved similarly.

Now, we shall prove Main Theorem. To this end we prepare several lemmas.

LEMMA 2.3. *Let u be the solution of problem* (0.1) *with the initial data f and g ($\in C_0^\infty(R^3)$) such that the support of f and g is contained in $|x| \leq \gamma^\alpha$. Then, the solution u may be written as*

$$u = R_0 + F_0\,,$$

where F_0 is the free space solution with the same initial data as u and

$$F_0 = 0 \qquad \text{for } |x| \leq t - a_0\,,$$

while R_0 has compact support of at most $|x| \leq a_0 + b_0$ at $t = b_0$, and is a solution of problem (0.1) *for $t > b_0$.*

Furthermore, we have

$$E(R_0; \infty, t, 0) \leq 4E(u; \infty, 0, 0)\,, \qquad t \geq 0\,.$$

Here a_0 and b_0 are the number defined by (2.3) *and* (2.4), *respectively, and $E(;,,,)$ is the notation introduced in* § 1.

Proof. It is clear that $F_0 = 0$ for $|x| \leq t - a_0$ ($a_0 = \gamma^\alpha$) by Huyghen's principle. Hence, by the definition of b_0, it follows that for $t > b_0$, $F_0 = 0$ in $\{x\,||x| \leq g(t)\}$, $g(t) = (t + \gamma)^\alpha$, so that F_0 is a solution of problem (0.1) for $t > b_0$. Since u is a solution of problem (0.1), R_0 is also a solution for $t > b_0$. Furthermore, by the argument of the dependence of domain[2)], it is easily seen that R_0 has compact support of at most $|x| \leq a_0 + b_0$ at $t = b_0$. Finally we have

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[2)] The equation $g(t) = t + \gamma^\alpha$ has no root in $t > 0$ since $\gamma > 1$. This means that the forward cone with bottom $\{0\} \times \{x\,||x| \leq \gamma^\alpha\}$ does not intersect the support of $b(t, x)$.

$$E(R_0;\infty,t,0)\le 2(E(u;\infty,t,0)+E(F_0;\infty,t,0))\le 4E(u;\infty,0,0)\ ,$$

since F_0 is the free space solution with the same initial data as u and since $E(u;\infty,t,0)\le E(u;\infty,0,0)$.

LEMMA 2.4. *Let* $\{T_k\}_{k=0}^{\infty}$, $\{S_k\}_{k=0}^{\infty}$, $\{a_k\}_{k=0}^{\infty}$ *and* $\{b_k\}_{k=0}^{\infty}$ *be the sequences defined above and let* R_0 *and* F_0 *be as in Lemma* 2.3. *Then, we can construct* $\{R_k\}_{k=1}^{\infty}$ *and* $\{F_k\}_{k=1}^{\infty}$ *with the following properties:*

(a) $R_{k-1}=R_k+F_k$, *for* $t\ge S_k$;

(b) F_k *is the free space solution with the same initial data as* R_{k-1} *at* $t=S_k$, *and*

$$F_k=0 \qquad \text{for } |x|<t-S_k-a_k\,;$$

(c) R_k *is a solution of problem* (0.1) *for* $t>S_k+b_k$, *and has compact support of at most* $|x|\le a_k+b_k$ *at* $t=S_k+b_k$.

(d) $E(R_k;\infty,0,S_k+b_k)\le 4E(R_{k-1};a_k+2b_k,T_k-b_{k-1},S_{k-1}+b_{k-1})$.

Proof. First, we consider the case of $k=1$. Let F_1 be the free space solution with the same initial data as R_0 at $t=S_1(S_1=T_1)$. We continue F_1 as $F_1=R_0$ for $t<S_1$. Then, $\Box F_1=0$ in the domain exterior to $\{(t,x)\,|\,0<t\le S_1,|x|\le g(t)\}$. We apply Huyghen's principle to F_1 in this domain. Let (t,x) be a point with $|x|<t-S_1-a_1$. Then, the backward cone with vertex at (t,x) does not intersect $\{(t,x)\,|\,0<t\le S_1,$ $|x|\le g(t)\}$, and intersect the plane $t=b_0$ outside the sphere $|x|=S_1+a_1-b_0$ $(=T_1+a_1-b_0)$ (see Fig. 1). By (2.6) in Lemma 2.2, $T_1+a_1-b_0>a_0+b_0$, and the support of R_0 at $t=b_0$ is contained in $|x|\le a_0+b_0$ by Lemma 2.3. Thus, we conclude that $F_1=0$ for $|x|<t-S_1-a_1$. Therefore, by the definition of b_1, $F_1=0$ in $|x|\le g(t)$ for $t\ge S_1+b_1$. This implies that R_1 is a solution of problem (0.1) for $t>S_1+b_1$. Similarly to the proof of Lemma 2.3, it is easily seen by the argument of the dependence of domain that R_1 has compact support of at most $|x|\le a_1+b_1$ at $t=S_1+b_1$.

It remains to prove the property (d). By property (c) and the standard method of energy estimate[3], we obtain

$$\begin{aligned}E(R_1;\infty,0,S_1+b_1)&=E(R_1;a_1+b_1,0,S_1+b_1)\\&\le 2(E(F_1;a_1+b_1,0,S_1+b_1)+E(R_0;a_1+b_1,0,S_1+b_1))\\&\le 4E(R_0;a_1+2b_1,T_1-b_0,b_0)\ ,\end{aligned}$$

[3] It is readily proved that $E(F_1(R_0);a_1+b_1,0,S_1+b_1)\le E(F_1(R_0);a_1+2b_1,0,S_1)$.

where the last inequality follows from the fact that R_0 is a solution of problem (0.1) for $t > b_0$. Following the above procedure and noting (2.6) in Lemma 2.2, we can construct F_k and R_k by induction on k.

THEOREM 2.1. *Suppose that Assumption* (A) *is satisfied and that* $0 \leq \alpha < \frac{1}{2}$. *Let* u *be the smooth solution of problem* (0.1) *with the initial data* f *and* g $(\in C_0^\infty(R^3))$ *such that the support of* f *and* g *is contained in* $|x| \leq \gamma^\alpha, \gamma > 1$, *and let* $h, h > 0$, *be fixed. Then, there exist constants* θ *and* β *such that*

$$E(u; h, t, 0) \leq 4 \exp(-\theta t^\beta) E_0(u) ,$$

where $\beta = (p+1)^{-1}$, p *being the constant defined by* (2.1), *and*

$$E_0(u) = \int (|\nabla f|^2 + |g|^2) dx .$$

Proof. According to Lemma 2.4, we can write

$$u = \sum_{j=0}^{n} F_j + R_n \qquad \text{for } t > S_n ,$$

where

$$F_j = 0 \qquad \text{for } |x| \leq t - S_j - a_j \tag{2.8}$$

and

(2.9) R_n is a solution of problem (0.1) for $t > S_n + b_n$.

Let $t > S_n + b_n + h$. Then, in view of (2.8) and the fact that $b_n \geq a_n$ (see, Lemma 2.1), $u = R_n$ in $|x| \leq h$, so that by (2.9) and Lemma 2.4,

$$\begin{aligned} E(u; h, t, 0) &\leq E(R_n; \infty, t - S_n - b_n, S_n + b_n) \leq E(R_n; \infty, 0, S_n + b_n) \\ &\leq 4E(R_{n-1}; a_n + 2b_n, T_n - b_{n-1}, S_{n-1} + b_{n-1}) . \end{aligned}$$

By (2.5) in Lemma 2.2, $a_n + 2b_n < \frac{1}{2}(T_n - b_{n-1})$. Hence, we can apply Lemma 2.5 to $E(R_{n-1}; a_n + 2b_n, T_n - b_{n-1}, S_{n-1} + b_{n-1})$ to obtain

$$\begin{aligned} &E(R_{n-1}; a_n + 2b_n, T_n - b_{n-1}, S_{n-1} + b_{n-1}) \\ &\quad \leq C((T_n - b_{n-1})^{-2} d(R_{n-1})^2 \\ &\qquad + (T_n - b_{n-1})^{-1} S_n^{\alpha(2+\delta)}) E(R_{n-1}; \infty, 0, S_{n-1} + b_{n-1}) , \end{aligned}$$

where $d(R_{n-1})$ denotes the radius of the ball with center at the origin containing the support of R_{n-1} at $t = S_{n-1} + b_{n-1}$ and satisfies

$$d(R_{n-1}) \leq a_{n-1} + b_{n-1} \leq (M+1)a_{n-1} \leq Cn^{\alpha(p+1)}T^{\alpha}$$

by (c) in Lemma 2.4 and Lemma 2.1. Furthermore, making use of (2.7) in Lemma 2.4 and recalling the definition of p given by (2.1), we have

$$\begin{aligned} E(R_{n-1}; a_n + 2b_n, T_n - b_{n-1}, S_{n-1} + b_{n-1}) \\ \leq CT^{\alpha(2+\delta)-1}E(R_{n-1}; \infty, 0, S_{n-1} + b_{n-1}) \end{aligned}$$

for C independent of n. We repeat this procedure and using Lemma 2.3, we finally have

(2.10) $\quad E(u; h, t, 0) \leq (CT^{\alpha(2+\delta)-1})^n E(R_0; \infty, 0, b_0) \leq 4\exp(-n\tilde{\theta})E_0(u)$,

where we take T, noting that $\alpha(2+\delta) < 1$, so large that $-\tilde{\theta} = \log(CT^{\alpha(2+\delta)-1}) < 0$. Thus, for given $t > 0$, we choose the maximal integer n such that $t > S_n + b_n + h$. Then, there exists a constant $C(T)$ such that $n \geq C(T)t^{\beta}$, $\beta = (p+1)^{-1}$. This, together with (2.10), completes the proof.

Remark. If, in addition to (i) ~ (iii) in Assumption (A), we assume that

$$b(t, x) \leq C(1 + |x|)^{-1-\varepsilon}, \qquad \varepsilon > 0,$$

then Theorem 2.1 holds for $\alpha < 1$ with $\beta = (p+1)^{-1}$, $p \geq \alpha(1-\alpha)^{-1}$. In fact, in this case we have

$$\begin{aligned} \int_0^{\infty}\int (1+r)^{-1-\delta}|v_r(t;s)|^2\,dxdt + \int_0^{\infty}\int (1+r)^{-3-\delta}|v(t;s)|^2\,dxdt \\ \leq CE(v; \infty, 0, s) \end{aligned}$$

instead of Lemma 1.4, so that Lemma 1.5 holds with $\alpha(2+\delta)$ replaced by α.

§ 3. Decays for wave equations with potentials

We consider the following equation in three dimension space R^3:

(3.1) $$u_{tt} - \Delta u + q(t, x)u = 0$$

with initial data $u(0, x) = f(x)$ and $u_t(0, x) = g(x)$ of compact support. Here we make the following assumptions on $q(t, x)$.

ASSUMPTION (B). (i) $q(t, x)$ is a smooth function with bounded derivatives. (ii) $q(t, x)$ is non-negative. (iii) For each $t \geq 0$, (3.2) the support of $q(t, x)$ is contained in $\{x \mid |x| \leq (t+\gamma)^\alpha\}$, $0 \leq \alpha < 1$, $\gamma > 1$.

(iv) $q_r(t, x) \leq 0$ for each $t \geq 0$. (v) There exists a constant β_0, $0 < \beta_0 < 1$, such that for $t \geq t_0$ and $|x| \geq R_0$, t_0 and R_0 being large enough,

$$q_t(t, x) + \beta_0 q_r(t, x) \leq 0 . \tag{3.3}$$

(vi) There exists a constant K such that for $t \geq t_0$

$$|q_t(t, x)| \leq \frac{K}{t} . \tag{3.4}$$

The constants β_0, t_0, R_0 and K are used with the meaning ascribed here throughout this section.

As in §1, let $s, s \geq 0$, be fixed and we consider the following equation:

$$v_{tt}(t; s) - \Delta v(t; s) + q(t; s)v(t; s) = 0 \tag{3.5}$$

with initial data $v(0; s)$ and $v_t(0; s)$ of compact support, where $q(t; s) = q(t+s, x)$, and by (3.2)

(3.6) the support of $q(t; s)$ is contained in $\{x \mid |x| \leq (t+s+\gamma)^\alpha\}$. Furthermore, by (3.4)

$$|q_t(t; s)| \leq \frac{K}{t+s} \qquad \text{for } t \geq t_0 - s . \tag{3.7}$$

We begin with the following identity.

LEMMA 3.1 (cf. [3], Lemma 1). *Let $u(t, x)$ and $\zeta(r)$, $r = |x|$, be smooth functions and let $\chi_i(x) = \zeta(r)\dfrac{x_i}{r}$. Let β, $\beta > 0$, be a constant. Then,*

$$(u_{tt} - u_{jj} + qu)(2u_t + 2\beta\chi_i u_i + \beta\chi_{ii}u) = X_t(u; t) + \nabla \cdot Y(u) + Z(u) ,$$

where

$$X(u; t) = (u_t^2 + |\nabla u|^2 + qu^2) + u_t(2\beta\chi_i u_i + \beta\chi_{ii}u)$$

$$Y_j(u) = -u_j(2u_t + 2\beta\chi_i u_i + \beta\chi_{ii}u) + \beta\chi_j(|\nabla u|^2 - u_t^2 + qu^2) + \frac{\beta}{2}\chi_{iij}u^2$$

$$Z(u) = 2\beta\chi_{ij}u_i u_j - \frac{\beta}{2}\chi_{iijj}u^2 - q_t u^2 - \beta q_i \chi_i u^2 .$$

Furthermore, using the notation

$$w_j = \zeta(r)u_j + \frac{1}{2}\left(\zeta_r(r) + \frac{2}{r}\zeta(r)\right)\frac{x_j}{r}u \quad and \quad w_r = w_j \cdot \frac{x_j}{r}$$

we can write $X(u;t)$ as follows:

$$X(u;t) = \varphi_1(u) + \varphi_2(u) + \varphi_3(u) + \varphi_4(u) + \varphi_5(u)\ ,$$

where

$$\begin{aligned}
\varphi_1(u) &= (1-\beta)(u_t^2 + |\nabla u|^2) + qu^2 \\
\varphi_2(u) &= \beta(1-\zeta^2)|\nabla u|^2 \\
\varphi_3(u) &= \beta(u_t^2 + |w|^2 + 2w_r u_t) \\
\varphi_4(u) &= -\frac{\beta}{2}\nabla\cdot\left(\zeta(r)\left(\zeta_r(r) + \frac{2}{r}\zeta(r)\right)\frac{x}{r}u^2\right) \\
\varphi_5(u) &= \frac{\beta}{4}\left(\zeta_r(r)^2 + 2\zeta(r)\left(\zeta_{rr}(r) + \frac{4}{r}\zeta_r(r)\right)\right)u^2\ .
\end{aligned}$$

Proof. The proof is elementary, so we omit it.

LEMMA 3.2. *Let $v(t) = v(t;s)$ be a smooth solution of problem* (3.5). *Then, there exists constants C and $t_1, t_1 \geq t_0$, such that for $s \geq t_1$ and $T \geq 0$,*

$$\hat{E}(v;\infty,T,s) \leq C\hat{E}(v;\infty,0,s)$$

and

$$\int_0^T \int (1+r)^{-3-\delta}v^2 dxdt \leq C\hat{E}(v;\infty,0,s)$$

where $0 < \delta < 1$ and

$$\hat{E}(v;h,T,s) = \int_{|x|<h} (v_t(T;s)^2 + |\nabla v(T;s)|^2 + q(T;s)v(T;s)^2)dx$$

for $0 < h \leq \infty$.

Proof. Let β_0 be the constant introduced in Assumption (B) and let $\beta = \beta_0 + \varepsilon, \varepsilon > 0$. We take ε so small that $0 < \beta < 1$. We use Lemma 3.1 with $\zeta(r) = 1 - (1+r)^{-\delta}$, $0 < \delta < 1$, $q = q(t;s)$ and β defined above. Then, following the same method as in the proof of Lemma 2.4, we have

$$\int X(v;T)dx + \int_0^T \int Z(v)dxdt = \int X(v;0)dx\ . \tag{3.8}$$

We claim that there exists constants C_1 and t_1 such that for $s \geq t_1 \geq t_0$

(3.9) $$\frac{\beta}{2}\chi_{iijj} + q_t(t;s) + \beta\zeta(r)q_r(t;s) \leq -C_1(1+r)^{-3-\delta}.$$

Indeed, by our choice of $\zeta(r)$ and the definition of β, $\beta\zeta(r) > \beta_0$ for $r \geq R_1 > R_0$. Hence, this, together with (3.3) and the non-positivity of $q_r(t;s)$ ((iv) of Assumption (B)), implies that for $r \geq R_1$

(3.10) $$q_t(t;s) + \beta\zeta(r)q_r(t;s) \leq q_t(t;s) + \beta_0 q_r(t;s) \leq 0.$$

On the other hand, by (1.10) and (3.4) and again by the non-positivity of q_r

$$\frac{\beta}{2}\chi_{iijj} + q_t(t;s) + \beta\zeta(r)q_r(t;s) \leq -\frac{\beta}{2}\delta(1+\delta)(1+r)^{-3-\delta} + \frac{K}{t+s},$$

so that for $s \geq t_1$ (large enough)

$$\frac{\beta}{2}\chi_{iijj} + q_t(t;s) + \beta\zeta(r)q_r(t;s) \leq -C_2$$

in $r \leq R_1$, which, together with (3.10) and (1.10), gives (3.9). Therefore, by (3.9) and (1.8), we obtain

(3.11) $$\int X(v;T)dx + C\int_0^T\int(1+r)^{-3-\delta}v^2 dxdt \leq \int X(v;0)dx.$$

We recall the expression of $X(v;T)$ in Lemma 3.1. Then, for our choice of $\zeta(r)$, $\varphi_5(v) \geq 0$, so that by the condition $0 < \beta < 1$ and $0 < \zeta(r) \leq 1$

(3.12) $$\int X(v;T)dx \geq C_3\hat{E}(v;\infty,T,s)$$

for $C_3 > 0$. Furthermore, by the Poincaré inequality, it is easily seen that

$$\int X(v;0)dx \leq C_5\hat{E}(v;\infty,0,s).$$

This completes the proof.

We use the following identity similar to (1.6) for the proof of the next lemma:

$$(3.13)\qquad \begin{aligned}(u_{tt} - u_{jj} + qu)((r^2 + t^2)u_t + 2tru_r + 2tu) \\ = \tilde{F}_t(u\,;t) + \nabla\cdot\tilde{G}(u) + \tilde{H}(u)\,,\end{aligned}$$

where

$$\begin{aligned}\tilde{F}(u\,;t) &= F(u\,;t) + \tfrac{1}{2}(r^2 + t^2)qu^2 \\ \tilde{G}_j(u) &= G_j(u) + tqu^2x_j \\ \tilde{H}(u) &= -(\tfrac{1}{2}(r^2 + t^2)q_t + 2tq + trq_r)u^2\end{aligned}$$

and $F(u\,;t)$ and $G_j(u)$ are as in Lemma 1.3.

LEMMA 3.3. *Let $v(t) = v(t\,;s)$ be a smooth solution of problem* (3.5) *and let t_1 be as in Lemma* 3.2. *Let $\alpha < \frac{1}{3}$ and let δ be so small that $\alpha(3 + \delta) < 1$. Then there exists a constant C such that for $s \geq t_1$ and $T \geq 1$*

$$\hat{E}(v\,;\tfrac{1}{2}T, T, s) \leq C(T^{-2}d(v(0\,;s))^2 + T^{-1}(T + s)^{\alpha(3+\delta)})\hat{E}(v\,;\infty, 0, s)\,,$$

where $d(v(0\,;s))$ denotes the radius of the ball with center at the origin containing the support of $v(0\,;s)$ and $\hat{E}(\,;\,,\,,)$ is the notation introduced in Lemma 3.2.

Proof. Integrating the identity (3.13) with $u = v(t\,;s)$ and $q = q(t\,;s)$ and using (3.7) and Lemma 3.2, we obtain in the same way as in the proof of Lemma 1.5 that

$$\tfrac{1}{8}T^2\hat{E}(v\,;\tfrac{1}{2}T, T, s) \leq C(d(v(0\,;s))^2 + s^{2\alpha} + T(T + s)^{\alpha(3+\delta)})\hat{E}(v\,;\infty, 0, s)\,.$$

The conclusion easily follows from the above estimate.

LEMMA 3.4. *Let u be the solution of problem* (3.1) *with the initial data f and g $(\in C_0^\infty(R^3))$ such that the support of f and g is contained in $|x| \leq \gamma^\alpha$. Then, the solution u may be written as*

$$u = R_0 + F_0\,,$$

where R_0 and F_0 have the same properties as in Lemma 2.3. *Furthermore,*

$$\hat{E}(R_0\,;\infty, t, 0) \leq C(t)\hat{E}(u\,;\infty, 0, 0)\,.$$

Proof. The proof is the same as that of Lemma 2.3 and the last assertion is easily verified.

LEMMA 3.5. *Let $\{T_k\}_{k=0}^\infty$, $\{S_k\}_{k=0}^\infty$, $\{a_k\}_{k=0}^\infty$ and $\{b_k\}_{k=0}^\infty$ be the sequences*

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defined in §2. *Let* R_0 *and* F_0 *be as in Lemma* 3.4. *Then, we can construct* $\{R_k\}_{k=1}^{\infty}$ *and* $\{F_k\}_{k=1}^{\infty}$ *with properties* (a), (b) *and* (c) *in Lemma* 2.4 *and* (d′) *stated below.*

$$\text{(d}'\text{)}\quad \begin{aligned}\hat{E}(R_k;\infty,0,S_k+b_k) \le 2(&\hat{E}(R_{k-1};a_k+2b_k,T_k-b_{k-1},S_{k-1}+b_{k-1})\\ &+\hat{E}(R_{k-1};a_k+b_k,T_k+b_k-b_{k-1},S_{k-1}+b_{k-1}))\,.\end{aligned}$$

Proof. The construction of R_k and F_k with properties (a), (b) and (c) is the same as in the proof of Lemma 2.4. We shall prove (d′). By property (c), we have

$$\begin{aligned}\hat{E}(R_k;\infty,0,S_k+b_k) &= \hat{E}(R_k;a_k+b_k,0,S_k+b_k)\\ &\le 2(\hat{E}(R_{k-1};a_k+b_k,0,S_k+b_k)\\ &\quad+\hat{E}(F_k;a_k+b_k,0,S_k+b_k))\,.\end{aligned}$$

Since F_k is the free space solution for $t > S_k$ with the same initial data as R_{k-1} at $t = S_k$ and since $F_k = 0$ on the support of $q(t,x)$ at $t = S_k + b_k$ (see (2.4)),

$$\begin{aligned}\hat{E}(F_k;a_k+b_k,0,S_k+b_k) &\le \hat{E}(F_k;a_k+2b_k,0,S_k)\\ &= \hat{E}(R_{k-1};a_k+2b_k,T_k-b_{k-1},S_{k-1}+b_{k-1})\,.\end{aligned}$$

This completes the proof.

Let $0 \le \alpha < \frac{1}{3}$ and let δ be so small that $\alpha(3+\delta) < 1$. We fix p, $p > 0$, as follows:

$$p \ge \alpha(3+\delta)(1-\alpha(3+\delta))^{-1} \tag{3.14}$$

so that $p \ge \alpha(3+\delta)(p+1)$. Then, the main result of this section can be stated as follows:

THEOREM 3.1. *Suppose that Assumption* (B) *is satisfied and that* $0 \le \alpha < \frac{1}{3}$. *Let* u *be the solution of problem* (3.1) *with the initial data* f *and* g ($\in C_0^\infty(R^3)$) *such that the support of* f *and* g *is contained in* $|x| \le \gamma^\alpha$. *Let* $h, h > 0$, *be fixed. Then, there exist constants* C, θ *and* β *such that*

$$\hat{E}(u;h,t,0) \le C\exp(-\theta t^\beta)\hat{E}_0(u)\,,$$

where $\beta = \dfrac{1}{p+1}$, p *being the constant defined by* (3.14), *and* $\hat{E}_0(u) = \int (g^2 + |\nabla f|^2 + q(0,x)f^2)dx$.

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Proof. The proof is done exactly in the same way as in the proof of Theorem 2.1.

EXAMPLE. Let $\chi(x)$ be a smooth function such that $C_1r^2 \leq \chi(x) \leq C_2r^2$, $r = |x|$, and that $\chi_r(x) \geq C_3r$ and let $\varphi(s)$, $0 \leq s \leq \infty$, be a nonnegative smooth function such that $\varphi(s) = 0$ for $s \geq 1$ and that $\varphi_s(s) \leq 0$. Then, consider the following function: $q(t, x) = \varphi\left(\frac{\chi(x)}{(t + \gamma)^{2\alpha}}\right)$, $0 < \alpha < 1$, $\gamma > 1$. We can easily show that the function $q(t, x)$ satisfies Assumption (B).

Remark. If, in addition to Assumption (B), we assume that

$$q(t, x) \leq C(1 + r)^{-2}$$

for a constant C independent of t and x, we easily see that the result of Theorem 3.1 holds for $0 \leq \alpha < 1$ with $\beta = (p + 1)^{-1}$, $p = \alpha(1 - \alpha)^{-1}$

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