

IDEMPOTENT-SEPARATING EXTENSIONS OF REGULAR SEMIGROUPS WITH ABELIAN KERNEL

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Abstract

Let S be a regular semigroup and $D(S)$ its associated category as defined in Loganathan (1981). We introduce in this paper the notion of an *extension* of a $D(S)$ -module A by S and show that the set $\text{Ext}(S, A)$ of equivalence classes of extensions of A by S forms an abelian group under a Baer sum. We also study the functorial properties of $\text{Ext}(S, A)$.

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1. Introduction

Let S be a regular semigroup and $D(S)$ its associated category (see Section 2 for the definition of $D(S)$). By a $D(S)$ -module we mean a functor $A: D(S) \rightarrow Ab$ from $D(S)$ to the category of abelian groups and by a $D(S)$ -homomorphism a natural transformation between such functors.

Let $\pi: T \rightarrow S$ be an idempotent-separating homomorphism from a regular semigroup T onto S . If the kernel of π is abelian then the kernel can be viewed as a $D(S)$ -module $\text{Ker } \pi: D(S) \rightarrow Ab$. This suggests the following definition: An *extension* of a $D(S)$ -module A by S is a triple (T, π, i) consisting of a regular semigroup T , an idempotent-separating surjective homomorphism of regular semigroups $\pi: T \rightarrow S$ with abelian kernel and an isomorphism $i: A \rightarrow \text{Ker } \pi$ of $D(S)$ -modules. Let $\text{Ext}(S, A)$ denote the set of equivalence classes of extensions of A by S . We define an addition in $\text{Ext}(S, A)$ and show that it makes $\text{Ext}(S, A)$ an abelian group. We also study the functorial properties of $\text{Ext}(S, A)$.

The cohomological interpretation of $\text{Ext}(S, A)$ will be given in another paper. Here we content ourselves with the following remarks. (For details see Loganathan (1981).) When G is a group the category $D(G)$ reduces to the group G itself and in this case our results are well known and classical. When S is an inverse semigroup then the concept of the $D(S)$ -module is equivalent to that of the S -module as defined by Lausch (1975). Consequently, in the case of semi-lattices of groups, our results turn out to be equivalent to those obtained by Sribala (1977). For regular semigroups the category $D(S)$ is equivalent to the category $\mathfrak{D}(S)$ introduced by Leech (1975). In particular, the type of extensions considered in this paper is a special case of the \mathfrak{K} -coextensions studied by Leech.

2. Preliminaries

Let S be a regular semigroup and $E(S)$ its set of idempotents. For $x \in S$, we denote by

$$V(x) = \{x' \in S \mid xx'x = x \text{ and } x'xx' = x'\}$$

the set of inverses of x in S . If $x' \in V(x)$ then (x, x') will be called a *regular pair* in S . For $e, f \in E(S)$, we denote by

$$S(e, f) = \{h \in E(S) \mid he = h = fh \text{ and } ehf = ef\}$$

the *sandwich set* of e and f (Nambooripad (1979)).

If ρ is a congruence on S then the *kernel* of ρ is the set of ρ -classes which contain idempotents of S . By the kernel of a homomorphism we mean the kernel of its associated congruence. If ρ is idempotent-separating then, since $\rho \subseteq \mathfrak{K}$, for each $e \in E(S)$ the ρ -class N_e containing the idempotent e is a subgroup of S . The kernel of an idempotent-separating congruence ρ is called *abelian* if the groups N_e are abelian.

A set $N = \{N_e \mid e \in E(S)\}$, of subgroups of S , is called a *group kernel normal system* if the following hold

- (i) $e \in N_e$ for each $e \in E(S)$;
- (ii) $af = fa$ for each $a \in N_e$ and $f \leq e, f \in E(S)$;
- (iii) $x'N_e x \subseteq N_{x'x}$ for each regular pair (x, x') of S with $xx' \leq e$.

Let $N = \{N_e \mid e \in E(S)\}$ be a group kernel normal system of S . We denote by ρ_N the relation on S defined by

$$\{(x, y) \in S \times S \mid \text{for some } x' \in V(x) \text{ and } y' \in V(y), \\ xx' = yy', x'x = y'y \text{ and } y'x \in N_{x'x}\}.$$

By Lallement (1967), Theorem 3.11, ρ_N is an idempotent-separating congruence on S with N as its kernel.

Next we recall the definition of the category $D(S)$ introduced in Loganathan (1981).

Let S be a regular semigroup. Let $C(S)$ denote the category whose objects are the idempotents of S , where a morphism from the idempotent e to the idempotent f is a triple (e, x, x') with (x, x') a regular pair in S satisfying $xx' \leq e$, $x'x = f$. Composition is defined by $(e, x, x')(x'x, y, y') = (e, xy, y'x')$. Composition is clearly associative and (e, e, e) is the identity morphism at the object e .

On $C(S)$ define a relation ρ as follows. If $(e, x, x'), (e, y, y') : e \rightarrow f$ are morphisms from e to f then

$$(e, x, x')\rho(e, y, y') \text{ if and only if } x = y \text{ or } x' = y'.$$

The relation ρ is clearly reflexive and symmetric and compatible with composition. Hence ρ' , the transitive closure of ρ , is a congruence on $C(S)$. We write $D(S)$ for the quotient category $C(S)/\rho'$. If (e, x, x') is a morphism of $C(S)$, we shall write $[e, x, x']$ for the image of (e, x, x') in $D(S)$. Let $\theta : S' \rightarrow S$ be a homomorphism of regular semigroups. Then the maps $e \mapsto e\theta, [e, x, x'] \mapsto [e\theta, x\theta, x'\theta]$, define a functor $D(\theta) : D(S') \rightarrow D(S)$.

DEFINITION 2.1. A map $x \mapsto x^* : S \rightarrow S$ is called an *inverse map* if (i) $x^* \in V(x)$ for each $x \in S$; (ii) $x^* \in H_e$ if $x \in H_e$. In particular, if $e \in E(S)$ then $e^* = e$.

Let S be a regular semigroup with an inverse map $x \mapsto x^*$. For $x, y \in S$, we denote by $K_{x,y}$ the morphism

$$[y^*y, y^*y(xy)^*xy, (xy)^*xy] : y^*y \rightarrow (xy)^*xy$$

and by $J_{x,y}$ the morphism

$$[x^*x, x^*xy, (xy)^*xh] : x^*x \rightarrow (xy)^*xy,$$

where h is any element of $S(x^*x, yy^*)$. Clearly $J_{x,y}$ does not depend on the choice of h and hence it is well-defined.

LEMMA 2.2. For $x, y, z \in S$, we have

- (i) $J_{x,y}J_{xy,z} = J_{x,yz}$;
- (ii) $K_{y,z}K_{x,yz} = K_{xy,z}$;
- (iii) $K_{x,y}J_{xy,z} = J_{y,z}K_{x,yz}$.

PROOF. (i) and (ii) are easy to verify and (iii) also follows by observing that

$$\begin{aligned} K_{x,y}J_{xy,z} &= [y^*y, y^*y(xy)^*xyz, (xyz)^*xyh] \\ &= [y^*y, y^*yz(xyz)^*xyz, (xyz)^*xyh] \\ &= [y^*y, y^*yz(xyz)^*xyz, (xyz)^*xyk] \\ &= J_{y,z}K_{x,yz}, \end{aligned}$$

where h, k are any elements of $S((xy)^*xy, zz^*)$ and $S(y^*y, zz^*)$ respectively.

3. The group $\text{Ext}(S, A)$

Let T, S be regular semigroups and let $\pi: T \rightarrow S$ be an idempotent-separating homomorphism from T onto S . For $e \in E(S)$, we denote by $(\text{Ker } \pi)_e$ the group $\{t \in T \mid t\pi = e\}$. For each morphism $[h, t, t']: h \mapsto k$ of $D(T)$ we define

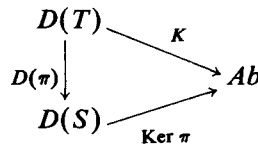
$$K([h, t, t']): (\text{Ker } \pi)_{hm} \rightarrow (\text{Ker } \pi)_{km}$$

by $aK([h, t, t']) = t'at, a \in (\text{Ker } \pi)_{hm}$. It is easy to see that $K([h, t, t'])$ is well-defined and that it is a homomorphism of groups. Therefore the maps

$$h \mapsto (\text{Ker } \pi)_{hm}, \quad [h, t, t'] \mapsto K([h, t, t'])$$

define a functor K from $D(T)$ to the category of groups.

Suppose now that the kernel of π is abelian, that is, the groups $(\text{Ker } \pi)_e, e \in E(S)$, are abelian. Then $K: D(T) \rightarrow Ab$ induces a $D(S)$ -module $\text{Ker } \pi: D(S) \rightarrow Ab$ such that the diagram



is commutative. By assigning to each regular pair (x, x') in S a regular pair $((x, x')j_1, (x, x')j_2)$ in T such that $((x, x')j_1)\pi, (x, x')j_2)\pi = (x, x')$, we see that $\text{Ker } \pi$ can be described, more explicitly, as the $D(S)$ -module which associates to each object e of $D(S)$ the group $(\text{Ker } \pi)_e$ and to each morphism $[e, x, x']: e \rightarrow f$ the homomorphism $\text{Ker } \pi([e, x, x']): (\text{Ker } \pi)_e \rightarrow (\text{Ker } \pi)_f$, given by

$$(a) \text{Ker } \pi([e, x, x']) = (x, x')j_2a(x, x')j_1, \quad a \in (\text{Ker } \pi)_e.$$

Motivated by the above observation we introduce the following

DEFINITION 3.1. Let S be a regular semigroup and A a $D(S)$ -module. An extension of A by S is a triple $E = (T, \pi, i)$ consisting of a regular semigroup T , an idempotent-separating surjective homomorphism $\pi: T \rightarrow S$ such that the kernel of π is abelian, and an isomorphism of $D(S)$ -modules $i: A \rightarrow \text{Ker } \pi$.

Consider the commutative diagram

$$\begin{array}{ccc} T' & \xrightarrow{\pi'} & S' \\ \downarrow \mu & & \downarrow \theta \\ T & \xrightarrow{\pi} & S \end{array}$$

of homomorphisms of regular semigroups such that π' and π are idempotent-separating surjective homomorphisms with abelian kernel. Let $D(\theta)$ ($\text{Ker } \pi$) be the composite

$$D(S') \xrightarrow{D(\theta)} D(S) \xrightarrow{\text{Ker } \pi} Ab.$$

Then μ induces a homomorphism $\phi_\mu: \text{Ker } \pi' \rightarrow D(\theta)(\text{Ker } \pi)$ of $D(S')$ -modules such that

$$\phi_\mu|_{(\text{Ker } \pi')_e} = \mu|_{(\text{Ker } \pi')_e}: (\text{Ker } \pi')_e \rightarrow D(\theta)(\text{Ker } \pi)_e = (\text{Ker } \pi)_{e\theta}.$$

Let $E' = (T', \pi', i')$ be an extension of A' by S' and $E = (T, \pi, i)$ an extension of A by S . A *morphism* $\Gamma: E' \rightarrow E$ of extensions is a triple $\Gamma = (\phi, \mu, \theta)$ consisting of homomorphisms $\mu: T' \rightarrow T$, $\theta: S' \rightarrow S$ of regular semigroups and a homomorphism $\phi: A' \rightarrow D(\theta)A$ of $D(S')$ -modules, such that: $\mu\pi = \pi'\theta$ and $\phi D(\theta)i = i'\phi_\mu$, where

$$D(\theta)i: D(\theta)A \rightarrow D(\theta)(\text{Ker } \pi)$$

is the homomorphism of $D(S')$ -modules induced by $i: A \rightarrow \text{Ker } \pi$. We say two extensions $E' = (T', \pi', i')$, $E = (T, \pi, i)$ of A by S are *equivalent* if there exists a homomorphism (necessarily an isomorphism) $\mu: T' \rightarrow T$ such that $(\text{Id}_A, \mu, \text{Id}_S): E' \rightarrow E$ is a morphism of extensions. We call μ an *equivalence of extensions*. We denote by $\text{Ext}(S, A)$ the equivalence classes of extensions of A by S and by $[E]$ the equivalence class containing the extension $E = (T, \pi, i)$.

DEFINITION 3.2. Let S be a regular semigroup and A a $D(S)$ -module. The *semi-direct product* of S and A with respect to an inverse map $x \mapsto x^*: S \rightarrow S$ is the regular semigroup

$$S \times A = \{(x, a) | x \in S, a \in A_{x^*x}\}$$

with the multiplication given by

$$(x, a)(y, b) = (xy, aA(J_{x,y}) + bA(K_{x,y})).$$

Associativity of the multiplication follows from Lemma 2.2. The set $E(S \times A)$ of idempotents of $S \times A$ is $\{(e, 0_e) \in S \times A | e \in E(S)\}$ and if $(x, a) \in S \times A$ then

$$V((x, a) = \{(y, (-a)A([x^*x, x^*xy, y^*yx])) \in S \times A | y \in V(x)\}.$$

Suppose now that $S \times A$ is the semi-direct product of S and A with respect to an inverse map $x \mapsto x^*$. Then the projection $\pi_0: S \times A \rightarrow S$, defined by $(x, a)\pi_0 = x$, and the isomorphism $i_0: A \rightarrow \text{Ker } \pi_0$, given by $(a)i_0 = (e, a)$, $a \in A_e$, $e \in E(S) = \text{Ob}(D(S))$, of $D(S)$ -modules define an extension, denoted $E_0 = (S \times A, \pi_0, i_0)$, of A by S . Also, there is a homomorphism $\nu_0: S \rightarrow S \times A$, given by $(x)\nu_0 = (x, 0_{x^*x})$, which satisfies $\nu_0\pi_0 = \text{Id}_S$.

We call an extension $E = (T, \pi, i)$ of A by S *split* if there exists a homomorphism $\nu: S \rightarrow T$ such that $\nu\pi = \text{Id}_S$; the homomorphism ν is then called a *splitting*. For example E_0 is a split extension with a splitting ν_0 .

If $E = (T, \pi, i)$ is a split extension of A by S with a splitting ν then the map $\mu: S \times A \rightarrow T$, given by $(x, a)\mu = (x\nu)(ai)$, is a homomorphism of regular semigroups and it is an equivalence of extensions. Conversely, if an extension E is equivalent to E_0 then, obviously, it is a split extension. In other words, an extension E of A by S is split if and only if it is equivalent to E_0 . In particular, the equivalence class determined by E_0 does not depend on the particular choice of the inverse map $x \mapsto x^*$ used to define the multiplication in $S \times A$.

Let $E_r = (T_r, \pi_r, i_r)$, $r = 1, 2$, be extensions of A by S . Consider the regular subsemigroup

$$T = \{(t_1, t_2) \in T_1 \times T_2 \mid t_1\pi_1 = t_2\pi_2\}$$

of $T_1 \times T_2$. Since π_1 and π_2 are idempotent-separating homomorphisms, the set $N = \{N_e \mid e \in E(S)\}$, where $N_e = \{((a)i_1, (-a)i_2) \mid a \in A_e\}$, is a group kernel normal system of T . Write $T_1 + T_2 = T/\rho_N$ and denote the element of $T_1 + T_2$ containing (t_1, t_2) by $\overline{(t_1, t_2)}$. It is easy to see that the map $\pi: T_1 + T_2 \rightarrow S$, given by $\overline{(t_1, t_2)}\pi = t_1\pi_1 (= t_2\pi_2)$, is an idempotent-separating homomorphism from $T_1 + T_2$ onto S . Also, it is easy to see that the map $i: A \rightarrow \text{Ker } \pi$, defined by $(a)i = \overline{((a)i_1, (0)i_2)}$, $a \in A_e$, $e \in E(S)$, is an isomorphism of $D(S)$ -modules. Thus we obtain an extension, denoted $E_1 + E_2 = (T_1 + T_2, \pi, i)$, of A by S . We call $E_1 + E_2$ the *Baer sum* of the extensions E_1 and E_2 .

THEOREM 3.3. *Ext(S, A) is an abelian group under the operation $[E_1] + [E_2] = [E_1 + E_2]$. The zero element in the abelian group $\text{Ext}(S, A)$ is the equivalence class $[E_0]$ determined by the split extensions.*

PROOF. *Commutativity of +.* Let $E_r = (T_r, \pi_r, i_r)$, $r = 1, 2$, be extensions of A by S . Then the map $\mu: T_1 + T_2 \rightarrow T_2 + T_1$, given by $\overline{(t_1, t_2)}\mu = \overline{(t_2, t_1)}$, is easily seen to be an equivalence of extensions so that $[E_1] + [E_2] = [E_1 + E_2] = [E_2] + [E_1]$.

Associativity of +. Let $E_r = (T_r, \pi_r, i_r)$, $r = 1, 2, 3$, be extensions of A by S . We have to show that $(E_1 + E_2) + E_3$ is equivalent to $E_1 + (E_2 + E_3)$. This

follows by noting that the map $\mu: (T_1 + T_2) + T_3 \rightarrow T_1 + (T_2 + T_3)$, defined by

$$\overline{((t_1, t_2), t_3)} \mu = \overline{(t_1, (t_2, t_3))},$$

is an equivalence of extensions.

Identity for +. Let $E = (T, \pi, i)$ be an extension of A by S . Then the map $\mu: T \rightarrow T + (S \times A)$, given by $t\mu = \overline{(t, (t\pi, 0_{(t\pi)*t\pi}))}$, is an equivalence of extensions between E and $E + E_0$. Hence $[E] + [E_0] = [E + E_0] = [E] = [E_0] + [E]$.

Inverse (relative to +). Let $E = (T, \pi, i)$ be an extension of A by S . Denote by $-E = (T, \pi, -i)$ the extension obtained from E by defining $a(-i) = (-a)i$, $a \in A$. We claim that $[-E]$ is the inverse of $[E]$. It clearly suffices to show that the extension $E + (-E)$ is split. Choose a map $j: S \rightarrow T$ such that $j\pi = \text{Id}_S$. Define $\nu: S \rightarrow T + T$ by $(x)\nu = \overline{(xj, xj)}$. It follows from the definition of $E + (-E)$ that the map ν is a homomorphism. Obviously $\nu\pi = \text{Id}_S$. Hence $E + (-E)$ is a split extension. This completes the proof of the theorem

Next we will study the functorial properties of $\text{Ext}(S, A)$.

PROPOSITION 3.4. *Let $E = (T, \pi, i)$ be an extension of A by S and let $\phi: A \rightarrow B$ be a homomorphism of $D(S)$ -modules. Then there is an extension $\phi E = (U, \pi', i')$ of B by S and a homomorphism $\mu: T \rightarrow U$ such that $(\phi, \mu, \text{Id}_S): E \rightarrow \phi E$ is a morphism of extensions. The pair $(\phi E, \mu)$ is unique upto an equivalence.*

PROOF. Let $T \times B'$ be the semi-direct product of T and B' with respect to an inverse map $t \mapsto t^*: T \rightarrow T$, where B' denote the composite $D(T) \xrightarrow{D(\pi)} D(S) \xrightarrow{B} Ab$. For $(e, 0_e) \in E(T \times B')$, let

$$N_{(e, 0_e)} = \{((a)i, (-a)\phi) \in T \times B' | a \in A_{em}\}.$$

In view of the bijection $(e, 0_e) \leftrightarrow e$ between $E(T \times B')$ and $E(T)$, we denote the group $N_{(e, 0_e)}$ simply by N_e . We shall prove that $N = \{N_e | e \in E(T)\}$ is a group kernel normal system of $T \times B'$.

- (i) Clearly $(e, 0_e) \in N_e$, for each idempotent $(e, 0_e)$ of $T \times B'$.
- (ii) Suppose that $((a)i, (-a)\phi) \in N_e$ and let $(f, 0_f) < (e, 0_e)$. Then $f < e$ in T so that $(ai)f = f(ai)$ and, by Definition 2.1(ii), $((ai)f)^* = ((-a)i)f$. It follows that

$$((a)i, (-a)\phi)(f, 0_f) = (f, 0_f)((a)i, (-a)\phi).$$

- (iii) Suppose that $((a)i, (-a)\phi) \in N_e$ and let

$$((t, b), (u, (-b)B'([t^*t, t^*tu, u^*ut]))$$

be a regular pair in $T \times B'$ such that

$$(t, b) (u, (-b)B'([t^*t, t^*tu, u^*ut])) = (tu, 0_u) < (e, 0_e).$$

Since $(u(a)i)^*u(a)i \mathcal{L} u^*u \mathcal{L} tu, tu \in S((u(a)i)^*u(a)i, tt^*)$. Therefore,

$$\begin{aligned} (u, (-b)B'([t^*t, t^*tu, u^*ut]))((a)i, (-a)\phi)(t, b) \\ = (u(ai)t, (-b)B'([t^*t, t^*(ai)t, u(u(ai))^*ut])) \\ + (-a)\phi B'([e, t, u]) + bB'([t^*t, t^*t, ut])) \\ = (u(ai)t, (-a)\phi B'([e, t, u])), \end{aligned}$$

since

$$B'([t^*t, t^*(ai)t, u(u(ai))^*ut]) = B'([t^*t, t^*t, ut]).$$

Now, by putting $a' = aA([e\pi, t\pi, u\pi])$, we see that

$$(u(ai)t, (-a)\phi B'([e, t, u])) = ((a')i, (-a')\phi) \in N_{u'}.$$

Hence $N = \{N_e | e \in E(T)\}$ is a group kernel normal system of $T \times B'$.

Write $U = T \times B' / \rho_N$. The composite $T \times B' \xrightarrow{\pi_0} T \xrightarrow{\pi} S$ induces a surjective homomorphism $\pi': U \rightarrow S$, which is idempotent-separating. If we define $i': B \rightarrow \text{Ker } \pi'$ by $(b)i' = (e, b)\rho_N, b \in B_e$, then, clearly, i' is a monomorphism. It is also an epimorphism. For, if $((t, b)\rho_N)\pi' = ((e, 0_e)\rho_N)\pi'$, for some $(e, 0_e)\rho_N \in E(U)$, then $t\pi = e\pi$ so that we can find an element $a \in A_{e\pi}$ such that $(a)i = t$. Hence $(t, b)\rho_N = (e, (a)\phi - b)\rho_N = ((a)\phi - b)i'$. Therefore i' is an isomorphism of $D(S)$ -modules. Consequently, (U, π', i') is an extension of B by S which we denote by ϕE . Finally, we define $\mu: T \rightarrow U$ by $(t)\mu = (t, 0_{t^*t})\rho_N$. It is easily seen that $(\phi, \mu, \text{Id}_S): E \rightarrow \phi E$ is a morphism of extensions.

Suppose that $((V, \pi'', i''), \mu')$ is any other such pair. Then the map $\mu'': T \times B' \rightarrow V$, defined by $(t, a)\mu'' = (t)\mu'(a)i''$, is easily seen to be a homomorphism of regular semigroups. If $((a)i, (-a)\phi) \in N_e$ then $((a)i, (-a)\phi)\mu'' = (ai)\mu'((-a)\phi)i'' = (a\phi)i''((-a)\phi)i'' = 0$. Therefore μ'' induces a homomorphism $\bar{\mu}: U \rightarrow V$. Clearly $\bar{\mu}$ is an equivalence of extensions and satisfies the equation $\mu\bar{\mu} = \mu'$. This completes the proof of the proposition.

PROPOSITION 3.5. *Let $E = (T, \pi, i)$ be an extension of A by S and let $\theta: S' \rightarrow S$ be a homomorphism of regular semigroups. Let $D(\theta)A$ denote the composite $D(S') \xrightarrow{D(\theta)} D(S) \xrightarrow{A} Ab$. Then there is an extension $E\theta = (T', \pi', i')$ of $D(\theta)A$ by S' and a homomorphism $\mu: T' \rightarrow T$ of regular semigroups such that $(\text{Id}_{D(\theta)A}, \mu, \theta): E\theta \rightarrow E$ is a morphism of extensions. The pair $(E\theta, \mu)$ is unique up to an equivalence.*

PROOF. Let $T' = \{(x, t) \in S' \times T \mid x\theta = t\pi\}$ be the regular subsemigroup of $S' \times T$. The projection $\pi': T' \rightarrow S'$, defined by $(x, t)\pi' = x$, and the homomorphism $i': D(\theta)A \rightarrow \text{Ker } \pi'$ of $D(S')$ -modules, given by $(a)i' = (e, (a)i)$, $e \in E(S')$, $a \in (D(\theta)A)_e = A_{e\theta}$, define an extension $E\theta = (T', \pi', i')$. If we define $\mu: T' \rightarrow T$ by $(x, t)\mu = t$, $(x, t) \in T'$, then, clearly $(\text{Id}_{D(\theta)A}, \mu, \theta): E\theta \rightarrow E$ is a morphism of extensions.

Suppose that $(E' = (T'', \pi'', i''), \mu'')$ is any other such pair. Then the homomorphism $\mu': T'' \rightarrow T'$, given by $(t)\mu' = (t\pi'', t\mu'')$, is an equivalence of extensions and satisfies the equation $\mu'\mu = \mu''$. Hence $(E\theta, \mu)$ is unique up to an equivalence.

The next result is immediate from Propositions 3.4 and 3.5.

PROPOSITION 3.6. *Let $\theta: S' \rightarrow S$ be a homomorphism of regular semigroups and let $\phi: A \rightarrow B$ be a homomorphism of $D(S)$ -modules. If $E = (T, \pi, i)$ is an extension of A by S then the extension $(D(\theta)\phi)(E\theta)$ is equivalent to the extension $(\phi E)\theta$. Here $D(\theta)\phi$ denote the homomorphism $D(\theta)A \rightarrow D(\theta)B$ of $D(S')$ -modules.*

We shall write \mathcal{C} for the following category: an object of \mathcal{C} is a pair (S, A) with S a regular semigroup and A a $D(S)$ -module; a morphism $(\theta, \phi): (S, A) \rightarrow (T, B)$ in \mathcal{C} consists of a homomorphism $\theta: S \rightarrow T$ of regular semigroups and a homomorphism $\phi: D(\theta)B \rightarrow A$ of $D(S)$ -modules.

THEOREM 3.7. *The mapping which associates to each object (S, A) of \mathcal{C} , the abelian group $\text{Ext}(S, A)$ and, to each morphism $(\theta, \phi): (S, A) \rightarrow (T, B)$, the homomorphism $\text{Ext}(\theta, \phi): \text{Ext}(T, B) \rightarrow \text{Ext}(S, A)$ given by the composite $\text{Ext}(T, B) \rightarrow \text{Ext}(S, D(\theta)B) \rightarrow \text{Ext}(S, A)$ is a contravariant functor from \mathcal{C} to the category of abelian groups.*

PROOF. If $\theta: S \rightarrow T$ is a homomorphism of regular semigroups and B is a $D(T)$ -module then, by Proposition 3.5, the map $[E] \mapsto [E\theta]: \text{Ext}(T, B) \rightarrow \text{Ext}(S, D(\theta)B)$ is a homomorphism of abelian groups. Similarly, if S is a regular semigroup, then by Proposition 3.4, $\text{Ext}(S, -)$ is a functor from the category of $D(S)$ -modules to the category of abelian groups. In particular, if $\phi: A \rightarrow A'$ is a homomorphism of $D(S)$ -modules then $[E] \mapsto [\phi E]: \text{Ext}(S, A) \rightarrow \text{Ext}(S, A')$ is a homomorphism of abelian groups. Therefore if $(\theta, \phi): (S, A) \rightarrow (T, B)$ is a morphism of \mathcal{C} then $\text{Ext}(\theta, \phi)$ is a homomorphism of abelian groups from $\text{Ext}(T, B)$ to $\text{Ext}(S, A)$. It remains to show that $\text{Ext}(-, -)$ is contravariant. Let

$(\theta, \phi): (S, A) \rightarrow (T, B)$ and $(\mu, \psi): (T, B) \rightarrow (U, C)$ be morphisms of \mathcal{C} . Let $[E] \in \text{Ext}(U, C)$ then

$$\begin{aligned} ([E]) \text{Ext}(\mu, \psi) \text{Ext}(\theta, \phi) &= [\phi\{(\psi(E\mu))\theta\}] \\ &= [\phi\{(D(\theta)\psi)((E\mu)\theta)\}], \text{ by Proposition 3.6} \\ &= ([E]) \text{Ext}(\theta\mu, (D(\theta)\psi)\phi) \\ &= ([E]) \text{Ext}((\theta, \phi)(\mu, \psi)). \end{aligned}$$

Therefore, $\text{Ext}(\mu, \psi) \text{Ext}(\theta, \phi) = \text{Ext}((\theta, \phi)(\mu, \psi))$. Hence $\text{Ext}(-, -)$ is a contravariant functor.

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