7

The X-ray Transform in Non-positive Curvature

Consider the geodesic X-ray transform I_m acting on symmetric m-tensor fields. We have proved in Theorem 4.4.1 that I_0 is injective on any simple surface. It follows from Theorem 4.4.2 also that I_1 is solenoidal injective. In this chapter we make the additional assumption that (M, g) has non-positive Gaussian curvature, and prove the classical result of Pestov and Sharafutdinov (1987) that I_m is solenoidal injective for any m. The proof at this point follows easily from the vertical Fourier analysis and the Guillemin–Kazhdan identity in Chapter 6. We will prove later in Chapter 10 the solenoidal injectivity of I_m on any simple surface, but this requires additional technology.

We will also use the assumption of non-positive curvature to improve the H^1 stability estimate for I_0 given in Theorem 4.6.4 to a sharper $H_T^{1/2}$ estimate, which parallels the classical Radon transform estimate in Theorem 1.1.8. A similar stability estimate will be given for I_m. Finally, on simple surfaces with strictly negative Gaussian curvature, we give rather strong Carleman estimates that, in particular, imply the injectivity of the attenuated geodesic X-ray transform. All these results are based on the Guillemin–Kazhdan identity, considered as a frequency localized version of the Pestov identity and shifted to a different Sobolev scale. The stability estimates were first given in Paternain and Salo (2021) and the Carleman estimates in Paternain and Salo (2018).

7.1 Tensor Tomography

Recall from Section 6.4 that the geodesic X-ray transform I_m acting on symmetric m-tensor fields is said to be s-injective if any $h \in C^\infty(S^m(T^*M))$ with $I_m h = 0$ is a potential tensor, i.e. $h = d_s p$ where $p \in C^\infty(S^{m-1}(T^*M))$ with $p|_{\partial M} = 0$. The following result settles the uniqueness question for I_m on simple surfaces with non-positive curvature.

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Theorem 7.1.1 *Let (M, g) be a simple surface with non-positive curvature. Then I_m is s-injective for any $m \geq 0$.*

The case $m = 0$ was already established in Theorem 4.4.2 so we will assume that $m \geq 1$. Using the reduction to a transport equation problem given in Proposition 6.4.4, it is sufficient to prove the following result.

Theorem 7.1.2 *Let (M, g) be a simple surface with non-positive curvature. If $u \in C^\infty(SM)$ satisfies $Xu = f$ in SM and $u|_{\partial SM} = 0$, and if f has degree $m \geq 1$, then u has degree $m - 1$.*

The proof relies on the following basic fact stating that the equation $Xu = f$ can be written in terms of the Fourier coefficients of u and f using the splitting $X = \eta_+ + \eta_-$.

Lemma 7.1.3 (Fourier coefficients of Xu) *Let (M, g) be a compact oriented surface with smooth boundary, and let $u \in C^\infty(SM)$ satisfy $Xu = f$. Then*

$$\eta_+ u_{k-1} + \eta_- u_{k+1} = f_k, \qquad k \in \mathbb{Z}.$$

In particular, if f has degree m, then

$$\eta_+ u_{k-1} + \eta_- u_{k+1} = 0, \qquad |k| \geq m + 1.$$

Proof We use the following facts from Lemma 6.1.3 and Lemma 6.1.5:

- $u = \sum_{k=-\infty}^{\infty} u_k$ with convergence in $C^\infty(SM)$;
- $Xu = \eta_+ u + \eta_- u$ where $\eta_\pm \colon \Omega_k \to \Omega_{k\pm 1}$;
- $f = \sum_{k=-\infty}^{\infty} f_k$ with convergence in $C^\infty(SM)$.

Using these facts and collecting terms of the same order, the equation $Xu = f$ implies that

$$\eta_+ u_{k-1} + \eta_- u_{k+1} = f_k.$$

The result follows. □

The main result now follows by using the Guillemin–Kazhdan identity, or more precisely its consequence (Beurling contraction property) in Proposition 6.5.2.

Proof of Theorem 7.1.2 By Lemma 7.1.3 one has

$$\eta_+ u_{k-1} + \eta_- u_{k+1} = 0, \qquad |k| \geq m + 1. \tag{7.1}$$

Assume first that $k \geq m + 1$. Since the Gaussian curvature is non-positive, the Beurling contraction property (Theorem 6.5.2) implies that

$$\|\eta_- u_{k-1}\| \leq \|\eta_+ u_{k-1}\|.$$

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Combining this with (7.1) yields

$$\|\eta_- u_{k-1}\| \leq \|\eta_- u_{k+1}\|, \qquad k \geq m+1. \tag{7.2}$$

Iterating (7.2) N times yields

$$\|\eta_- u_{k-1}\| \leq \|\eta_- u_{k-1+2N}\|, \qquad k \geq m+1.$$

We now note that since $u \in C^\infty(SM)$, one has $\eta_- u \in L^2(SM)$. This implies that $\sum \|\eta_- u_l\|^2 < \infty$, which in particular gives $\|\eta_- u_l\| \to 0$ as $l \to \pm\infty$. We can thus let $N \to \infty$ above to obtain that

$$\eta_- u_{k-1} = 0, \qquad k \geq m+1. \tag{7.3}$$

We may combine (7.3) and (7.1) to obtain that

$$\eta_- u_l = \eta_+ u_l = 0, \qquad l \geq m.$$

Since $X = \eta_+ + \eta_-$, we thus have for $l \geq m$ that

$$X u_l = 0, \qquad u_l|_{\partial SM} = 0.$$

This shows that u_l is constant along geodesics and vanishes at the boundary. Thus we must have

$$u_l = 0, \qquad l \geq m.$$

A similar argument for $k \leq -m-1$, using the second part of Theorem 6.5.2, yields that

$$u_l = 0, \qquad l \leq -m.$$

This concludes the proof. □

Remark 7.1.4 The proof above has historical significance as it is virtually identical to the original proof in Guillemin and Kazhdan (1980a) of solenoidal injectivity for closed surfaces of negative curvature. Guillemin and Kazhdan were originally interested in the problem of infinitesimal spectral rigidity.

7.2 Stability for Functions

Let (M, g) be a compact simple surface, and let I_0 be the geodesic X-ray transform. Recall from Theorem 4.6.4 that the X-ray transform enjoys the stability estimate

$$\|f\|_{L^2(M)} \leq C \|I_0 f\|_{H^1(\partial_+ SM)}$$

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for any $f \in C^\infty(M)$. We compare this with the stability estimate for the Radon transform in $\mathbb{R}^2$ from Theorem 1.1.8, which states that

$$\|f\|_{L^2(\mathbb{R}^2)} \leq \frac{1}{\sqrt{2}} \|Rf\|_{H_T^{1/2}(\mathbb{R}\times S^1)}$$

for $f \in C_c^\infty(\mathbb{R}^2)$. Note that the estimate for Rf is stated in parallel-beam geometry, whereas the estimate for $I_0 f$ is stated in fan-beam geometry.

There are two important differences between the above stability estimates: the latter estimate involves an $H^{1/2}$ norm instead of H^1, and the $H_T^{1/2}$ norm is only taken with respect to the s-variable in $Rf(s,\omega)$ in the sense that

$$\|Rf\|_{H_T^{1/2}(\mathbb{R}\times S^1)} = \|(1+\sigma^2)^{1/4}(Rf)\tilde{}(\sigma,\omega)\|_{L^2(\mathbb{R}\times S^1)}.$$

In this section we will improve the stability estimate for $I_0 f$ and replace the H^1 norm with a suitable $H_T^{1/2}$ norm. This will be done by using vertical Fourier expansions and the Guillemin–Kazhdan identity. However, we will need the additional assumption that (M,g) has non-positive curvature.

We introduced in Section 4.5 the vector field T that is tangent to ∂SM. Define the $H_T^1(\partial SM)$ norm via

$$\|w\|^2_{H_T^1(\partial SM)} = \|w\|^2_{L^2(\partial SM)} + \|Tw\|^2_{L^2(\partial SM)}.$$

Note that this is different from the $H^1(\partial SM)$ norm, which was given by

$$\|w\|^2_{H^1(\partial SM)} = \|w\|^2_{L^2(\partial SM)} + \|Tw\|^2_{L^2(\partial SM)} + \|Vw\|^2_{L^2(\partial SM)}.$$

Thus the H_T^1 norm only involves the horizontal tangential derivatives along ∂M, but not the vertical derivatives.

The space $H_T^{1/2}(\partial SM)$ is defined as the complex interpolation space between $L^2(\partial SM)$ and $H_T^1(\partial SM)$ (for interpolation spaces, see Bergh and Löfström (1976)). The spaces $H_T^1(\partial_+ SM)$ and $H_T^{1/2}(\partial_+ SM)$ are defined in a similar way. The following stability estimate is the main result in this section.

Theorem 7.2.1 *Let (M,g) be a compact simple surface with non-positive Gaussian curvature. Then*

$$\|f\|_{L^2(M)} \leq \frac{1}{\sqrt{2\pi}} \|I_0 f\|_{H_T^{1/2}(\partial_+ SM)}, \qquad f \in C^\infty(M).$$

The first step in the proof is to rewrite the boundary term in the Guillemin–Kazhdan identity in terms of the tangential vector field T from Definition 4.5.2.

Proposition 7.2.2 *Let (M,g) be a compact surface with smooth boundary. For any $u \in C^\infty(SM)$ one has*

$$\|\eta_- u\|^2 = \|\eta_+ u\|^2 - \frac{i}{2}(KVu,u) + \frac{i}{2}(Tu,u)_{\partial SM}.$$

Proof From Lemma 6.5.1 we have

$$\|\eta_- u\|^2 = \|\eta_+ u\|^2 - \frac{i}{2}(KVu,u) - (\eta_- u, \mu_{-1} u)_{\partial SM} + (\eta_+ u, \mu_1 u)_{\partial SM}.$$

Since $\mu = \langle v, \nu\rangle$ and $V\mu = -\langle v_\perp, \nu\rangle$, so that $\mu_{\pm 1} = \frac{1}{2}(\mu \mp iV\mu)$, the boundary terms become

$$\frac{1}{2}\left[(\eta_+ u, (\mu - iV\mu)u)_{\partial SM} - (\eta_- u, (\mu + iV\mu)u)_{\partial SM}\right].$$

Using that $\eta_\pm = \frac{1}{2}(X \pm iX_\perp)$, the boundary terms further simplify to

$$\frac{1}{2}\left[-(Xu, i(V\mu)u)_{\partial SM} + i(X_\perp u, \mu u)_{\partial SM}\right] = \frac{i}{2}((V\mu)Xu + \mu X_\perp u, u)_{\partial SM}.$$

By Lemma 4.5.4 the last expression is equal to $\frac{i}{2}(Tu,u)_{\partial SM}$. □

Next we consider a version of the Beurling contraction property with boundary terms on surfaces with non-positive curvature.

Proposition 7.2.3 *Let (M,g) be a compact surface with smooth boundary. Suppose that $K \le -\kappa_0$ for some $\kappa_0 \ge 0$, and let $u \in \Omega_k$. If $k \ge 0$ then*

$$\|\eta_- u\|^2 + \frac{\kappa_0}{2}k\|u\|^2 \le \|\eta_+ u\|^2 + \frac{i}{2}(Tu,u)_{\partial SM},$$

whereas if $k \le 0$ one has

$$\|\eta_+ u\|^2 + \frac{\kappa_0}{2}|k|\|u\|^2 \le \|\eta_- u\|^2 - \frac{i}{2}(Tu,u)_{\partial SM}.$$

Proof This follows directly from Proposition 7.2.2. □

Given $f \in C^\infty(M)$, we wish to apply the Beurling contraction property to the Fourier coefficients of u^f. The function u^f is not, in general, in $C^\infty(SM)$, so we will work in slightly smaller sets as in Section 4.5. Let $\rho \in C^\infty(M)$ satisfy $\rho(x) = d(x,\partial M)$ near ∂M with $\rho > 0$ in M^{int} and $\partial M = \rho^{-1}(0)$. Define $\nu(x) = \nabla\rho(x)$ for $x \in M$, let $\mu(x,v) := \langle v, \nu(x)\rangle$ for $(x,v) \in SM$, and define

$$T := (V\mu)X + \mu X_\perp.$$

Thus T extends the tangential vector field from ∂SM into SM. By Exercise 4.5.6 it satisfies $[V,T] = 0$ in SM. Define $M_\varepsilon := \{x \in M\,;\, \rho(x) \ge \varepsilon\}$.

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We start the proof of Theorem 7.2.1 with the following result, which estimates f in terms of an inner product on ∂SM involving $u^f|_{\partial SM}$, the tangential vector field T, and the fibrewise Hilbert transform H (see Section 6.2). Recall that in (4.5) we proved the estimate

$$\|f\|^2_{L^2(SM)} \leq -(Tu^f, Vu^f)_{\partial SM}.$$

The estimate below is better, since the right-hand side does not involve vertical derivatives of u.

Lemma 7.2.4 *Let (M,g) be a compact simple surface with non-positive curvature. For any $f \in C^\infty(M)$, one has*

$$\|f\|^2_{L^2(SM)} \leq \left(Tu^f, Hu^f\right)_{\partial SM}.$$

Proof Let $f \in C^\infty(M)$ and let $u = u^f$, so that $Xu = -f$ and u is smooth in SM_ε for $\varepsilon > 0$ small. Since the curvature is non-positive, for any $k \geq 0$ Proposition 7.2.3 gives that

$$\|\eta_- u_k\|^2_{SM_\varepsilon} \leq \|\eta_+ u_k\|^2_{SM_\varepsilon} + \frac{i}{2}(Tu_k, u_k)_{\partial SM_\varepsilon}. \tag{7.4}$$

Notice also that the equation $Xu = -f$ gives $\eta_+ u_k + \eta_- u_{k+2} = 0$ for $k \geq 0$ (see Lemma 7.1.3). Combining this with the inequality above yields

$$\|\eta_- u_k\|^2_{SM_\varepsilon} \leq \|\eta_- u_{k+2}\|^2_{SM_\varepsilon} + \frac{i}{2}(Tu_k, u_k)_{\partial SM_\varepsilon}. \tag{7.5}$$

We iterate (7.5) for $k = 1, 3, 5, \ldots$ and use the fact that $\|\eta_- u_l\|_{SM_\varepsilon} \to 0$ as $l \to \infty$ (which follows since $\eta_- u \in L^2(SM_\varepsilon)$). This gives that

$$\|\eta_- u_1\|^2_{SM_\varepsilon} \leq \frac{i}{2} \sum_{j=0}^{\infty} (Tu_{1+2j}, u_{1+2j})_{\partial SM_\varepsilon}.$$

A similar argument for $k \leq -1$, using the second part of Proposition 7.2.3, shows that

$$\|\eta_+ u_{-1}\|^2_{SM_\varepsilon} \leq -\frac{i}{2} \sum_{j=0}^{\infty} (Tu_{-1-2j}, u_{-1-2j})_{\partial SM_\varepsilon}.$$

Combining the above estimates and using the equation $Xu = -f$ again gives

$$\begin{aligned}\|f\|^2_{SM_\varepsilon} &= \|\eta_- u_1 + \eta_+ u_{-1}\|^2_{SM_\varepsilon} \leq 2(\|\eta_- u_1\|^2_{SM_\varepsilon} + \|\eta_+ u_{-1}\|^2_{SM_\varepsilon}) \\ &\leq i \sum_{k \text{ odd}} (Tu_k, w_k)_{\partial SM_\varepsilon},\end{aligned}$$

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where

$$w_k := iHu_k = \begin{cases} u_k, & k > 0, \\ -u_k, & k < 0. \end{cases}$$

We next use the fact that $[T, V] = 0$, which implies that T maps Ω_k to Ω_k. Hence the estimate for f may be rewritten as

$$\|f\|^2_{SM_\varepsilon} \leq (Tu, Hu)_{\partial SM_\varepsilon}.$$

Since $u^f|_{\partial_+ SM} = I_0 f$ and $u^f|_{\partial_- SM} = 0$, one has $u|_{\partial SM} \in L^2(\partial SM)$. By Corollary 4.5.8 one also has $Tu^f|_{\partial SM} \in L^2(\partial SM)$. In particular, $u|_{\partial SM} \in H^1_T(\partial SM)$. One also has $Hu|_{\partial SM} \in H^1_T(\partial SM)$, since

$$\|Hu\|^2_{\partial SM} \leq \sum \|u_k\|^2_{\partial SM} = \|u\|^2_{\partial SM}, \tag{7.6}$$

$$\|Hu\|^2_{H^1_T(\partial SM)} \leq \sum \left(\|u_k\|^2_{\partial SM} + \|Tu_k\|^2_{\partial SM}\right) = \|u\|^2_{H^1_T(\partial SM)}. \tag{7.7}$$

The last identity used again that $[V, T] = 0$. Taking the limit as $\varepsilon \to 0$ as in Exercise 4.5.9 gives that

$$\|f\|^2_{SM} \leq (Tu, Hu)_{\partial SM}. \qquad \square$$

Next we give an estimate for the right-hand side of the previous lemma.

Lemma 7.2.5 *Let (M, g) be a compact surface with smooth boundary. For any $u, w \in H^1_T(\partial SM)$ one has*

$$|(Tu, Hw)_{\partial SM}| \leq \|u\|_{H^{1/2}_T(\partial SM)} \|w\|_{H^{1/2}_T(\partial SM)}.$$

Proof Given $s > 0$, let $H^{-s}_T(\partial SM)$ be the dual space of $H^s_T(\partial SM)$. We first use the estimate

$$|(Tu, Hw)_{\partial SM}| \leq \|Tu\|_{H^{-1/2}_T(\partial SM)} \|Hw\|_{H^{1/2}_T(\partial SM)}.$$

Interpolating (7.6) and (7.7) shows that H satisfies

$$\|Hw\|_{H^{1/2}_T(\partial SM)} \leq \|w\|_{H^{1/2}_T(\partial SM)}.$$

It remains to estimate the norm of Tu. First note that

$$\|Tu\|_{L^2(\partial SM)} \leq \|u\|_{H^1_T(\partial SM)}.$$

Next we estimate the H^{-1}_T norm using that T is skew-adjoint (see Lemma 4.5.4):

$$\begin{aligned}\|Tu\|_{H^{-1}_T(\partial SM)} &= \sup_{\|w\|_{H^1_T} = 1} (Tu, w)_{\partial SM} = - \sup_{\|w\|_{H^1_T} = 1} (u, Tw)_{\partial SM} \\ &\leq \|u\|_{L^2(\partial SM)}.\end{aligned}$$

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Interpolating the two estimates above gives

$$\|Tu\|_{H_T^{-1/2}(\partial SM)} \leq \|u\|_{H_T^{1/2}(\partial SM)}$$

as required. □

Combining Lemma 7.2.4 and Lemma 7.2.5, we obtain a stability estimate for f in terms of u^f:

Lemma 7.2.6 *Let (M, g) be a compact simple surface with non-positive curvature. For any $f \in C^\infty(M)$, one has*

$$\|f\|_{L^2(SM)} \leq \|u^f\|_{H_T^{1/2}(\partial SM)}.$$

We can now prove the main stability result.

Proof of Theorem 7.2.1 Recall that $u^f|_{\partial_+ SM} = I_0 f$ and $u^f|_{\partial_- SM} = 0$. Thus $u^f|_{\partial SM} = E_0(I_0 f)$ where E_0 is the operator that extends a function by zero from $\partial_+ SM$ to ∂SM. It follows from Lemma 7.2.6 that

$$\|f\|_{L^2(SM)} \leq \|E_0(I_0 f)\|_{H_T^{1/2}(\partial SM)}. \tag{7.8}$$

We clearly have

$$\|E_0 h\|_{L^2(\partial SM)} \leq \|h\|_{L^2(\partial_+ SM)}, \qquad h \in L^2(\partial_+ SM).$$

Let $H^1_{T,0}(\partial_+ SM)$ be the closure of $C_c^\infty((\partial_+ SM)^{\text{int}})$ in $H^1_T(\partial_+ SM)$. Then

$$\|E_0 h\|_{H^1_T(\partial SM)} \leq \|h\|_{H^1_{T,0}(\partial_+ SM)}$$

first for $h \in C_c^\infty((\partial_+ SM)^{\text{int}})$ and then for $h \in H^1_{T,0}(\partial_+ SM)$ by density. Let $H^{1/2}_{T,0}(\partial_+ SM)$ be the complex interpolation space between $L^2(\partial_+ SM)$ and $H^1_{T,0}(\partial_+ SM)$. Interpolation gives that

$$\|E_0 h\|_{H_T^{1/2}(\partial SM)} \leq \|h\|_{H^{1/2}_{T,0}(\partial_+ SM)}.$$

Since $I_0 f \in H^1_0(\partial_+ SM)$ by Proposition 4.1.3, it follows in particular that $I_0 f \in H^{1/2}_{T,0}(\partial_+ SM)$. Thus

$$\|E_0(I_0 f)\|_{H_T^{1/2}(\partial SM)} \leq \|I_0 f\|_{H_T^{1/2}(\partial_+ SM)}. \tag{7.9}$$

Combining (7.8) and (7.9) gives the desired estimate

$$\sqrt{2\pi}\,\|f\|_{L^2(M)} = \|f\|_{L^2(SM)} \leq \|I_0 f\|_{H_T^{1/2}(\partial_+ SM)}.$$ □

7.3 Stability for Tensors

We will now give a stability estimate for I_m where $m \geq 1$. Recall that solenoidal injectivity of I_m means that the only symmetric m-tensors satisfying $I_m f = 0$ are of the form $f = d_s h$ where $h \in C^\infty(S^{m-1}(T^*M))$ and $h|_{\partial M} = 0$. This means that from the knowledge of $I_m f$ one only expects to recover the solenoidal part f^s of f (see Theorem 6.4.7). The following result gives a stability estimate for this problem. A very similar estimate was obtained in Boman and Sharafutdinov (2018) for Euclidean domains, but phrased using parallel-beam geometry.

Theorem 7.3.1 *Let (M,g) be a compact simple surface with non-positive Gaussian curvature. For any $m \geq 1$ one has*

$$\|f^s\|_{L^2(M)} \leq C\|I_m f\|_{H_T^{1/2}(\partial_+ SM)}, \qquad f \in C^\infty(S^m(T^*M)).$$

The proof will be similar to that of Theorem 7.2.1. As in Section 6.3, it will be convenient to identify a symmetric m-tensor field f on M with a function $f \in C^\infty(SM)$ having degree m and to work with the transport equation $Xu^f = -f$ in SM. We begin with an analogue of Lemma 7.2.4 for m-tensors.

Lemma 7.3.2 *Let (M,g) be a compact simple surface with non-positive curvature. For any $f \in C^\infty(SM)$ having degree $m \geq 1$, one has*

$$\|f + X(u_{-(m-1)} + \cdots + u_{m-1})\|^2_{L^2(SM)} \leq \frac{1}{2}(Tu, Hu)_{\partial SM},$$

where $u = u^f$.

Proof We work in a slightly smaller set M_ε as in the proof of Lemma 7.2.4, so that u is smooth in SM_ε. Since $Xu = -f$ and f has degree m, one has $\eta_+ u_k + \eta_- u_{k+2} = 0$ for $k \geq m$. Thus from (7.4) we obtain an analogue of (7.5):

$$\|\eta_- u_k\|^2_{SM_\varepsilon} \leq \|\eta_- u_{k+2}\|^2_{SM_\varepsilon} + \frac{i}{2}(Tu_k, u_k)_{\partial SM_\varepsilon}, \qquad k \geq m.$$

Iterating this for $k = m, m+2, \ldots,$ and using that $\eta_- u_k \to 0$ in $L^2(SM_\varepsilon)$ as $k \to \infty$, gives

$$\|\eta_- u_m\|^2_{SM_\varepsilon} \leq \frac{i}{2}\sum_{j=0}^{\infty}(Tu_{m+2j}, u_{m+2j})_{\partial SM_\varepsilon}.$$

Starting with $k = m+1$ instead, and adding the resulting estimates, yields that

$$\|\eta_- u_m\|^2_{SM_\varepsilon} + \|\eta_- u_{m+1}\|^2_{SM_\varepsilon} \leq \frac{i}{2}\sum_{j=0}^{\infty}(Tu_{m+j}, u_{m+j})_{\partial SM_\varepsilon}.$$

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A similar argument for $k \leq -m$, using the second part of Proposition 7.2.3, shows that

$$\|\eta_+ u_{-m}\|^2_{SM_\varepsilon} + \|\eta_+ u_{-m-1}\|^2_{SM_\varepsilon} \leq -\frac{i}{2}\sum_{j=0}^{\infty}(Tu_{-m-j}, u_{-m-j})_{\partial SM_\varepsilon}.$$

The equation $Xu = -f$, where f has degree m, and the fact that both T and H map Ω_k to Ω_k, imply that

$$\begin{aligned}
&\|f + X(u_{-(m-1)} + \cdots + u_{m-1})\|^2_{SM_\varepsilon} \\
&\quad = \|\eta_- u_m\|^2_{SM_\varepsilon} + \|\eta_- u_{m+1}\|^2_{SM_\varepsilon} + \|\eta_+ u_{-m}\|^2_{SM_\varepsilon} + \|\eta_+ u_{-m-1}\|^2_{SM_\varepsilon} \\
&\quad \leq \frac{1}{2}(Tu, Hu)_{\partial SM_\varepsilon}.
\end{aligned}$$

Taking the limit as $\varepsilon \to 0$ as in the end of proof of Lemma 7.2.4 proves the result. □

Combining Lemma 7.3.2 and Lemma 7.2.5 gives the desired stability estimate for f in terms of u^f:

Lemma 7.3.3 *Let (M, g) be a compact simple surface with non-positive curvature. For any $f \in C^\infty(SM)$ having degree $m \geq 1$, one has*

$$\|f + X(u_{-(m-1)} + \cdots + u_{m-1})\|_{L^2(SM)} \leq \frac{1}{\sqrt{2}}\|u\|_{H^{1/2}_T(\partial SM)}$$

where $u = u^f$.

Theorem 7.3.1 will now follow by rewriting the above estimate in a form that involves the solenoidal part f^s.

Proof of Theorem 7.3.1 Given $f \in C^\infty(S^m(T^*M))$, we will use the isomorphism ℓ_m in Proposition 6.3.5 and write $\tilde{f} := \ell_m f$, $\tilde{u} := u^{\tilde{f}}$ and

$$\tilde{q} := -\sum_{\substack{|k|\leq m-1 \\ k \text{ is odd/even}}} \tilde{u}_k, \tag{7.10}$$

where the sum is over odd k if m is even, and over even k if m is odd.

Using Lemma 7.3.3 and the parity of $\tilde{f}$ and $X\tilde{q}$, we have

$$\left\|\tilde{f} - X\tilde{q}\right\|^2 \leq \left\|\tilde{f} + X(\tilde{u}_{-(m-1)} + \cdots + \tilde{u}_{m-1})\right\|^2 \leq \frac{1}{2}\|\tilde{u}\|^2_{H^{1/2}_T(\partial SM)}.$$

Let $q = \ell_{m-1}^{-1}\tilde{q}$, so that $X\tilde{q} = \ell_m d_s q$ by Lemma 6.3.2. Using (6.15), we obtain that

$$\|f - d_s q\|_{L^2(M)} \leq C_m\|\tilde{f} - X\tilde{q}\|_{L^2(SM)} \leq C_m\|\tilde{u}\|_{H^{1/2}_T(\partial SM)}. \tag{7.11}$$

Let f have solenoidal decomposition $f = f^s + d_s p$. Writing $w := p - q$, we have

$$\|f - d_s q\|^2 = \|f^s + d_s w\|^2 = \|f^s\|^2 + 2\,\mathrm{Re}(f^s, d_s w) + \|d_s w\|^2.$$

Since f^s is symmetric and solenoidal and $p|_{\partial M} = 0$, an integration by parts gives that

$$(f^s, d_s w) = (f^s, \nabla w) = (i_\nu f^s, q)_{\partial M},$$

where $i_\nu f^s(v_1, \ldots, v_{m-1}) := f^s(v_1, \ldots, v_{m-1}, \nu)$. Thus

$$\|f - d_s q\|^2 \geq \|f^s\|^2 - 2|(i_\nu f^s, q)_{\partial M}|.$$

Combining this with (7.11) and using Young's inequality with $\varepsilon > 0$, yield that

$$\|f^s\|^2_{L^2(M)} \leq C\|\tilde{u}\|^2_{H_T^{1/2}(\partial SM)} + \varepsilon\|i_\nu f^s\|^2_{H^{-1/2}(\partial M)} + \frac{1}{\varepsilon}\|q\|^2_{H^{1/2}(\partial M)}.$$

By Lemma 7.3.4 we have $\|i_\nu f^s\|_{H^{-1/2}(\partial M)} \leq C\|f^s\|_{L^2(M)}$, and choosing $\varepsilon > 0$ small enough allows us to absorb this term to the left-hand side. In addition, using Lemma 7.3.5 gives that

$$\|f^s\|_{L^2(M)} \leq C\|\tilde{u}\|_{H_T^{1/2}(\partial SM)}.$$

It remains to note that $\tilde{u}|_{\partial SM} = E_0(I_m f)$ where E_0 denotes extension by zero from $\partial_+ SM$ to ∂SM. Using (7.9) with $I_0 f$ replaced by $I_m f$ concludes the proof. □

Lemma 7.3.4 *If (M, g) is compact with smooth boundary and $f \in C^\infty(S^m(T^*M))$ is solenoidal, then*

$$\|i_\nu f\|_{H^{-1/2}(\partial M)} \leq C\|f\|_{L^2(M)}.$$

Proof The idea is that since f solves $\delta_s f = 0$ in M, the boundary value $i_\nu f|_{\partial M}$ can be interpreted weakly as an element of $H^{-1/2}(\partial M)$. Let $E\colon H^{1/2}(\partial M) \to H^1(M)$ be a bounded extension operator on tensors (such a map can be constructed from a corresponding extension map for functions by working in local coordinates and using a partition of unity). Then, since $\delta_s f = 0$,

$$\begin{aligned}
\|i_\nu f\|_{H^{-1/2}(\partial M)} &= \sup_{\|r\|_{H^{1/2}(\partial M)=1}} (i_\nu f, r)_{\partial M} \\
&= \sup_{\|r\|_{H^{1/2}(\partial M)=1}} -(f, \nabla E r)_M \\
&\leq \sup_{\|r\|_{H^{1/2}(\partial M)=1}} \|f\|_{L^2}\|Er\|_{H^1} \leq C\|f\|_{L^2}.
\end{aligned}$$

□

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Lemma 7.3.5 *If* $q = \ell_{m-1}^{-1}\tilde{q}$ *where* $\tilde{q}$ *is defined by* (7.10), *then*

$$\|q\|_{H^{1/2}(\partial M)} \leq C\|\tilde{u}\|_{H_T^{1/2}(\partial SM)}.$$

Proof We prove the statement by interpolation. Since $\tilde{q} = \ell_{m-1}q$, (6.15) and orthogonality imply that

$$\|q\|_{L^2(\partial M)}^2 \leq C\|\tilde{q}\|_{L^2(\partial SM)}^2 \leq C\|\tilde{u}\|_{L^2(\partial SM)}^2. \tag{7.12}$$

Consider now the $H^1(\partial M)$ norm. In local coordinates we may write $q = q_{j_1\cdots j_{m-1}}dx^{j_1} \otimes \cdots \otimes dx^{j_{m-1}}$, and the $H^1(\partial M)$ norm involves the $L^2(\partial M)$ norms of the components $q_{j_1\cdots j_{m-1}}$ and $\partial_T q_{j_1\cdots j_{m-1}}$, where ∂_T is the tangential derivative. Locally $\tilde{q} = q_{j_1\cdots j_{m-1}}v^{j_1}\cdots v^{j_{m-1}}$. By Definition 4.5.2, we have

$$T\tilde{q} = (\partial_T q_{j_1\cdots j_{m-1}})v^{j_1}\cdots v^{j_{m-1}} + \cdots$$

where $\cdots$ denotes terms whose L^2 norms can be controlled by $\|q\|_{L^2(\partial M)}$. Thus, using (6.15) again,

$$\|q\|_{H^1(\partial M)} \leq C\|\tilde{q}\|_{H_T^1(\partial SM)}.$$

Finally, by Lemma 4.5.4, the operators V and T commute on ∂SM. This implies that $(Tw_k, Tw_l)_{\partial SM} = 0$ if $w_k \in \Omega_k$, $w_l \in \Omega_l$ and $k \neq l$. Thus

$$\|T\tilde{q}\|_{L^2(\partial SM)}^2 = \sum_{\substack{|k|\leq m-1 \\ k \text{ is odd/even}}} \|T\tilde{u}_k\|_{L^2(\partial SM)}^2 \leq \|T\tilde{u}\|_{L^2(\partial SM)}^2.$$

Using the definition of the H_T^1 norm, this shows that

$$\|\tilde{q}\|_{H_T^1(\partial SM)}^2 \leq \|\tilde{u}\|_{H_T^1(\partial SM)}^2.$$

Thus we have proved that

$$\|q\|_{H^1(\partial M)} \leq C\|\tilde{u}\|_{H_T^1(\partial SM)}. \tag{7.13}$$

Interpolating (7.12) and (7.13) proves the statement. □

7.4 Carleman Estimates

In Sections 7.1–7.3, we used the Guillemin–Kazhdan identity to prove uniqueness and stability results for the X-ray transform on simple surfaces with non-positive Gaussian curvature. Here we show that if the curvature is strictly negative, one can apply weights to the Guillemin–Kazhdan identity and obtain stronger Carleman estimates that are robust under certain perturbations. We will use this to prove uniqueness for an attenuated X-ray transform.

Let (M, g) be a simple surface, and let $\mathcal{A} \in C^\infty(SM)$. In Section 5.3 we introduced the attenuated X-ray transform of $f \in C^\infty(SM)$ as

$$I_{\mathcal{A}} f = u^f|_{\partial_+ SM},$$

where u^f is the solution of

$$Xu + \mathcal{A}u = -f \text{ in } SM, \qquad u|_{\partial_- SM} = 0.$$

Clearly $I_{\mathcal{A}}$ is the standard geodesic X-ray transform when $\mathcal{A} = 0$. We will specialize to the case where $f = f(x) \in C^\infty(M)$, so that $I_{\mathcal{A}}$ is acting on 0-tensors, and

$$\mathcal{A} = a_{-1} + a_0 + a_1 \in \Omega_{-1} \oplus \Omega_0 \oplus \Omega_1.$$

Thus the attenuation $\mathcal{A}$ is the sum of a scalar function $a_0(x)$ and a 1-form $a_1 + a_{-1}$.

Theorem 7.4.1 *Let (M, g) be a simple surface with negative Gaussian curvature. If $\mathcal{A} = a_{-1} + a_0 + a_1$ with $a_k \in \Omega_k$, then $I_{\mathcal{A}}$ is injective on $C^\infty(M)$.*

This is a consequence of the following energy estimate:

Theorem 7.4.2 *Let (M, g) be a simple surface with Gaussian curvature $K \leq -\kappa_0$ for some $\kappa_0 > 0$. For any $m \geq 0$ and $\tau \geq 1$, one has*

$$\sum_{|k| \geq m} |k|^{2\tau} \|u_k\|^2 \leq \frac{2}{\kappa_0 \tau} \sum_{|k| \geq m+1} |k|^{2\tau} \|(Xu)_k\|^2,$$

whenever $u \in C^\infty(SM)$ with $u|_{\partial SM} = 0$.

The previous theorem involves a large parameter τ, and the constant on the right is of the form C/τ, which becomes very small when τ is chosen large. As discussed in Paternain and Salo (2018) this behaviour is typical of Carleman estimates, and in fact the weights $|k|^{2\tau}$ can be written as $e^{2\tau\varphi(k)}$ where $\varphi(k) = \log |k|$ corresponds to a logarithmic Carleman weight. Adjusting the parameter $\tau > 0$ will allow us to deal with a possibly large attenuation and prove injectivity of the attenuated X-ray transform. The estimate in Theorem 7.4.2 can also be understood as a version of the Pestov identity shifted to a different vertical Sobolev scale.

This argument based on Carleman estimates is quite robust and it immediately extends to complex matrix-valued attenuations (even some non-linear ones) and tensor fields. However, it requires the additional assumption that the Gaussian curvature is negative. We will remove this curvature assumption later in Chapter 12 (in the scalar case) and Chapters 13–14 (in the matrix case).

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Proof of Theorem 7.4.1 Let $f \in C^\infty(M)$ satisfy $I_\mathcal{A} f = 0$. By Theorem 5.3.6 one has $u := u^f \in C^\infty(SM)$, and u solves the equation

$$Xu + \mathcal{A}u = -f \text{ in } SM, \qquad u|_{\partial SM} = 0. \tag{7.14}$$

Note that for $|k| \geq 1$, since $f = f(x)$ one has

$$\begin{aligned}\|(Xu)_k\| = \|(\mathcal{A}u)_k\| &= \|a_1 u_{k-1} + a_0 u_k + a_{-1} u_{k+1}\| \\ &\leq C(\|u_{k-1}\| + \|u_k\| + \|u_{k+1}\|).\end{aligned}$$

We now insert u in the estimate of Theorem 7.4.2, which yields that

$$\begin{aligned}\sum_{|k|\geq m} |k|^{2\tau} \|u_k\|^2 &\leq \frac{C}{\tau} \sum_{|k|\geq m+1} |k|^{2\tau} \left(\|u_{k-1}\|^2 + \|u_k\|^2 + \|u_{k+1}\|^2\right) \\ &\leq \frac{C}{\tau} \sum_{|k|\geq m} (|k|+1)^{2\tau} \|u_k\|^2.\end{aligned}$$

If we additionally assume that $m \geq 2\tau$, then for $|k| \geq m$ one has

$$(|k|+1)^{2\tau} = |k|^{2\tau}(1 + 1/|k|)^{2\tau} \leq e|k|^{2\tau}.$$

Thus, whenever $m \geq 2\tau$ we have

$$\sum_{|k|\geq m} |k|^{2\tau} \|u_k\|^2 \leq \frac{C_1}{\tau} \sum_{|k|\geq m} |k|^{2\tau} \|u_k\|^2,$$

where C_1 is independent of τ and u. Choosing τ so that $\tau \geq 2C_1$ implies that

$$u_k = 0, \qquad |k| \geq 4C_1.$$

It follows that u must have finite degree.

Finally we need to show that $u \equiv 0$. Suppose that u has degree $l \geq 0$. Then $u_k = 0$ for $k \geq l+1$. Using the equation (7.14), u_l satisfies

$$\eta_+ u_l + a_1 u_l = 0, \qquad u_l|_{\partial SM} = 0.$$

Using the special coordinates (x, θ) and Lemma 6.1.8, so that $M = \overline{\mathbb{D}}$, we have $u_l(x,\theta) = \tilde{u}_l(x) e^{il\theta}$ and $a_1 = \tilde{a}_1(x) e^{i\theta}$ where $\tilde{u}_l \in C^\infty(\overline{\mathbb{D}})$ solves the equation

$$e^{(l-1)\lambda} \partial_z \left(\tilde{u}_l e^{-l\lambda}\right) + \tilde{a}_1 \tilde{u}_l = 0 \text{ in } \mathbb{D}, \qquad \tilde{u}_l|_{\partial\mathbb{D}} = 0.$$

We choose an integrating factor $h \in C^\infty(\overline{\mathbb{D}})$ (for instance by using the Cauchy transform) that solves

$$\partial_z h = e^\lambda \tilde{a}_1 \text{ in } \mathbb{D}.$$

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Then

$$\partial_z\left(e^h \tilde{u}_l e^{-l\lambda}\right) = 0 \text{ in } \mathbb{D}, \qquad \tilde{u}_l|_{\partial\mathbb{D}} = 0.$$

The only solution of this equation is $\tilde{u}_l = 0$. Thus we must have $u_l \equiv 0$. This argument shows that $u_k = 0$ for $k \geq 0$, and similarly one obtains that $u_k = 0$ for $k \leq 0$. □

Proof of Theorem 7.4.2 Let $u \in C^\infty(SM)$ with $u|_{\partial SM} = 0$. We begin with the Guillemin–Kazhdan identity: for any $k \geq 0$, Proposition 6.5.2 gives that

$$\|\eta_- u_k\|^2 + \frac{\kappa_0}{2} k \|u_k\|^2 \leq \|\eta_+ u_k\|^2.$$

In order to get the term $\|(Xu)_{k+1}\|^2$ on the right, we write

$$\begin{aligned}
\|\eta_+ u_k\|^2 &= \|(Xu)_{k+1} - \eta_- u_{k+2}\|^2 \\
&= \|(Xu)_{k+1}\|^2 - 2\,\mathrm{Re}((Xu)_{k+1}, \eta_- u_{k+2}) + \|\eta_- u_{k+2}\|^2 \\
&\leq \left(1 + \frac{1}{\varepsilon_k}\right) \|(Xu)_{k+1}\|^2 + (1 + \varepsilon_k)\|\eta_- u_{k+2}\|^2,
\end{aligned}$$

where the parameter $\varepsilon_k > 0$ will be chosen soon. Inserting this estimate in the previous inequality yields that

$$\|\eta_- u_k\|^2 + \frac{\kappa_0}{2} k \|u_k\|^2 \leq \left(1 + \frac{1}{\varepsilon_k}\right) \|(Xu)_{k+1}\|^2 + (1 + \varepsilon_k)\|\eta_- u_{k+2}\|^2.$$

We multiply this inequality with a weight $\gamma_k > 0$, which will be fixed later, and add up the resulting inequalities over $k \geq m$. This shows that

$$\begin{aligned}
&\sum_{k=m}^{\infty} \gamma_k \left(\|\eta_- u_k\|^2 + \frac{\kappa_0}{2} k \|u_k\|^2\right) \\
&\qquad \leq \sum_{k=m}^{\infty} \gamma_k \left(\left(1 + \frac{1}{\varepsilon_k}\right) \|(Xu)_{k+1}\|^2 + (1 + \varepsilon_k)\|\eta_- u_{k+2}\|^2\right). \qquad (7.15)
\end{aligned}$$

In order to get an estimate with only $\|(Xu)_{k+1}\|^2$ terms on the right, we would like to absorb the $\|\eta_- u_{k+2}\|^2$ terms from the right to the left. This is possible if the parameters are chosen so that

$$(1 + \varepsilon_k)\gamma_k \leq \gamma_{k+2}.$$

In particular, we need to assume $\gamma_{k+2} > \gamma_k$ for this to work. To keep the weights $\gamma_k(1 + \frac{1}{\varepsilon_k})$ on the right as small as possible, we fix the choice

$$\varepsilon_k = \frac{\gamma_{k+2} - \gamma_k}{\gamma_k}.$$

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With this choice, (7.15) takes the form

$$\sum_{k=m}^{\infty} \frac{\kappa_0}{2} k\gamma_k \|u_k\|^2 \leq \sum_{k=m}^{\infty} \left(1 + \frac{1}{\varepsilon_k}\right) \gamma_k \|(Xu)_{k+1}\|^2$$
$$= \sum_{k=m+1}^{\infty} \frac{\gamma_{k+1}\gamma_{k-1}}{\gamma_{k+1} - \gamma_{k-1}} \|(Xu)_k\|^2. \qquad (7.16)$$

The estimate (7.16) is true for any weights $\gamma_k > 0$ with $\gamma_{k+2} > \gamma_k$, and by taking limits also whenever $\gamma_k \geq 0$ and $\gamma_{k+2} > \gamma_k$. However, the weights can grow at most polynomially if we want the left-hand side to be well defined (recall that $Xu \in C^\infty$, so $V^N(Xu) \in L^2$ showing that $\sum |k|^{2N} \|(Xu)_k\|^2$ is finite for $N > 0$). We let $s > 0$ and fix the choice

$$\gamma_k = k^s.$$

To estimate the coefficient $\frac{\gamma_{k+1}\gamma_{k-1}}{\gamma_{k+1}-\gamma_{k-1}}$, we note the following elementary bounds for $t \in (0, 1)$:

$$\log(1 + t) \geq t \log(2), \qquad \log(1 - t) \leq -t \leq -t \log(2).$$

Hence

$$(1 + t)^s - (1 - t)^s \geq 2 \sinh(st \log(2)) \geq 2 \log(2)st \geq st.$$

This yields for $k \geq 1$ the bound

$$\frac{\gamma_{k+1}\gamma_{k-1}}{\gamma_{k+1} - \gamma_{k-1}} = \frac{(k^2 - 1)^s}{k^s((1 + 1/k)^s - (1 - 1/k)^s)} \leq \frac{1}{s} k^{s+1}.$$

Using the last estimate in (7.16) gives that

$$\frac{\kappa_0}{2} \sum_{k=m}^{\infty} k^{s+1} \|u_k\|^2 \leq \frac{1}{s} \sum_{k=m+1}^{\infty} k^{s+1} \|(Xu)_k\|^2.$$

Analogously, using the second part of Proposition 6.5.2 gives the estimate

$$\frac{\kappa_0}{2} \sum_{k=-\infty}^{-m} |k|^{s+1} \|u_k\|^2 \leq \frac{1}{s} \sum_{k=-\infty}^{-m-1} |k|^{s+1} \|(Xu)_k\|^2.$$

Combining these two estimates and setting $2\tau = s + 1$, prove the theorem. □

7.5 The Higher Dimensional Case

The results in this chapter were proved by using vertical Fourier analysis and the Beurling contraction property, which was a consequence of the Guillemin–Kazhdan identity. Since these results have higher dimensional counterparts as described in Section 6.6, all the results in this chapter extend to higher dimensional manifolds whose sectional curvatures are non-positive. We state the results below and refer to Paternain and Salo (2021, 2018) for the proofs.

Let (M, g) be a compact simple manifold of dimension $n \geq 2$. The first result gives the solenoidal injectivity of the X-ray transform I_m on symmetric m-tensor fields.

Theorem 7.5.1 *Let (M, g) be a simple manifold with non-positive sectional curvature. Then I_m is s-injective for any $m \geq 0$.*

In order to state the stability results we need to discuss the $H_T^{1/2}$ space in higher dimensions. Given $u \in C^\infty(SM)$, we first define the full horizontal gradient

$$\overset{\mathtt{h}}{\overline{\nabla}} u := \overset{\mathtt{h}}{\nabla} u + (Xu)v.$$

Note that $\overset{\mathtt{h}}{\overline{\nabla}} u$ is the horizontal part of $\nabla_{SM} u$ (the gradient of u with respect to Sasaki metric) in the splitting $\xi = (\xi_H, \xi_V)$ for $\xi \in TSM$ given in (3.12). The tangential part of $\overset{\mathtt{h}}{\overline{\nabla}} u$ on ∂SM is defined by

$$\overset{\mathtt{h}}{\overline{\nabla}}{}^{\|} u := \overset{\mathtt{h}}{\overline{\nabla}} u - \langle \overset{\mathtt{h}}{\overline{\nabla}} u, \nu \rangle \nu,$$

where ν is the inner unit normal for ∂M. Next we define the H_T^1 norm on $\partial_+ SM$ by

$$\|u\|^2_{H_T^1(\partial_+ SM)} = \|u\|^2_{L^2(\partial_+ SM)} + \|\overset{\mathtt{h}}{\overline{\nabla}}{}^{\|} u\|^2_{L^2(\partial_+ SM)}.$$

The space $H_T^{1/2}(\partial_+ SM)$ is defined as the complex interpolation space halfway between $L^2(\partial_+ SM)$ and $H_T^1(\partial_+ SM)$.

The following result states the stability estimates for the X-ray transform on tensor fields.

Theorem 7.5.2 *Let (M, g) be a simple manifold with non-positive sectional curvature. Then*

$$\|f\|_{L^2(M)} \leq C \|I_0 f\|_{H_T^{1/2}(\partial_+ SM)}, \qquad f \in C^\infty(M).$$

For any $m \geq 1$ one has

$$\|f^s\|_{L^2(M)} \leq C \|I_m f\|_{H_T^{1/2}(\partial_+ SM)}, \qquad f \in C^\infty(S^m(T^* M)).$$

The injectivity result for the attenuated X-ray transform takes the following form. We consider attenuations $\mathcal{A}$ that are sums of scalar functions and 1-forms, which is written as $\mathcal{A} \in \Theta_0 \oplus \Theta_1$ in the notation of Section 6.6.

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Theorem 7.5.3 *Let (M, g) be a simple manifold whose sectional curvatures are all negative. If $\mathcal{A} = a_0 + a_1$ with $a_k \in \Theta_k$, then $I_{\mathcal{A}}$ is injective on $C^\infty(M)$.*

The Carleman estimate required for proving the previous theorem is as follows.

Theorem 7.5.4 *Let (M, g) be a simple manifold whose sectional curvatures satisfy $K \leq -\kappa_0$ for some $\kappa_0 > 0$. For any $m \geq 1$ and $\tau \geq 1$, one has*

$$\sum_{l=m}^{\infty} l^{2\tau} \|u_l\|^2 \leq \frac{(n+4)^2}{\kappa_0 \tau} \sum_{l=m+1}^{\infty} l^{2\tau} \|(Xu)_l\|^2,$$

whenever $u \in C^\infty(SM)$ with $u|_{\partial SM} = 0$.