BULL. AUSTRAL. MATH. SOC. MOS 46J35
VOL. 3 (1970), 39-47.

On Lebesgue-type decompositions for Banach algebras

Howard Anton

If the maximal ideal space of a commutative complex unitary Banach algebra, X , is equipped with a nonnegative, finite, regular Borel measure, m , then for each element, x , in X , a complex regular Borel measure, m_x , can be obtained by integrating the Gelfand transform of x with respect to m over the Borel sets. This paper considers the possibility of direct sum decompositions of the form $X = A_x \oplus P_x$ where

$$A_x = \{z \in X\colon m_z << m_x\} \text{ and } P_x = \{z \in X\colon m_z \perp m_x\} .$$

1. Introduction

Let X be a commutative complex Banach algebra with identity and let M designate the maximal ideal space of X . Suppose also that M has the Gelfand topology, m is a nonnegative, finite, regular Borel measure on M , and $x \to \hat{x}$ is the Gelfand mapping from X into $C(M)$. Since $\hat{x}$ is continuous, it follows from the compactness of M that $\hat{x} \in L^1(m)$. Let m_x denote the complex regular Borel measure defined by $m_x(E) = \int_E \hat{x}(M)dm$ where E varies over the Borel sets of M . By means of the mapping $x \to m_x$ we can associate a complex regular Borel measure with each element of X .

It will be shown that the sets $A_x = \{z \in X\colon m_z << m_x\}$ and

Received 21 March 1970.

$P_x = \{z \in X\colon m_z \perp m_x\}$ form closed ideals in X. If a mild condition is imposed on m, then $A_x \cap P_x = \{0\}$. The results of this paper provide solutions to the following problems:

(a) find sufficient conditions for an element of X to lie in the subspace $A_x \oplus P_x$;

(b) find necessary and sufficient conditions for X to admit a Lebesgue-type decomposition $X = A_x \oplus P_x$ for some x in X, with A_x and P_x non-zero ideals;

(c) for a given x in X, find necessary and sufficient conditions for X to admit the decomposition $X = A_x \oplus P_x$ with A_x and P_x non-zero ideals.

We will adhere throughout to the following conventions and notation:

e denotes the identity of X and is assumed to have a norm of 1;

$\operatorname{rad}(x)$ denotes the spectral radius of x;

S denotes the sigma-algebra of Borel sets in M; and

$B(M)$ denotes the Banach space of complex regular Borel measures on M with the total variation norm.

Our measure theoretic terminology follows that used in [4].

2. Lebesgue-type decompositions

The map $\theta\colon X \to B(M)$ defined by $\theta(x) = m_x$ is a continuous linear transformation with norm$m(M)$. If θ is one-to-one, then the measure m is said to have *property* α. Since property α is equivalent to the assertion that $\hat{x}(M) = 0$ a.e. (m) implies $x = 0$, it is clear that in the Banach algebra $C(\Omega)$ where Ω is a compact Hausdorff space, every nonnegative Borel measure on Ω which assumes positive values on the non-empty open sets has property α. In particular, Lebesgue measure on $[a, b]$ has property α. Further, the Gelfand transform $x \to \hat{x}$ may be regarded as a continuous embedding of X into $L^\infty(M, m)$. If m has property α , then it follows that X is semi-simple and the Gelfand transform is an isomorphism into $L^{\infty}(M, m)$.

LEMMA 1. *If* x *and* $y \in X$ *then*

(i) $m_{xy}(E) = \int_E \hat{y}(M)dm_x = \int_E \hat{x}(M)dm_y$,

(ii) $m_{xy} << m_x$ *and* $m_{xy} << m_y$.

Proof. Using Radon-Nikodym derivatives we can write $d_{m_y} = \hat{y}(M)dm$ and $d_{m_x} = \hat{x}(M)dm$. Since $m_{xy}(E) = \int_E \hat{x}(M)\hat{y}(M)dm$ for all E in S , *(i)* follows. *(ii)* is an immediate consequence of *(i)*.

LEMMA 2.

(i) $A_x = \{z \in X: m_z << m_x\}$ *is a closed ideal containing* x *;*

(ii) $P_x = \{z \in X: m_z \perp m_x\}$ *is a closed ideal;*

(iii) $A_x \cap P_x = \{z \in X: m_z = 0\}$ *;*

(iv) if m *has property* α *,* $A_x \cap P_x = \{0\}$.

Proof. *(i)* Let $y, z \in A_x$. If $|m_x|(E) = 0$ then $m_y(E) = 0$ and $m_z(E) = 0$. By linearity of θ , $m_{y+z}(E) = 0$. It now follows that $y + z \in A_x$. Let $y \in A_x$ and $w \in X$. By part *(ii)* of Lemma 1, $m_{yw} << m_y << m_x$ so that $yw \in X$. To see that A_x is closed, let $z_j \to z$ where (z_j) is a sequence in A_x . It follows from the continuity of θ that $m_{z_j}(E) \to m_z(E)$ for all E in S . If $|m_x|(E) = 0$, then $m_z(E) = 0$ so that $z \in A_x$. It is clear that $x \in A_x$.

(ii) Let $y, z \in P_x$. We can find sets A and B in S such that $|m_y|(A) = |m_x|(A^c) = 0$ and $|m_z|(B) = |m_x|(B^c) = 0$. Let $C = A \cap B$. $|m_{y+z}|(C) \le |m_y|(C) + |m_z|(C) \le |m_y|(A) + |m_z|(B) = 0$. Also

$|m_x|(C^c) = |m_x|(A^c \cup B^c) \leq |m_x|(A^c) + |m_x|(B^c) = 0$. It now follows that $m_{y+z} \perp m_x$ and $y + z \in P_x$. Let $y \in P_x$ as before and let $w \in X$. By part *(ii)* of Lemma 1, $m_{wy} << m_y$ so that $|m_{wy}| << m_y$. Since $|m_y|(A) = 0$ it follows that $|m_{wy}|(A) = 0$. Since $|m_x|(A^c) = 0$, $m_{wy} \perp m_x$ so that $wy \in P_x$. To see that P_x is closed, let (z_j) be a sequence in P_x such that $z_j \to z$. There exist sets B_j in S such that $|m_{z_j}|(B_j) = 0$ and $|m_x|(B_j^c) = 0$. Let $B = \bigcap_{j=1}^{\infty} B_j$. Clearly $|m_{z_j}|(B) = 0$ for $j = 1, 2, \ldots$. From the continuity of θ , $|m_{z_j}|(B) \to |m_z|(B)$ so that $|m_z|(B) = 0$. But

$$|m_x|(B^c) \leq \sum_{j=1}^{\infty} |m_x|(B_j^c) = 0 .$$

It now follows that $z \in P_x$ so that P_x is closed.

(iii) By a standard measure theory result, $m_z << m_x$ and $m_z \perp m_x$ both hold if and only if $m_z = 0$.

(iv) If m has property α , $m_z = 0$ implies $z = 0$ so that this part is immediate from *(iii)*.

The next few results describe the relationship between the absolute continuity statement $m_x << m_y$ and the behavior of m on M . In the ensuing discussion, $j(x)$ denotes the compact set $\hat{x}^{-1}\{0\} = \{M \in M : x \in M\}$ and $m_x \equiv m_y$ denotes that $m_x << m_y$ and $m_y << m_x$.

LEMMA 3. *The following statements are equivalent:*

(i) $|m_x|(E) = 0$;

(ii) $m(E - j(x)) = 0$;

(iii) $m(E) = m(E \cap j(x))$.

Proof. The equivalence follows directly from the observation that

https://doi.org/10.1017/S0004972700045627 Published online by Cambridge University Press

$|m_x|(E) = \int_E |\hat{x}(M)|\,dm$ so that $|m_x|(E) = 0$ if and only if $\hat{x}(M) = 0$ a.e. on E.

LEMMA 4. *$m_x \ll m_y$ if and only if $m\big(j(y) - j(x)\big) = 0$.*

Proof. Let $m\big(j(y)-j(x)\big) = 0$ and assume $|m_y|(E) = 0$. From part *(ii)* of Lemma 3, $m\big(E-j(y)\big) = 0$. Since $E - j(x) \subset \big(E-j(y)\big) \cup \big(j(y)-j(x)\big)$, we obtain $m\big(E-j(x)\big) \le m\big(E-j(y)\big) + m\big(j(y)-j(x)\big) = 0$ so that $m\big(E-j(x)\big) = 0$. From part *(i)* of Lemma 3, $|m_x|(E) = 0$. It now follows that $m_x \ll m_y$. Conversely, assume $m_x \ll m_y$ or equivalently $|m_x| \ll m_y$. Since it is evident that $|m_y|\big(j(y)\big) = 0$, we have $|m_x|\big(j(y)\big) = 0$ and consequently $m\big(j(y)-j(x)\big) = 0$.

PROPOSITION 1. *The following are equivalent:*

(i) $m_x \equiv m_y$;

(ii) $A_x = A_y$;

(iii) $m\big(j(x) \,\Delta\, j(y)\big) = 0$, *where* Δ *represents set symmetric difference.*

Proof. By Lemma 4, $m_x \equiv m_y$ if and only if $m\big(j(x)-j(y)\big) = 0$ and $m\big(j(y)-j(x)\big) = 0$. This is clearly equivalent to *(iii)*. The equivalence of *(i)* and *(ii)* is an immediate consequence of the fact that $x \in A_x$ and $y \in A_y$.

COROLLARY 1. *If x is an invertible element of X, then*

(i) $m_x \equiv m$;

(ii) $A_x = X$;

(iii) *if m has property α, then* $P_x = \{0\}$.

Proof. If x is invertible, then $j(x) = \emptyset$. In particular $j(e) = \emptyset$, so that $m\big(j(x) \,\Delta\, j(e)\big) = 0$. By Proposition 1 and the fact

https://doi.org/10.1017/S0004972700045627 Published online by Cambridge University Press

$m_e = m$, we obtain $m_x \equiv m$. Since $A_e = X$, *(ii)* also follows from Proposition 1. Part *(iii)* follows from the last part of Lemma 2.

If $y = x_1 + x_2$ where $x_1 \in A_x$ and $x_2 \in P_x$ then we say that y is *decomposable with respect to* x . It should be noted that if m has property α , and if y is decomposable with respect to x , then part *(iv)* of Lemma 2 guarantees that the decomposition of y is unique.

PROPOSITION 2. *If* m *has property* α *and if there exist elements* w *and* z *in* X *such that* $m_y(E \cap j(x)) = m_z(E)$ *and* $m_y(E \cap j(x)^c) = m_w(E)$ *for all* E *in* S *, then* y *is decomposable with respect to* x *. Further, the decomposition is* $y = w + z$ *where* $w \in A_x$ *and* $z \in P_x$.

Proof. Let $A = j(x)$ and suppose w and z are elements with the above stated properties. Since $m_z(E) = 0$ for every measurable subset of A^c , we have $|m_z|(A^c) = 0$. By Lemma 3, $|m_x|(A) = 0$ so that $m_z \perp m_x$. If $|m_x|(E) = 0$ then since $\int_E |\hat{x}(M)|dm = 0$ and $|\hat{x}(M)| > 0$ for M in $E \cap A^c$, we obtain $m(E \cap A^c) = 0$. Since $m_y \ll m$, we have $m_y(E \cap A^c) = 0$ so that $m_w(E) = 0$ and consequently $m_w \ll m_x$. Clearly $m_y = m_w + m_z = m_{w+z}$. Since m has property α , we have the decomposition $y = w + z$.

The remaining theorems depend on the following two well-known results on direct-sum decompositions of Banach algebras. (See [3] pp. 95-96 or [5].)

Result 1. Let X be a commutative complex Banach algebra with identity. If I_1 and I_2 are non-zero ideals and $X = I_1 \oplus I_2$, then $\mathcal{M}$ is disconnected. Further, if $e = e_1 + e_2$ is the representation of e , then $\mathcal{M}$ is the disjoint union of the non-empty closed sets $\mathcal{M}_1 = \{M : \hat{e}_1(M) = 1\}$ and $\mathcal{M}_2 = \{M : \hat{e}_2(M) = 1\}$.

Result 2. Let X be a commutative complex Banach algebra with

https://doi.org/10.1017/S0004972700045627 Published online by Cambridge University Press

identity. If $M = M_1 \cup M_2$ is a partition of M into disjoint non-empty closed sets, there exist non-zero idempotents e_1 and e_2 such that $M_1 = \{M : \hat{e}_1(M) = 1\}$, $M_2 = \{M : \hat{e}_2(M) = 1\}$, and $e = e_1 + e_2$. Further, X admits the decomposition $X = I_1 \oplus I_2$ where I_1 and I_2 are the non-zero ideals e_1X and e_2X respectively.

For the duration of this paper, we assume that X is a commutative complex Banach algebra with identity and that m has property α . Also $X = A \oplus B$ will be called a non-trivial decomposition if A and B are non-zero ideals.

PROPOSITION 3. *X admits a non-trivial decomposition $X = A_x \oplus P_x$ for some x in X if and only if M is disconnected.*

Proof. If X has a non-trivial decomposition $X = A_x \oplus P_x$ for some x in X , then M is disconnected by Result 1. Conversely, assume M is disconnected and that $M = M_1 \cup M_2$ is a partition of M into disjoint non-empty closed sets. Let e_1 and e_2 be the idempotents described in Result 2 and let $X = I_1 \oplus I_2$ be the direct sum decomposition described there. To complete the proof we will show $I_1 = A_{e_1}$ and $I_2 = P_{e_1}$.

If $z \in I_1$ then $z = e_1z$. By part *(ii)* of Lemma 1, it follows that $m_z \ll m_{e_1}$ so that $I_1 \subset A_{e_1}$. Let $z \in A_{e_1}$. We will show $m_z = m_{e_1z}$. By property α , it will follow that $z = e_1z$ and consequently that $A_{e_1} \subset I_1$. Since $z \in A_{e_1}$ and since m_{e_1} vanishes on every measurable subset of M_2 , we have $m_z(E \cap M_2) = 0$ for all E in S . Consequently, if $F = E \cap M_1$ and $G = E \cap M_2$ then

$$m_z(E) = \int_F \hat{z}(M)dm + \int_G \hat{z}(M)dm = \int_E \hat{e}_1(M)\hat{z}(M)dm = m_{e_1z}(E)$$ for all E in S , that is $m_z = m_{e_1z}$. We will now show $I_2 = P_{e_1}$. If $z \in I_2$, then $z = e_2z$ so that $|m_z|(M_1) = \int_{M_1} |\hat{e}_2(M)\hat{z}(M)|dm = 0$. Clearly $|m_{e_1}|(M_2) = 0$ so that $m_z \perp m_{e_1}$ and consequently $I_2 \subset P_{e_1}$. Let $z \in P_{e_1}$. We will show $m_z = m_{e_2z}$. By property α , it will follow that $z = e_2z$ and consequently that $P_{e_1} \subset I_2$. Since $z \in P_{e_1}$,

$|m_{e_1}|(A) = |m_z|(A^c) = 0$ for some $A \in S$. Now let $H = E \cap A^c$, $J = E \cap A \cap M_1$, and $K = E \cap A \cap M_2$. Straightforward calculations show that for all E in S, $m_{e_2z}(E) = \int_H \widehat{e_2z}(M)dm + \int_K \widehat{e_2z}(M)dm$ and $m_z(E) = \int_H \hat{z}(M)dm + \int_J \widehat{e_1z}(M)dm + \int_K \widehat{e_2z}(M)dm$. Since $|m_{e_1z}| << m_{e_1}$ and $|m_{e_2z}| << m_z$ it follows that $m_{e_2z}(E) = m_z(E) = \int_K \widehat{e_2z}(M)dm$, that is, $m_{e_2z} = m_z$.

PROPOSITION 4. *For a given* x *in* X, X *admits a non-trivial decomposition* $X = A_x \oplus P_x$ *if and only if there exists an idempotent* q *other than* 0 *or* e *such that* $m_x \equiv m_q$.

Proof. Assume $X = A_x \oplus P_x$ is a non-trivial decomposition and $e = e_1 + e_2$ is the representation of e. Since $x \in A_x$, it follows that $e_1x = x$ so that $m_x << m_{e_1}$. The reverse relation is clear so that $m_x \equiv m_{e_1}$. e_1 is thus the desired idempotent. Conversely, let $M_1 = \{M : \hat{q}(M) = 1\}$ and $M_2 = \{M : \widehat{e-q}(M) = 1\}$. $M = M_1 \cup M_2$ is a partition of M into non-empty disjoint closed sets. Proceeding as in the proof of Proposition 3 with q and $e-q$ in place of e_1 and e_2 respectively, it follows that $X = A_q \oplus P_q$ is a non-trivial decomposition. Since $m_x \equiv m_q$, we have $A_x = A_q$ and $P_x = P_q$ so that $X = A_x \oplus P_x$.

References

[1] Howard Anton, "Borel measures on the maximal ideals of a Banach algebra", Ph.D. dissertation, Polytechnic Institute of Brooklyn, 1968.

[2] Nelson Dunford and Jacob T. Schwartz, *Linear operators, Part I: General theory* (Interscience Publishers, New York, London, 1958).

[3] I. Gelfand, D. Raikov and G. Shilov, *Commutative normed rings* (Chelsea, New York, 1964).

https://doi.org/10.1017/S0004972700045627 Published online by Cambridge University Press

[4] Edwin Hewitt and Karl Stromberg, *Real and abstract analysis. A modern treatment of the theory of functions of a real variable* (Springer-Verlag, Berlin, Göttingen, Heidelberg, New York, 1965).

[5] G.E. Šilov, "On the decomposition of a commutative normed ring into a direct sum of ideals", *Mat. Sb.* 32 (74) (1953), 353-364, (Russian). *Amer. Math. Soc. Transl.* (2) 1 (1955), 37-48.

Drexel University,
Philadelphia, USA.

https://doi.org/10.1017/S0004972700045627 Published online by Cambridge University Press