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A fixed point theorem for a family of nonexpansive mappings

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Let E be a separated, locally convex topological vector space and F a commutative family of nonexpansive mappings defined on a quasi-complete convex (not necessarily bounded) subset X of E. In this paper, it is proved that if one of the mappings in F is condensing with a bounded range then the family F has a common fixed point in X. This result improves several well-known results and supplements a recent result of E. Tarafdar (*Bull. Austral. Math. Soc.* 13 (1975), 241-254) for such mappings.

Introduction

Let E be a separated, locally convex topological vector space and $\mathcal{U}$ a neighborhood basis of the origin consisting of absolutely convex subsets of E. For each $U \in \mathcal{U}$, let p_U be the Minkoswki's functional of U. Let $P = \{p_U : U \in \mathcal{U}\}$. Let X be a nonempty subset of E. A mapping $f : X \to X$ is called P-nonexpansive (see Tarafdar [7]) if for each $p \in P$ and for all $x, y \in X$, $p(f(x)-f(y)) \leq p(x-y)$. In a recent paper [7], Tarafdar considered a commutative family of such nonexpansive mappings and proved the following extension of an earlier result of Belluce and Kirk [2].

THEOREM 1. *Let E be quasi-complete and X a nonempty, bounded, closed, and convex subset of E and M a compact subset of X. If F is a commutative family of P-nonexpansive mappings on X having the property: there exists a $g \in F$ such that for each $x \in X$,*

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(1) $$\mathrm{cl}\{g^n(x) : n \in I \text{ (positive integers)}\} \cap M \neq \emptyset ,$$

then the family F *has a common fixed point in* M .

In the present paper, we prove a common fixed point theorem for the family F in Theorem 1, when X therein is not necessarily bounded and instead of (1) we assume the existence of a $g \in F$ which is condensing with a bounded range. The main result of this paper generalizes a well-known result of DeMarr [3] and extends to locally convex spaces a result of Bakhtin [1], supplementing a recent result of Tarafdar [7] for such a family F .

1.

Recall that a closed subset X of E is called quasi-complete if its closed and bounded subsets are complete. Furthermore, a subset T of E is totally bounded if for each $U \in \mathcal{U}$, there exists a finite subset F of T such that $T \subseteq F + U = \{a+b : a \in F, b \in U\}$. For a subset S of E , let

$$Q(S) = \{U \in \mathcal{U} : S \subseteq T+U, \text{ for some totally bounded subset } T \text{ of } E\} .$$

Let X be a nonempty subset of E . Following Su and Sehgal [6] (see also Himmelberg, Porter and Van Vleck [4]) a mapping $f : X \to E$ is called condensing if for each bounded but not totally bounded subset $A \subseteq X$, $Q(A) \subsetneq Q(f(A))$. Note, that if A is a totally bounded subset of a quasi-complete subset X of E , then the closure of A (cl A) is compact.

2.

The following result is basic to the main result of this paper.

THEOREM 2. *Let* X *be a non-empty quasi-complete convex subset of* E *and* $g : X \to X$ *be a continuous condensing mapping such that* $g(X)$ *is a bounded subset of* X . *Then* $G = \{x \in X : g(x) = x\}$ *is a nonempty compact subset of* X .

Proof. That the set G is non-empty is a consequence of a result of Su and Sehgal ([6], Lemma 1). Furthermore, the continuity of g implies that G is closed and being bounded, it follows that G is complete. Since

$$Q(g(G)) = Q(G)$$

and g is condensing, therefore G is totally bounded and hence a compact subset of X .

In [7] (Lemma 2.2), Tarafdar has proved the following results.

LEMMA 1. *Let* M *be a compact subset of* E . *If for some* $p \in P$,

(2) $$d_p = \sup\{p(x-y) : x, y \in M\} > 0 ,$$

then there is a u *in the convex hull of* M $(\text{co}(M))$ *such that*

(3) $$r = \sup\{p(x-u) : x \in M\} < d_p .$$

The following is the main result of this paper and is related to the lines of argument in [1].

THEOREM 3. *Let* X *be a nonempty, quasi-complete convex subset of* E *and* F *a commutative family of* P*-nonexpansive self mappings of* X *satisfying the condition:*

(4) *there exists a* $g \in F$ *such that* g *is condensing and* $g(X)$ *is bounded.*

Then the family F *has a common fixed point in* X .

Proof. Let

$$A = \{S \subseteq X : S \text{ is nonempty, convex, and } f(S) \subseteq S \text{ for each } f \in F\} .$$

Clearly $X \in A$. Define a partial order $<$ in A by $S_1 < S_2$ iff $S_2 \subseteq S_1$. We show that any chain in A has an upper bound. Let $\{S_\alpha : \alpha \in \Delta\}$ be a chain in A . Let $A = \cap\{S_\alpha : \alpha \in \Delta\}$. Since a P-nonexpansive mapping is continuous, it follows by Theorem 2 that, for each $\alpha \in \Delta$,

$$F_\alpha = \{x \in S_\alpha : g(x) = x\} ,$$

is a nonempty compact subset of S_α . Thus $F = \cap\{F_\alpha : \alpha \in \Delta\}$ is a non-empty subset of A . It is clear now that $A \in A$ and that A is an upper bound of the chain $\{S_\alpha : \alpha \in \Delta\}$. Therefore, by Zorn's Lemma, there exists a minimal nonempty convex set $S_0 \subseteq X$ such that $f(S_0) \subseteq S_0$ for each $f \in F$. Let

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$$F = \{x \in S_0 : g(x) = x\} .$$

Then F is a nonempty compact subset of S_0 and since for any $f \in \mathcal{F}$ and $x \in F$,

$$f(x) = f\big(g(x)\big) = g\big(f(x)\big) ,$$

it follows that $f(F) \subseteq F$ for each $f \in \mathcal{F}$. Let

$$\mathcal{B} = \{C \subseteq X : C \text{ is nonempty, compact, and } f(C) \subseteq C \text{ for each } f \in \mathcal{F}\} .$$

Then $F \in \mathcal{B}$. Define the same partial order in $\mathcal{B}$ as in $\mathcal{A}$. Then it is easy to show by Zorn's Lemma that there is a minimal nonempty compact set $M \subseteq X$ such that $f(M) \subseteq M$ for each $f \in \mathcal{F}$. Clearly

(5) $$M \subseteq S_0 ,$$

and the minimality of M in $\mathcal{B}$ implies that

(6) $$f(M) \equiv M \text{ for each } f \in \mathcal{F} .$$

We show that M consists of exactly one element of X. Suppose not. Then, since E is separated, there is a $p \in \mathcal{P}$ satisfying (2), and hence by Lemma 1, there is a $u \in \operatorname{co}(M)$ satisfying (3). Now, S_0 being convex, it follows by (5) that $u \in S_0$. Let for each $x \in M$ and r given by (3),

$$V(x) = \{z \in E : p(x-z) \le r\} .$$

Then $V(x)$ is convex and $u \in V(x)$ for each $x \in M$. Set

(7) $$V = \{V(x) : x \in M\} \text{ and } S = S_0 \cap V .$$

Clearly S is convex and $u \in S$. We show that $f(S) \subseteq S$ for each $f \in \mathcal{F}$. Since $f\big(S_0\big) \subseteq S_0$, it suffices to show that $f(V) \subseteq V$ for each $f \in \mathcal{F}$. Let $z \in V$ and $f \in \mathcal{F}$. Then, by (7),

(8) $$p(x-z) \le r$$

for each $x \in M$. Now for each $x \in M$, it follows by (6) that there is a $y = y(x) \in M$ such that $f(y) = x$ and hence, by (8),

$$p\big(f(z)-x\big) = p\big(f(z)-f(y)\big) \le p(z-y) \le r$$

for each $x \in M$. Thus $f(z) \in V(x)$ for each $x \in M$; that is $f(S) \subseteq S$

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for each $f \in F$. Thus $S \in A$, and by (7) and the minimality of S_0 in A,

(9) $$S = S_0 .$$

Now p being continuous and M compact, there are elements x and y in M such that $p(x-y) = d_p$. This equality implies that $y \notin V(x)$ and consequently $y \notin S$. However, by (5), $y \in S_0$. This contradicts (8). Thus $M = \{x\}$ for some $x \in X$ and hence $f(x) = x$ for each $f \in F$. This completes the proof of Theorem 3.

If E is a Banach space and U_0 is the collection of spherical neighborhoods of the origin with $P_0 = \{P_U : U \in U_0\}$, then a mapping f on a subset X of E is P_0-nonexpansive iff f is nonexpansive; that is $\|f(x)-f(y)\| \leq \|x-y\|$ for all $x, y \in X$. Further, if $f : X \to E$ is condensing in the sense of Sadovskiĭ [5], then f is condensing with respect to U_0 (see [4]). Therefore, as a consequence of Theorem 3, we have the following extension of a result of Bakhtin [2].

COROLLARY 1. *Let X be a nonempty, closed and convex subset of a Banach space E and F a commutative family of nonexpansive self mappings of X satisfying the condition: there exists a $g \in F$ such that g is condensing in the sense* [5] *with $g(X)$ bounded. Then the family F has a common fixed point in X.*

COROLLARY 2. *Let X be a nonempty, closed, and convex subset of a Banach space E and F a commutative family of nonexpansive self mappings of X. If for some $g \in F$, $g(X)$ is contained in a compact subset of X, then the family F has a common fixed point in X.*

It may be remarked that Corollary 2 contains a result of DeMarr [3].

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