RELATIVISTIC HIERARCHY OF REFERENCE SYSTEMS AND TIME SCALES

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ABSTRACT. Relativistic hierarchy of reference systems (RS) developed in recent years by different authors is examined in detail. Metric expressions and transformation relations for solar system barycentric RS (BRS), heliocentric RS (HRS), Earth-Moon local RS (LRS), geocentric RS (GRS), topocentric RS (TRS) and Earth satellite RS (SRS) may be obtained explicitly in harmonic coordinates of GRT. The time coordinate of any RS involves the corresponding time scale. Particular attention is given to the closed form representation of GRS avoiding expansions in powers of the geocentric coordinates. GRS has been constructed in both versions of dynamically non-rotating GRS (DGRS) or kinematically non-rotating GRS (KGRS). DGRS and KGRS differ in their space axes orientation by the amount of the geodesic precession. Similarly, taking into account the motion of the Sun around the center of the Galaxy one should distinguish between dynamically non-rotating BRS (DBRS) and kinematically non-rotating BRS (KBRS) differing in their space axes orientation by the amount of the galactic precession. Reduction to the galactic time and the galactic space axes may be needed in the nearest future.

1. INTRODUCTION

Presently, any theory of astronomical reference systems and time scales adequate to the precision of modern observations may be developed only within the GRT framework. However, the preliminary discussion by the IAU Working Group on Reference Systems of the corresponding relativistic formulations suggested in (Brumberg and Kopejkin, 1989a, 1990) has shown that many astronomers are not ready to apprehend these formulations and regard them as "too technical". The difficulties of apprehension increase when considering not only one approach to develop relativistic reference systems but a whole set of the alternative approaches based, for instance, on PPN formalism (Will, 1981), generalization of Fermi normal coordinates (Ashby and Bertotti, 1986; Fukushima, 1988), tetrad formalism (Soffel, 1989), etc. The aim of the present paper is to elucidate the key statements of relativistic formulations and to simplify as much as possible their "technical" aspects not diminishing

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the level of necessary experimental precision and mathematical accuracy. The paper is based on the technique to construct the hierarchy of reference systems in harmonic coordinates exposed in (Brumberg and Kopejkin, 1989a, 1990).

2. REFERENCE SYSTEM IN GRT

A reference (coordinate) system in GRT is given by the symmetric quadratic form ds^2 determining the metric of the four dimensional pseudo-Riemannian space of events of GRT

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad , \quad x^0 = ct \quad , \quad \mu, \nu = 0,1,2,3 \quad , \qquad (1)$$

c being the velocity of light. The quantity t is called the coordinate time of the system. The spatial coordinates of the system are designated by x^i (i=1,2,3). Einstein summation rule over repeated greek or latin index is used everywhere. In absence of the gravitating masses the reference system may be chosen so that the components of the metric tensor $g_{\mu\nu}$ (the gravitation potentials) take the values $\eta_{\mu\nu}$ with

$$\eta_{00} = 1 \ , \ \eta_{0i} = 0 \ , \ \eta_{ij} = -\delta_{ij}, \quad i, j = 1,2,3 \qquad (2)$$

(pseudo-Euclidean or Minkowskian or Galilean metric). These values determine the inertial reference system of special relativity theory. In presence of the gravitating masses the components $g_{\mu\nu}$ determined by the Einstein field equations are represented by expansions in powers of v/c (v being the characteristic velocity of bodies)

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad , \qquad (3)$$

$$h_{00} = h_{00}^{(2)} + h_{00}^{(4)} + \ldots \quad , \qquad (4)$$

$$h_{0i} = h_{0i}^{(1)} + h_{0i}^{(3)} + \ldots \quad , \qquad (5)$$

$$h_{ij} = h_{ij}^{(2)} + \ldots \quad . \qquad (6)$$

The upper index in parentheses indicates the order of smallness with respect to v/c. The quantity $h_{0i}^{(1)}$ is caused not by the gravitating masses but by the motion of the reference system. Two cases are typical. The first one is related with a system in rotation. If a system is originated by rotation of the spatial axes of the inertial system and the angular velocity components on the moving axes x^i are designated by ω^i then the metric (1) will be characterized by the occurrence of the term

$$h_{0i}^{(1)} = -c^{-1} \varepsilon_{ijk} \omega^j x^k = -c^{-1} (\omega \times x)^i \quad . \qquad (7)$$

ε_{ijk} is the three dimensional fully antisymmetric Levi-Civita symbol (ε_{123}= +1). ω and x are the triplets of components ω^i and x^i respectively. In the second case a reference system is resulted from formal application to the inertial system of three dimensional Galileo transformation characterized by the translatory velocity v^i. Then the metric (1) will contain a typical term

$$h^{(1)}_{oi} = - c^{-1} v^i \quad . \tag{8}$$

In what follows we shall be interested only in reference systems transforming in absence of the gravitating masses into inertial systems of special relativity theory. Metric (1) of such systems cannot contain the terms (7) or (8) and expansion (5) begins only with the third order terms $h^{(3)}_{oi}$.

No matter where to stop in expansions (4)-(6) one obtains different accuracies for astrometric problems (based on the equations of light propagation) and for celestial mechanics problems (based on the equations of motion of celestial bodies). In fact, in neglecting $h_{\mu\nu}$ at all, metric (1) yields the Newtonian equations of light propagation. Retaining only $h^{(2)}_{oo}$ enables one to derive the Newtonian equations of motion of celestial bodies. This permits also to take into account the post-Newtonian terms in the problem of clock synchronization and time scale relations. Considering $h^{(2)}_{oo}$ and $h^{(2)}_{ij}$ results in the post-Newtonian equations of light propagation in the static field (ignoring the motion of the gravitating masses). In taking into account $h^{(2)}_{oo}$, $h^{(2)}_{ij}$ and $h^{(3)}_{oi}$ one yields the complete post-Newtonian equations of light propagation (considering the motion of the masses). Moreover, the terms $h^{(3)}_{oi}$ are crucial to describe the relativistic rotation of the spatial axes of the reference system. Finally, including $h^{(2)}_{oo}$, $h^{(2)}_{ij}$, $h^{(3)}_{oi}$ and $h^{(4)}_{oo}$ one obtains the post-Newtonian equations of motion.

Nowadays, one may meet suggestion to define relativistic reference systems by fixing only $h^{(2)}_{oo}$ and $h^{(2)}_{ij}$. This is sufficient for present day astrometry. But this is quite inadequate for modern celestial mechanics using the post-Newtonian equations of motion of the solar system bodies and will be soon insufficient for high-precision astrometry (millisecond pulsar timing, POINTS project, etc.) demanding the complete post-Newtonian equations of light propagation. It seems to us that one should define a reference system with some excess of accuracy and therefore it is reasonable to fix all the terms indicated in expansions (4)-(6).

The dots in (4)-(6) mean the terms of more high order, in particular, the terms responsible for gravitational radiation of the system of bodies. For ephemeris astronomy problems these terms may be ignored as yet.

3. BRS IN HARMONIC COORDINATES AND IN PPN FORMALISM

Dynamically non-rotating barycentric reference system for the solar system (DBRS) may be represented in the post-Newtonian approximation of GRT in arbitrary quasi-Galilean coordinates in the form

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$$h^{(2)}_{oo} = -2c^{-2}U \quad , \quad h^{(2)}_{ij} = -2c^{-2}U\delta_{ij} + a_{i,j} + a_{j,i} \quad , \qquad (9)$$

$$h^{(3)}_{oi} = 4c^{-3}U^{i} + a_{o,i} + a_{i,o} \quad . \qquad (10)$$

$$h^{(4)}_{oo} = 2c^{-4}(U^{2} - W) + 2a_{o,o} + 2c^{-2}U_{,k}a_{k} + 2c^{-2}\sum_{A}\frac{\partial U}{\partial x^{k}_{A}}(\tilde{a}_{k})_{A} \qquad (11)$$

with

$$W = \frac{3}{2}\Phi_{1} - \Phi_{2} + \Phi_{3} + 3\Phi_{4} + \frac{\partial^{2}\chi}{\partial t^{2}} \quad . \qquad (12)$$

Newtonian potential U, vector-potential U^{i} and complementary potentials Φ_{1}, Φ_{2}, Φ_{3}, Φ_{4} and χ satisfy Poisson equations

$$U_{,kk} = -4\pi G\rho \ , \ U^{i}_{,kk} = -4\pi G\rho v^{i} \ , \ \Phi_{1,kk} = -4\pi G\rho v^{2} \ ,$$
$$\Phi_{2,kk} = -4\pi G\rho U \ , \ \Phi_{3,kk} = -4\pi G\rho\Pi \ , \ \Phi_{4,kk} = -4\pi Gp, \ \chi_{,kk} = U \ . \qquad (13)$$

Comma with subsequent index denotes the derivative with respect to the corresponding variable. ρ is the conserved density, v^{i} is the velocity of the matter, p is the pressure and Π is the internal energy. G is the gravitational constant. a_{o} and a_{i} are four arbitrary functions (a_{i} are of the second order of smallness, a_{o} is of the third order). $x^{k}_{A}(t)$ are the spatial coordinates of body A, $(\tilde{a}_{k})_{A}$ means the regular part of a_{k} in substituting $x^{k} = x^{k}_{A}$. It is of importance to note that function a_{o} entering in the Lagrangian in form of total derivative has no influence on the post-Newtonian equations of motion of bodies.

Functions a_{o} and a_{i} are specified by the coordinate conditions represented by four differential or algebraic relations for the metric tensor components and their first derivatives. These functions determine the type of the coordinates employed. One uses rather often the harmonic coordinate conditions. Their main mathematical advantage is the existence of explicit mathematical formulation

$$[(-g)^{1/2}g^{\mu\nu}]_{,\nu} = 0 \quad , \quad g = det(g_{\mu\nu}) \quad , \quad g^{\mu\alpha}g_{\nu\alpha} = \delta^{\mu}_{\nu} \quad . \qquad (14)$$

If the tilde denotes harmonic coordinates then the relationship of the arbitrary coordinates used above with the harmonic ones is of the form

$$\tilde{x}^{o} = x^{o} + a_{o} \ , \ \tilde{x}^{k} = x^{k} - a_{k} \quad . \qquad (15)$$

This means that metric (1)-(6), (9)-(11) becomes harmonic with $a_{o}=a_{i}=0$.

Potential χ in the point mass approximation has the form

$$\chi = \frac{1}{2}\sum_{A} GM_{A}r_{A} \quad , \quad r_{A} = (r^{k}_{A}r^{k}_{A})^{1/2}, \quad r^{k}_{A} = x^{k} - x^{k}_{A} \qquad (16)$$

so that

$$\frac{\partial^{2}\chi}{\partial t^{2}} = -\frac{1}{2}\sum_{A} GM_{A}\{\frac{1}{r_{A}}[(n^{k}_{A}v^{k}_{A})^{2} - v^{2}_{A}] + n^{k}_{A}a^{k}_{A}\} \quad , \quad n^{k}_{A} = r^{k}_{A}/r_{A} \ , \qquad (17)$$

v^{k}_{A} and a^{k}_{A} being velocity and acceleration of the center of mass of body A. In Newtonian approximation

$$a^k_A = \sum_{B \neq A} GM_B \frac{x^k_B - x^k_A}{r^3_{AB}} \quad , \qquad r_{AB} = [(x^k_B - x^k_A)(x^k_B - x^k_A)]^{1/2} \quad .$$

At infinity (but let us remember that BRS is not valid for too large distances from the solar system) all potentials except for χ vanish and coefficients (9)-(11) will contain only one non-zero term

$$h^{(4)}_{00} = c^{-4} \sum_A GM_A n^k_A a^k_A \quad . \tag{18}$$

Hence, ar infinity the spatial part of the BRS metric in harmonic coordinates takes the Euclidean form but there remains in g_{00} relativistic term (18).

Along with harmonic representation the BRS metric in PPN formalism coordinates is also widely used (Misner et al., 1973; Will, 1981). The PPN system is characterized by its spatial isotropy (as well as the harmonic representation) and its Galilean form at infinity. This involves the choice of the coordinate functions as follows:

$$a_i = 0 \quad , \quad a_0 = c^{-3} \frac{\partial \chi}{\partial t} \quad . \tag{19}$$

In so doing, function (17) disappears from (11) and in the right-hand side of (10) there will be an additive term

$$a_{0,i} = c^{-3} \frac{\partial^2 \chi}{\partial t \, \partial x^i} = - \frac{1}{2} c^{-3} \sum_A \frac{GM_A}{r_A} v^k_A (\delta_{ik} - n^i_A n^k_A) \quad .$$

It is easy to see that at infinity all coefficients (9)-(11) for the BRS metric in PPN coordinates vanish resulting in Galilean form for the BRS metric.

Hence, the spatial harmonic and PPN coordinates are the same

$$\tilde{x}^i = x^i \tag{20}$$

whereas the time coordinates are related by the equation

$$\tilde{t} = t + c^{-4} \frac{\partial \chi}{\partial t} \quad , \quad \frac{\partial \chi}{\partial t} = - \frac{1}{2} \sum_A GM_A n^k_A v^k_A \quad . \tag{21}$$

Using physical terminology one says that PPN and harmonic coordinate systems belong to one and the same reference frame differing only by the time coordinates.

Relations (20) and (21) enable one to conclude that PPN formulation and harmonic representation of BRS are practically equivalent within the present accuracy (v^2/c^2). Only in dealing with the higher order effects (v^4/c^4 as in discussing POINTS observations) this difference might be taken into account. As pointed above, the advantage of harmonic coordinates is due to their explicit mathematical formulation. In transforming to any other reference system (geocentric, topocentric, etc.) it is easy to impose the harmonic conditions on new metric coefficients and to ensure therewith the harmonic form of the coordinate transformation. On the contrary, the PPN formulation has been developed only for BRS and, moreover, only in the post-Newtonian approximation. The transformation within the PPN framework to other systems has not been elaborated so far. There is no definite PPN procedure to take into account the higher order terms. With respect to these two aspects the harmonic representation is more advantageous.

Let us make one technical remark useful for comparison with formulations of (Misner et al., 1973; Will, 1981). One often uses the density ρ^* satisfying the relation

$$(-g)^{1/2}\rho^* dx^0 = \rho ds$$

or

$$\rho = \rho^*[1 + c^{-2}(\frac{1}{2} v^2 + 3U)] \quad .$$

Therefore,

$$U = U^* + c^{-2}(\frac{1}{2}\Phi_1 + 3\Phi_2)$$

where U^* is the same Newtonian potential as U but expressed with the aid of ρ^* (all other potentials are the same within the adopted accuracy). Hence, in virtue of (3), (9), (11) and (19) coefficient g_{00} of the PPN formalism BRS metric takes the form

$$g_{00} = 1 - 2c^{-2}U^* + c^{-4}(2U^2 - 4\Phi_1 - 4\Phi_2 - 2\Phi_3 - 6\Phi_4) \quad .$$

In (Misner et al., 1973; Will, 1981) ρ (designated by ρ^*) is used in the equations of motion and ρ^* (designated by ρ) is used in the field metric. Taking into account the difference in designations and signature we obtain the expressions of these papers (with the GRT values for the PPN parameters). We think it reasonable to use one and the same conserved density ρ both for the field metric and the equations of motion.

4. HIERARCHY OF ASTRONOMICAL REFERENCE SYSTEMS

For astronomical purposes it is suitable to have the hierarchy of relativistic reference systems (RS) as follows:

HRS LRS SRS
↑ ↗ ↓ ↗ ↑
BRS → GRS → TRS

with BRS (solar system barycentric RS), HRS (heliocentric RS), LRS (local Earth-Moon barycentric RS), GRS (geocentric RS), TRS (topocentric RS) and SRS (satellite RS). Such an hierarchy (except for HRS and LRS which may be constructed by analogy) has been constructed in (Brumberg and Kopejkin, 1989a,b) conforming with three principles:

1) harmonic coordinates are used for each system;

2) the principle of equivalence is met in each system, i.e. the influence of the external masses is described only by tidal terms (in particular, the terms like (8) are absent);

3) each system is dynamically non-rotating (there are no terms like (7) or similar third order terms in $h_{01}^{(3)}$).

For BRS there are no external bodies (in ignoring the influence of the Galaxy) and all the solar system bodies taken into account are considered as internal ones. For LRS the internal bodies are the Earth and the Moon whereas all remaining bodies are considered as external ones. Both HRS and GRS have one internal body, the Sun or the Earth respectively. For TRS and SRS all the bodies are external.

The matching procedure of two reference systems used in (Brumberg and

Kopejkin, 1989a,b) starts from the known metric coefficients of one system and permits to determine 1) metric coefficients of the second system, 2) transformation formulae relating two metrics and 3) the equations of motion of the origin of the second system with respect to the first system.

Obviously, a similar hierarchy may be constructed using alternative approaches mentioned above. But the use of one and the same type of coordinate conditions for all systems in combination with the matching procedure has definite advantages, for instance:

1) the finite transformation formulae for spatial coordinates of two systems (quadratic functions in contrast to the power series of the alternative approaches);

2) unambiguous determination of the required functions;

3) no difficulties in considering figure characteristics and proper rotation of the bodies.

So far, in all papers indicated above the GRS metric has been constructed in form of the series in powers of the geocentric spatial coordinates. It is to be noted that to derive the GRS post-Newtonian rigorous (avoiding expansions in powers of geocentric coordinates) equations of motion of Earth distant satellites it is sufficient to apply to the known rigorous BRS equations the finite transformation formulae from BRS to GRS. Just in the same manner one can get the post-Newtonian HRS equations of motion of the planets. It is not evident how to solve these two problems with the alternative approaches based on power series expansions. For astrometric purposes (reduction of observations) GRS within the level of $h^{(2)}_{00}$, $h^{(2)}_{ij}$ and $h^{(3)}_{0i}$ is quite adequate. As shown in the next section, the technique of (Brumberg and Kopejkin, 1989a,b) permits easily to construct GRS within this level of accuracy in the closed form. Needless to say that the same is valid for any other reference system.

Let us add that each system entering in the hierarchy under consideration may be subjected to the rigid body spatial rotation. The resulting system (RS^{+}) is not harmonic anymore. Its metric will contain a term like (7). Such a system is used for solving astrometric problems.

5. CLOSED FORM OF GRS METRIC (DGRS AND KGRS)

Transformation from DBRS determined by relations (1)-(6) and (9)-(11) with $a_0 = a_1 = 0$ to GRS defined by

$$ds^2 = \hat{g}_{\mu\nu} dw^{\mu} dw^{\nu} \quad , \quad w^0 = cu \tag{22}$$

and having its origin in the center of mass of the Earth E is given by the formulae generalizing the Lorentz transformation of special relativity theory (Brumberg and Kopejkin, 1989a,b)

$$u = t - c^{-2}[S(t) + v^k_E r^k_E] + c^{-4}[B(t,\mathbf{r}_E) - \frac{1}{2} v^2_E v^k_E r^k_E] + \dots , \tag{23}$$

$$w^i = r^i_E + c^{-2}\{[\frac{1}{2} v^i_E v^k_E + qF^{ik}(t) + D^{ik}(t)]r^k_E + D^{ijk}(t) r^j_E r^k_E\} + \dots \tag{24}$$

In transformation (24) of the spatial coordinates functions $D^{ik} = D^{ki}$

and $D^{ijk} = D^{ikj}$ determine the relativistic contraction of length. Antisymmetric function $F^{ik} = -F^{ki}$ acts as the angular velocity of rotation of the spatial axes. Indeed, introducing the triplet $\mathrm{F} = (F^1, F^2, F^3)$ with

$$F^i = \frac{1}{2}\,\varepsilon_{ijk}F^{jk} \qquad (25)$$

one has

$$F^{ik}r^k_E = \mathbf{r}_E \times \mathrm{F} \qquad (26)$$

resulting in rotation of the spatial axes. q is the scalar parameter to be defined below. The inverse transformation is of the form

$$t = u + c^{-2}[S(t) + v^k_E(t)w^k] - c^{-4}[B(u, \mathrm{w}) + v^i_E(qF^{ik} + D^{ik})w^k + v^i_E D^{ijk}w^j w^k] + \ldots \qquad (27)$$

$$x^i = x^i_E(t) + w^i - c^{-2}[(\frac{1}{2}\,v^i_k v^k_E + qF^{ik} + D^{ik})w^k + D^{ijk}w^j w^k] + \ldots (28)$$

Substituting the derivatives of t and x^i with respect to u and w^k into the tensor transformation of metric matching

$$\hat{g}_{\alpha\beta}(u, \mathrm{w}) = g_{\mu\nu}(t, \mathrm{x})\,\frac{\partial x^\mu}{\partial w^\alpha}\,\frac{\partial x^\nu}{\partial w^\beta} \qquad (29)$$

one gets

$$\hat{g}_{oo} = 1 + c^{-2}(-\,2U + 2\dot{S} - v^2_E + 2a^k_E w^k) \quad , \qquad (30)$$

$$\hat{g}_{oi} = c^{-3}[4U^i - 4Uv^i_E - B_{,i} + v^i_E(\dot{S} - \frac{1}{2}\,v^2_E) + (\frac{3}{2}\,v^i_E a^k_E + \frac{1}{2}\,v^k_E a^i_E + q\dot{F}^{ik} + \dot{D}^{ik})w^k + \dot{D}^{ijk}w^j w^k] \quad , \qquad (31)$$

$$\hat{g}_{ij} = -\,\delta_{ij} - 2c^{-2}[U\delta_{ij} - D^{ij} - (D^{ijk} + D^{jik})w^k] \quad . \qquad (32)$$

Function B may be presented in the form

$$B(u, \mathrm{w}) = B^{(o)}(u) + B^{(1)}_i w^i + B^{(2)}_{ij} w^i w^j + B^{(3)}(u, \mathrm{w}) \quad . \quad (33)$$

Function $B^{(o)}$ cannot be determined within the adopted accuracy of the matching procedure. Its expression may be found in (Kopejkin, 1989a,b; Brumberg and Kopejkin, 1990). Functions S, $B^{(1)}_i$, $B^{(2)}_{ij}$, F^{ik}, D^{ik}, D^{ijk} and a^i_E are determined on the basis of the conditions resulted from the tidal form of the external mass action. Separating Newtonian potential U and vector-potential U^i into the parts U_E and U^i_E due to the Earth alone and the parts $\bar{U}_E$ and $\bar{U}^i_E$ caused by the external masses one has

$$U = U_E + \bar{U}_E \quad , \quad U^i = U^i_E + \bar{U}^i_E \quad , \qquad (34)$$

$$\dot{S} = \frac{1}{2}\,v^2_E + \bar{U}_E(\mathrm{x}_E) \quad , \qquad (35)$$

$$D^{ik} = \delta_{ik}\bar{U}(\mathrm{x}_E) \quad , \quad D^{ijk} = \frac{1}{2}\,(\delta_{ij}a^k_E + \delta_{ik}a^j_E - \delta_{jk}a^i_E), \quad (36)$$

$$a^i_E = \bar{U}_{E,i}(\mathrm{x}_E) - Q_i \ , \quad Q_i = -\,\frac{1}{2}\,M^{-1}_E I^{jk}_E \bar{U}_{E,ijk}(\mathrm{x}_E) \ , \qquad (37)$$

$$B^{(1)}_i = 4\bar{U}^i_E(\mathrm{x}_E) - 3v^i_E\,\bar{U}_E(\mathrm{x}_E) \quad , \qquad (38)$$

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$$B^{(2)}_{ik} = \frac{1}{2}\,\dot{D}^{ik} + \bar{U}^{i}_{E,k}(x_E) + \bar{U}^{k}_{E,i}(x_E) - (v^i_E Q_k + v^k_E Q_i) - $$
$$- \frac{1}{2}\,(v^i_E a^k_E + v^k_E a^i_E) \quad , \tag{39}$$

$$\dot{F}^{ik} = -\,2[\bar{U}^{i}_{E,k}(x_E) - \bar{U}^{k}_{E,i}(x_E)] + 2(v^i_E Q_k - v^k_E Q_i) + $$
$$+ \frac{3}{2}\,(v^i_E a^k_E - v^k_E a^i_E) \quad . \tag{40}$$

In distinction to other quantities functions S, F^{ik} and $B^{(0)}$ are determined by differential relations. Function S yields (in the post-Newtonian approximation) the difference of BRS and GRS coordinate time scales in the geocenter ($r^k_E = x^k - x^k_E = 0$). Function F^{ik} describes in its main part the geodesic precession. Equations (37) represent the BRS Newtonian equations of motion of the Earth (at the next step of approximation the matching procedure results in the post-Newtonian equations). Function Q_i is the correction for the non-geodesic motion of the Earth due to the interaction of the Earth quadrupole moment and the external masses. One has therewith

$$M_E = \oint_{(E)} \rho\, d^3x \quad , \qquad I^{jk}_E = \oint_{(E)} \rho\, r^j_E r^k_E\, d^3x \quad .$$

In consequence, the closed form GRS metric for approximation (30)-(32) will be

$$\hat{g}_{oo} = 1 - 2c^{-2}[\hat{U}_E + Q_k w^k + \bar{U}_E(x_E+w) - \bar{U}_E(x_E) - \bar{U}_{E,k}(x_E)w^k] + \ldots, \tag{41}$$

$$\hat{g}_{oi} = c^{-3}\{4\hat{U}^i_E + (q-1)\dot{F}^{ik}w^k + 4[\bar{U}^i_E(x_E+w) - \bar{U}^i_E(x_E) - \bar{U}^i_{E,k}(x_E)w^k] - $$
$$- 4v^i_E[\bar{U}_E(x_E+w) - \bar{U}_E(x_E) - \bar{U}_{E,k}(x_E)w^k] - B^{(3)}_{,i} + \dot{D}^{ijk}w^j w^k\} + \ldots, \tag{42}$$

$$\hat{g}_{ij} = -\,\delta_{ij} - 2c^{-2}[\hat{U}_E + Q_k w^k + \bar{U}_E(x_E+w) - \bar{U}_E(x_E) - \bar{U}_{E,k}(x_E)w^k]\delta_{ij} + $$
$$+ \ldots \tag{43}$$

with

$$\hat{U}_E = U_E \quad , \qquad \hat{U}^i_E = U^i_E - v^i_E U_E \quad . \tag{44}$$

Function $B^{(3)}$ is determined by the harmonic conditions for the GRS metric. These conditions within the adopted accuracy result in the equations (with $\hat{g}_{\mu\nu} = \eta_{\mu\nu} + \hat{h}_{\mu\nu}$)

$$\hat{h}_{oo,i} - \hat{h}_{kk,i} + 2\hat{h}_{ik,k} = 0 \quad , \tag{45}$$

$$\hat{h}_{oo,o} + \hat{h}_{kk,o} - 2\hat{h}_{ok,k} = 0 \quad . \tag{46}$$

Equation (45) is satisfied identically in virtue of the structure of expressions (41) and (43). Equation (46) involves the equation

$$c\hat{U}_{E,o} + \hat{U}^k_{E,k} = 0 \tag{47}$$

satisfied in virtue of the equation of continuity and the Poisson equation determining function $B^{(3)}$

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$$B^{(3)}_{,kk} = 4[\bar{U}^k_{E,k}(x_E+w) - \bar{U}^k_{E,k}(x_E)] - 4v^k_E[\bar{U}_{E,k}(x_E+w) - \bar{U}_{E,k}(x_E)] + + 4[c\bar{U}_{E,0}(x_E+w) - \dot{\bar{U}}_E(x_E)] - \dot{a}^k_E w^k \quad . \tag{48}$$

Function $B^{(3)}$ and its derivatives of the first and second order with respect to the spatial coordinates should vanish with $w = 0$. Using explicit expressions of scalar and vector potentials it is easy to verify the identity

$$\bar{U}^k_{E,k}(x_E+w) - v^k_E\bar{U}_{E,k}(x_E+w) + c\bar{U}_{E,0}(x_E+w) = 0 \tag{49}$$

(relation (46) is imposed on expressions (41)-(43) and the substitution $x = x_E + w$ is performed before differentiating). Hence, equation (48) takes the simple form

$$B^{(3)}_{,kk} = - \dot{a}^k_E w^k \quad . \tag{50}$$

As a particular solution of this equation one may choose, for example

$$B^{(3)} = - \frac{1}{10} \dot{a}^i w^i w^j w^j \tag{51}$$

so that

$$B^{(3)}_{,i} = - \frac{1}{5} \dot{a}^k_E w^k w^i - \frac{1}{10} \dot{a}^i_E w^k w^k \quad . \tag{52}$$

Construction of the GRS metric in the closed form is completed by substituting (52) into (42).

Numerical parameter q plays an important role. Value $q = 1$ corresponds to dynamically non-rotating GRS (DGRS). In this case there are no terms due to geodesic precession in $\hat{g}_{0i}$. But the DGRS spatial axes rotate with respect to the BRS spatial axes as seen from transformation (24). Value $q = 0$ corresponds to kinematically non-rotating GRS (KGRS). In this case the spatial coordinate transformation (24) between BRS and GRS does not involve terms due to geodesic precession. Such terms occur in $\hat{g}_{0i}$. In investigating Earth satellite motion it is suitable to use DGRS (Brumberg and Kopejkin, 1989b). For reduction of observations both systems might be useful.

Let us add a technical remark. Function (51) differs from the function used in (Kopejkin, 1988; Brumberg and Kopejkin, 1989a,b). Function $B^{(3)}$ has been constructed in these papers in form of the series in powers of the geocentric coordinates starting with the terms

$$B^{(3)} = B_{ijk} w^i w^j w^k + \dots , \tag{53}$$

$$B_{ijk} = \frac{2}{9}[\bar{U}^i_{E,jk}(x_E) + \bar{U}^j_{E,ki}(x_E) + \bar{U}^k_{E,ij}(x_E)] - \frac{2}{9}[v^i_E\bar{U}_{E,jk}(x_E) + + v^j_E\bar{U}_{E,ki}(x_E) + v^k_E\bar{U}_{E,ij}(x_E)] + \frac{4}{45}[\delta_{jk}\dot{\bar{U}}_{E,i}(x_E) + \delta_{ki}\dot{\bar{U}}_{E,j}(x_E) + + \delta_{ij}\dot{\bar{U}}_{E,k}(x_E)] - \frac{1}{30}(\delta_{jk}\dot{a}^i_E + \delta_{ki}\dot{a}^j_E + \delta_{ij}\dot{a}^k_E) \quad . \tag{54}$$

It is easy to see that $B_{ikk} = - \frac{1}{6} \dot{a}^i_E$ and equation (50) is satisfied. Such

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a choice of $B^{(3)}$ provides the fulfillment of some relations of symmetry in the expansion of $\hat{g}_{oi}$, in particular, the condition $C_{jm} = C_{mj}$ in quadratic terms

$$g_{oi} = 4c^{-3}[\dot{U}^i_E + \frac{1}{4}(q-1)\dot{F}^{ik}w^k + \varepsilon_{ijk}C_{jm}w^k w^m - \frac{3}{10}\dot{Q}_k w^k w^i + \\ + \frac{1}{10}\dot{Q}_i w^k w^k + \ldots] \quad . \qquad (55)$$

6. CONSIDERATION OF THE INFLUENCE OF THE GALAXY

In ignoring the influence of the Galaxy BRS described by expressions (9)-(11) (with $a_o = a_1 = 0$) is non-rotating system both dynamically and kinematically. But in not so distant future consideration of the influence of the Galaxy may become necessary. Ignoring all local irregularities and considering the mass M of the Galaxy as being concentrated at its center one may relate galactic time T and galactic spatial coordinates X^i with BRS time-space coordinates t and x^i just in the same way as BRS-GRS transformation

$$t = T - c^{-2}[S_G(T) + V^k_B R^k_B] + \ldots, \quad R^k_B = X^k - X^k_B(T) \quad , \qquad (56)$$

$$x^i = R^i_B + c^{-2}\{[\frac{1}{2}V^i_B V^k_B + q_G F^{ik}_G(T) + D^{ik}_G(T)]R^k_B + D^{ijk}_G(T)R^j_B R^k_B\} + \\ + \ldots \qquad (57)$$

with $X^k_B(T)$ and $V^k_B(T) = dX^k_B(T)/dT$ being galactic coordinates and velocity components of the solar system barycenter B. q_G is a constant leading to dynamically ($q_G = 1$) or kinematically ($q_G = 0$) non-rotating BRS respectively (DBRS or KBRS). Assuming the galactic circular motion of the solar system barycenter with radius $X_B = (X^k_B X^k_B)^{1/2}$ and mean motion $N = 2\pi/P = (GM)^{1/2}/X_B^{3/2}$ one has $V^2_B = GM/X_B$ and

$$\dot{S}_G = \frac{3}{2}\frac{GM}{X_B} \quad , \quad \dot{F}^{ik}_G = \frac{3}{2}\frac{GM}{X^3_B}(X^i_B V^k_B - X^k_B V^i_B) \quad , \qquad (58)$$

$$D^{ik}_G = \delta_{ik}\frac{GM}{X_B} \, , \quad D^{ijk}_G = \frac{1}{2}\frac{GM}{X^3_B}(\delta_{jk}X^i_B - \delta_{ij}X^k_B - \delta_{ik}X^j_B) \quad . \qquad (59)$$

Function F^{ik}_G is responsible for the galactic precession caused by the motion of the Sun around the center of the Galaxy (quite similar to the geodesic precession due to the heliocentric motion of the Earth). Introducing as in (25) triplet $\mathbf{F}_G$ one has

$$\dot{\mathbf{F}}_G = \frac{3}{2}\frac{GM}{X^3_B}(\mathbf{X}_B \times \mathbf{V}_B) = \frac{3}{2}N\frac{GM}{X_B}\mathbf{k} \quad , \qquad (60)$$

k being a unit vector normal to the plane of the orbit of the Sun. In vector notation the transformation (57) is rewritten in the form

$$x = R_B + c^{-2}[\ \frac{1}{2}(V_B R_B)V_B + \frac{GM}{X_B} R_B + q_G(R_B \times F_G) + \frac{1}{2}\frac{GM}{X_B^3}(R_B^2)\ X_B -$$

$$- \frac{GM}{X_B^3}(X_B R_B)\ R_B] + \ldots \quad . \tag{61}$$

Using numerical values $M = 1.6 \cdot 10^{11}\ M_\odot$, $X_B = 2.5 \cdot 10^{22}$ cm, $P = 2.2 \cdot 10^{8}$ years = $6.6 \cdot 10^{15}$ s, $c^{-2}GM_\odot = 1.5$ km, $c = 3 \cdot 10^{10}$ cm/s one finds that the DBRS spatial axes rotate with respect to the galactic axes with angular velocity $|c^{-2}\dot{F}_G| = 0.85'' \cdot 10^{-6}$ per century. The term $c^{-2}V_B^k R_B^k$ in the time transformation (56) applied to the Earth represents an annual periodic term with the amplitude 0.4 s. The coefficient in the secular term of this transformation is $c^{-2}\dot{S}_G = 1.44 \cdot 10^{-6}$.

Component g_{oi} of BRS metric (9)-(11) with $a_o = a_1 = 0$ is represented now in the form

$$h^{(3)}_{oi} = c^{-3}[4U^i + (q_G - 1)\dot{F}_G^{ik}\ x^k] \quad . \tag{62}$$

Let us repeat once again that with $q_G = 1$ we have DBRS whose spatial axes rotate with respect to the galactic axes. With $q_G = 0$ we have KBRS whose spatial axes do not rotate with respect to the galactic axes. But the KBRS equations of motion of the solar system bodies contain Coriolis terms. For celestial mechanics purposes DBRS is preferable. For astrometric purposes both systems are useful.

7. TIME SCALES

The problem of time scales is to be solved finally by the IAU recommendations. But irrespective of definitions given by these recommendations the problem is based actually on the relations between the coordinate times of BRS (t), GRS (u) and TRS (τ) (in future the galactic time scale T may be needed). Relation between t and u is given in the form (23) (presently, $O(c^{-4})$ terms therewith may be omitted). Function S determined by (35) is presented in the form

$$S(t) = S^* t + S_p(t) \tag{63}$$

where S^* is a constant and $S_p(t)$ includes both periodic terms as well as non-periodic terms due to the secular evolution of the planetary orbits. Therefore, relation (23) is rewritten in the form

$$u = (1 - c^{-2}S^*)t - c^{-2}[S_p(t) + v_E^k r_E^k] + \ldots \quad . \tag{64}$$

Just in the same manner the relation between u and τ is

$$\tau = u - c^{-2}[V(u) + v_T^k(w^k - w_T^k)] + \ldots \tag{65}$$

with w_T^k and v_T^k being the GRS coordinates and velocity components of the TRS origin. Function $V(u)$ is determined by the differential relation

$$\frac{dV}{du} = \frac{1}{2} v_T^2 + \hat{U}_E(w_T) + Q_k w_T^k + \bar{U}_E(x_E+w_T) - \bar{U}_E(x_E) - \bar{U}_{E,k}(x_E) w_T^k \quad . (66)$$

Its solution is represented in the form

$$V(u) = V^* u + V_p(u) \quad (67)$$

with V^* being a constant one and the same for all possible TRS and hence independent both of time and coordinates of the TRS origin. Function $V_p(u)$ includes corrections for the height of the ground station above the surface of geoid, for lunar and solar tidal influence and for the geophysical factors (deviation of the Earth rotation from the rigid body rotation). Substitution of (67) into (65) yields

$$\tau = (1 - c^{-2} V^*)\ u - c^{-2}[V_p(u) + v_T^k(w^k - w_T^k)] \quad . \quad (68)$$

Relations (64) and (68) are crucial for the relativistic theory of time scales (Brumberg and Kopejkin, 1990).

8. CONCLUSION

The aim of this paper is to elucidate some key questions of relativistic theory of reference systems avoiding technical aspects as much as possible. The main results of the paper may be formulated as follows:

1. BRS metrics in the PPN formalism coordinates and in the harmonic representation are practically equivalent. Harmonic coordinates have advantage of being used for constructing the hierarchy of astronomical reference systems.

2. To elaborate the definitions of time scales it is presently sufficient to retain in the BRS metric only $O(c^{-2})$ terms in g_{00} (Newtonian potential). For relativistic reduction of observations in the static field (ignoring the motion of the masses) the $O(c^{-2})$ terms in g_{00} and g_{ij} are sufficient. For relativistic reduction of observations taking into account the motion of the masses and for rigorous formulation of dynamically or kinematically non-rotating reference system (DRS or KRS) the terms $O(c^{-3})$ in g_{0i} should be added. For unambiguous formulation of the post-Newtonian equations of motion of celestial bodies the terms $O(c^{-4})$ in g_{00} should be also taken into account.

3. Each system entering in the hierarchy of relativistic reference systems may be considered in version of dynamically non-rotating system (DRS) or kinematically non-rotating system (KRS). For celestial mechanics problems it is suitable to use DRS implying the absence of the Coriolis terms in the equations of motion. For astrometric problems both types are of importance.

4. The metric of each system may be presented in the closed form (as illustrated by the GRS metric taking into account only $O(c^{-2})$ and $O(c^{-3})$ terms).

Technical details of constructing reference systems and their relationships discussed here may be found in (Kopejkin, 1988, 1989a,b; Brumberg and Kopejkin, 1989a,b, 1990; Voinov, 1990; Brumberg, 1991).

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