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A CHARACTERIZATION OF A SPARSE BLASCHKE PRODUCT

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ABSTRACT. We give a characterization of a sparse Blaschke product b in terms of the separation of support sets of its zeros in $M(H^\infty + C)$ and the structure of the nonanalytic points. We use this characterization to give a sufficient condition on an interpolating Blaschke product q to have the following property: there exists a non trivial Gleason part P on which q is nonzero and less than one.

Introduction. Let D be the open unit disk in the complex plane and T be its boundary. Let L^∞ by the space of essentially bounded measurable functions on T with respect to Lebesgue measure. By H^∞ we mean the class of all bounded analytic functions in D. Via boundary function identification, H^∞ can be considered as a uniformly closed subalgebra of L^∞. A uniformly closed subalgebra B between H^∞ and L^∞ is called a Douglas algebra. Let C be the family of complex-valued continuous functions on T. It is well known that $H^\infty+C$ is the smallest Douglas algebra containing H^∞ properly. For any Douglas algebra B, we denote by $M(B)$ the space of nonzero multiplicative linear functionals on B, that is, the set of all maximal ideals in B.

An interpolating sequence $\{z_n\}_{n=1}^\infty$ is a sequence in D with the property that for any bounded sequence of complex numbers $\{\lambda_n\}_{n=1}^\infty$, there exists f in H^∞ such that $f(z_n) = \lambda_n$ for all n. A well-known condition states that a sequence $\{z_n\}_{n=1}^\infty$ is interpolating if and only if

$$\inf_n \prod_{n \neq m} \left| \frac{z_m - z_n}{1 - \bar{z}_n z_m} \right| > 0.$$

A Blaschke product

$$b(z) = \prod_{n=1}^{\infty} \frac{|z_n|}{z_n} \left(\frac{z_n - z}{1 - \bar{z}_n z} \right)$$

is called an interpolating Blaschke product if its zero set $\{z_n\}_{n=1}^\infty$ is an interpolating sequence ($|z_n|/z_n = 1$ is understood whenever $z_n = 0$). A sequence $\{z_n\}_{n=1}^\infty$ is said to be sparse (or thin) if it is an interpolating sequence and

$$\lim_{n\to\infty} \prod_{n \neq m} \left| \frac{z_m - z_n}{1 - \bar{z}_n z_m} \right| = 1.$$

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For a function f in $H^\infty+C$, we let $Z(f)=\{x\in M(H^\infty+C):f(x)=0\}$ be the zero set of f in $M(H^\infty+C)$. An inner function is a function in H^∞ of modulus 1 almost everywhere on T. We denoted by $H^\infty[\bar{b}]$ the Douglas algebra generated by H^∞ and the complex conjugate of the inner function b. For any x in $M(H^\infty+C)$, the closed support of the probability measure μ_x on $M(L^\infty)$ that represents x will be denoted by $\operatorname{supp}\mu_x$. It is known that if two support sets intersect, then one is contained in the other. This result is due to K. Hoffman.

At first I believed that if an interpolating Blaschke product b had the property of being small on a nontrivial Gleason part but not zero there, then b was small on some trivial part (a part consisting of one point). We will see that this is not true (Lemma 4 below). Lemmas 1 and 2 give us some information about the invertible elements in the Douglas algebra $H^\infty[\bar{b}]$. Let us call an interpolating Blaschke product b almost sparse if it satisfies the following two conditions.

(i) $M(H^\infty+C)\backslash M(H^\infty[\bar{b}])=\bigcup_{x\in Z(b)}P_x$ where P_x is the nontrivial Gleason part containing x;

(ii) If q is an invertible Blaschke product in $H^\infty[\bar{b}]$, then the set $Z(b)\cap M(H^\infty[\bar{q}])$ is an open-closed subset of $Z(b)$.

The points in $M(H^\infty+C)\backslash M(H^\infty[\bar{b}])$ are referred to as the nonanalytic points of b in $M(H^\infty+C)$.

We say that an interpolating Blaschke product b is dense in the algebra $H^\infty[\bar{b}]$ if for any Blaschke product q invertible in $H^\infty[\bar{b}]$, there is a factor b_1 of b in H^∞ such that $H^\infty[\bar{q}]=H^\infty[\bar{b}_1]$. Sparsity implies both almost sparse and density [Proposition 3.9 of [2], Theorem 1 of [3], and the proof of Lemma 1 of [4]], and while condition (ii) and density are closely related, in general, they are not equivalent.

We have the following:

LEMMA 1. *If b is an almost sparse Blaschke product, then b is dense in $H^\infty[\bar{b}]$.*

PROOF. Since b is almost sparse, the set $Z(b)\cap M(H^\infty[\bar{q}])$ is both an open and closed subset of $Z(b)$ for any Blaschke product q invertible in $H^\infty[\bar{b}]$. By the result of K. Izuchi [5, Corollary 3] there is a factor b_0 of b in H^∞ such that $Z(b_0)=Z(b)\cap M(H^\infty[\bar{q}])$. Factor b as b_0b_1. We claim that $H^\infty[\bar{b}_1]=H^\infty[\bar{q}]$. Since $b_1\neq 0$ on $M(H^\infty[\bar{q}])$, we have that $\bar{b}_1$ is in $H^\infty[\bar{q}]$, hence $H^\infty[\bar{b}_1]\subseteq H^\infty[\bar{q}]$.

To show that $H^\infty[\bar{q}]\subseteq H^\infty[\bar{b}_1]$, it suffices to show that $M(H^\infty[\bar{b}_1]\subseteq M(H^\infty[\bar{q}])$. It is enough to show that if $x\in M(H^\infty+C)$ and $|q(x)|<1$, then $|b_1(x)|<1$. If $|q(x)|<1$, then by (i) there is an x_0 in $Z_{(b)}$ such that $x\in P_{x_0}$. Since $Z(b_0)\subseteq M(H^\infty[\bar{q}])$, we have that $x_0\in Z(b_1)$. Thus $|b_1(x)|<1$ and we're done.

An immediate consequence of this lemma is the following. If b is almost sparse and B is any Douglas algebra contained in $H^\infty[\bar{b}]$ such that $M(B)\cap Z(b)$ is an open-closed subset of $Z(b)$, then $B=H^\infty[\bar{q}]$ for some interpolating Blaschke product q. This is Theorem 2 of [3].

Let us say that an interpolating Blaschke product b satisfying condition (i) has the uniqueness property if the union in (i) is unique, that is, for all x and y in $Z(b)$

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with $x \neq y$, we have that P_x and P_y are distinct Gleason parts containing x and y respectively. We say b satisfies the strong uniqueness property if $x, y \in Z(b)$, $x \neq y$, implies $\operatorname{supp} \mu_x \cap \operatorname{supp} \mu_u = \phi$.

To characterize those inner functions that are invertible in $H^\infty[\bar{b}]$ (when b is almost sparse), we observe that Budde [2] has shown that if P is a nontrivial Gleason part, then $\bar{P}$ contains a trivial point. We can now characterize our invertible elements in $H^\infty[\bar{b}]$.

LEMMA 2. *Let b be an interpolating Blaschke product that satisfies condition* (i) *of almost sparsity. If q is an inner function that is invertible in $H^\infty[\bar{b}]$, then q is the product of a finite number of interpolating Blaschke products.*

PROOF. By condition (i) and the remark above, q does not vanish on any nontrivial Gleason part. In the proof of Corollary 1 of [4], it is shown that these conditions imply that q is the product of a finite number of interpolating Blaschke products.

An example due to D. Sarason [3] of an interpolating Blaschke product b has the following property. There is a nontrivial Gleason part P on which $b \neq 0$, but $|b| < 1$ on P. This implies that

$$M(H^\infty + C)\backslash M(H^\infty[\bar{b}]) \neq \bigcup\{P_x : x \in Z(b)\}.$$

First we give examples of a class of interpolating Blaschke products that are almost sparse, and show that among these, there are some that have a similar property to the property mentioned above. I am grateful to Pamela Gorkin for pointing out this class to me.

LEMMA 3. *Let b be an interpolating Blaschke product that is the product of a finite number of sparse Blaschke products. Then b is almost sparse.*

Let $b = b_1 b_2 \ldots b_k$ where b_i, $1 = 1, 2, \ldots, k$ are sparse Blaschke products. Let q be an invertible Blaschke product in $H^\infty[\bar{b}]$. Since b_1 is sparse by [3, Theorem 1] the sets $Z(b_i) \cap M(H^\infty[\bar{q}])$ are open and closed subsets, $i = 1, 2, \ldots, k$. Therefore we have

$$Z(b) \cap M(H^\infty[\bar{q}]) = \bigcup_{i=1}^{k} [Z(b_i) \cap M(H^\infty[\bar{q}])].$$

Hence $Z(b) \cap M(H^\infty[\bar{q}])$ is an open and closed subset of $Z(b)$. So condition (ii) holds. It is clear that if $|b(x)| < 1$, then $|b_i(x)| < 1$ for some i. So there is an $x_0 \in Z(b_i)$ with $x \in P_{x_0}$. Thus

$$M(H^\infty + C)\backslash M(H^\infty[\bar{b}]) \subseteq \bigcup\{P_x : x_0 \in Z(b)\}$$

and so condition (i) holds.

THEOREM 1. *Let b be an interpolating Blaschke product that satisfies both condition* (i) *of almost sparsity and the strong uniqueness property. Then b is a sparse Blaschke product.*

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PROOF. Assume that b is not sparse. Then we have

$$\liminf_{k\to\infty} \prod_{n\neq k} \left| \frac{z_n - z_k}{1 - \bar{z}_k z_n} \right| \neq 1$$

where the sequence $\{z_n\}$ are the zeros of b in D. So there is a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that

$$\lim_{k\to\infty} \prod_{n\neq n_k} \left| \frac{z_n - z_{n_k}}{1 - \bar{z}_{n_k} z_n} \right| = \alpha < 1.$$

Let b_0 be a factor of b whose zeros in D are the set $\{z_{n_k}\}$ and the set $b_1 = b\bar{b}_0$. We show that this will imply that b violates the strong uniqueness property. Let x_0 be a cluster point of the subsequence $\{z_{n_k}\}$ in $M(H^\infty + C)$. Because of the second equation above, we have that

$$\lim_{k\to\infty} |b_1(z_{n_k})| = \alpha < 1.$$

Thus $|b_1(x_0)| < 1$. Let $S = \operatorname{supp} \mu_{x_0}$ and

$$H_S^\infty = \{f \in L^\infty : f_{|_S} \in H^\infty_{|_S}\}.$$

Since S is a weak peak set for H^∞, we have that H_S^∞ is a Douglas algebra [1, page 10]. It is also a well-known fact that if $x \in M(H^\infty + C)$, then $x \in M(H_S^\infty)$ if and only if $x \in M(L^\infty)$ or $\operatorname{supp} \mu_x \subseteq S$. In particular, $x_0 \in M(H_S^\infty)$. From the way we choose x_0, $|b_1(x_0)| < 1$, thus b_1 is not invertible in H_S^∞. So there is a y in $M(H_S^\infty)$ such that $b_1(y) = 0$. This implies that $\operatorname{supp} \mu_y \subseteq \operatorname{supp} \mu_{x_0}$ and $y \neq x_0$. This contradicts the strong uniqueness property of b. So b is sparse.

LEMMA 4. *Let b be an interpolating Blaschke product that satisfies both condition* (i) *of almost sparse and the uniqueness property, but b is not sparse. Then there is a nontrivial Gleason part P and a factorization of b as $b_0 b_1$ such that $b_0 \neq 0$ on P, but $|b_0| < 1$ on P.*

To see this note that there are x and y in $Z(b)$ with $P_x \cap P_y = \phi$ but $\operatorname{supp} \mu_x \cap \operatorname{supp} \mu_y \neq \phi$. Without loss of generality, we can assume $\operatorname{supp} \mu_x \subsetneq \operatorname{supp} \mu_y$ (see introduction). Factor b as $b_0 b_1$ with $x \in Z(b_0)$ and $y \in Z(b_1)$. Then $|b_0(y)| < 1$ (by Lemma 3.6 of [2]). So we have $|b_0| < 1$ on P_y, but $b_0 \neq 0$ there.

It is very difficult to understand the connection between support sets and nonanalytic points of an interpolating Blaschke product. But one can use one property of almost sparsity to characterize sparseness in terms of nonanalytic points. We state and prove this property of an almost sparse Blaschke product.

LEMMA 5. *Let b be an almost sparse Blaschke product. Then for every factor b_0 of b in H^∞, there is a factor b_0' of b in H^∞ such that*

$$M(H^\infty + C) \backslash M(H^\infty[\bar{b}_0]) = \bigcup \{P_x : x \in Z(b_0')\}.$$

PROOF. Let b_0 be any factor of b in H^∞. Then we certainly have

$$\bigcup_{x \in Z(b_0)} P_x \subseteq M(H^\infty + C) \backslash M(H^\infty[\bar{b}_0]).$$

The containment may be proper. To show that the factor of b can be found, consider the set $Z(b) \cap M(H^\infty[\bar{b}_0])$, which is an open and closed subset of $Z(b)$. Hence there is a factor b_1' of b in H^∞ such that $Z(b_1') = Z(b) \cap M(H^\infty[\bar{b}_0])$. Set $b_0' = b\bar{b}_1'$. By Lemma 1, $H^\infty[\bar{b}_0] = H^\infty[\bar{b}_0']$. To show that

$$M(H^\infty + C) \backslash M(H^\infty[\bar{b}_0]) \subseteq \bigcup \{P_y : y \in Z(b_0')\}, \tag{1}$$

let $x \in M(H^\infty + C)$ with $|b_0(x)| < 1$. Then by condition (i) of almost sparsity, there is a y_0 in $Z(b)$ such that $x \in P_{y_0}$. By the definition of b_0', $y_0 \in Z(b_0')$, hence $x \in \bigcup_{y \in Z(b_0')} P_y$. Since $M(H^\infty + C) \backslash M(H^\infty[\bar{b}_0]) = M(H^\infty + C) \backslash M(H^\infty[b_0'])$ we conclude that

$$M(H^\infty + C) \backslash M(H^\infty[\bar{b}_0]) = \bigcup \{P_y : y \in Z(b_0')\}.$$

The fact that $Z(b_0)$ may be properly contained in $Z(b_0')$ (even if b satisfies the uniqueness property) is the key difference between almost sparse and sparse as the next theorem indicates.

THEOREM 2. *Let b be an interpolating Blaschke product. If every factor b_0 of b in H^∞ has the property that*

$$M(H^\infty + C) \backslash M(H^\infty[\bar{b}_0]) = \bigcup \{P_{x_0} : x_0 \in Z(b_0)\}, \tag{2}$$

and also satisfies the uniqueness property, then b is sparse.

PROOF. Assume that b is not sparse. Then we have

$$\liminf_{k \to \infty} \prod_{n \neq k} \left| \frac{z_n - z_k}{1 - \bar{z}_k z_n} \right| \neq 1$$

where the sequence $\{z_n\}$ are the zeros of b in D. As in the proof of Theorem 1, there is a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that

$$\lim_{k \to \infty} \prod_{n \neq n_k} \left| \frac{z_n - z_{n_k}}{1 - \bar{z}_{n_k} z_n} \right| = \alpha < 1. \tag{3}$$

Let b_0 be the factor of b whose zeros in D are the subsequence $\{z_{n_k}\}$ and set $b_1 = b\bar{b}_0$. We will show that the factor b_1 violates the property in (2) above. To see this let x_0 be a cluster point of $\{z_{n_k}\}$ in $M(H^\infty + C)$. Because of (3) we have $|b_1(x_0)| < 1$. Let

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P_{x_0} be the nontrivial Gleason part containing x_0. Then $|b_1| < 1$ on P_{x_0}, but by the uniqueness property of b, $|b_1| \neq 0$ on P_{x_0}. Hence

$$\bigcup\{P_y : y \in Z(b_1)\} \subseteq M(H^\infty + C)\backslash M(H^\infty[\bar{b}_1]).$$

This contradicts (2) above. So b must be sparse.

Here are three questions that I have been unable to answer.

QUESTION 1. Construct an interpolating Blaschke product b that satisfies both condition (i) of almost sparsity and the uniqueness property, but is not sparse.

QUESTION 2. Suppose b is almost sparse and satisfies the uniqueness property. Must b be sparse?

One problem here comes from Lemma 5. That is, there is a factorization of b as $b_0 b_1$, with $|b_0| < 1$ on some nontrivial Gleason part P where $b_0 \neq 0$. If the converse of Lemma 3 were true, then this would give us a positive answer to this question.

QUESTION 3. Suppose the conditions mentioned in Question 1 imply sparsity and those in Question 2 also imply sparsity. How are these conditions related?

REFERENCES

1. A. Browder, *Introduction of Function Algebras,* W. A. Benjamin, Inc., 1969.
2. P. Budde, Ph.D. Dissertation, Univ. of California, Berkeley, 1982.
3. C. J. Guillory, *Douglas Algebras of the Form* $H^\infty[\bar{q}]$, to appear in the J. of Math. Anal.
4. C. J. Guillory, K. Izuchi and D. Sarason, *Interpolating Blaschke products and division in Douglas algebras*, Proc. Royal Irish Adac. **84A** (1984),1–7.
5. K. Izuchi, *Zero sets of interpolating Blaschke products*, to appear in the Pac. J. of Math.

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