A general proof of the Addition Theorems in Trigonometry.

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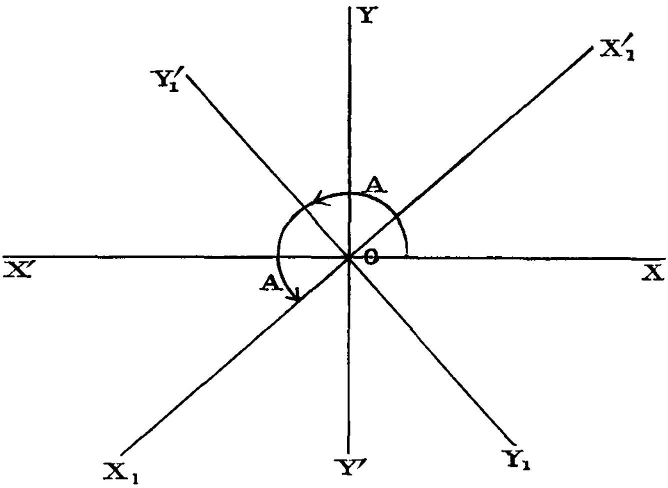
It is a very convenient method to begin the study of Trigonometry with one or two lessons on Coordinates and the Coordinate Diagram. This leads to a general definition of the Sine and Cosine of any angle which the beginner has little difficulty in comprehending. With the help of the Coordinate Diagram the theorems on projection which are of so much importance can be proved with great precision. It is in such a course that the present proof of the Addition Theorems might find a place.



Let X'OX, Y'OY be the axes of coordinates. Let them be rotated round O in the positive direction through an angle A which may be of any magnitude.

The new position of the x-axis is represented on the diagram by OX_1 which thus makes an angle A with OX.

The new position of OY is OY_1 which makes an angle $90^c + A$ with OX.

Any point P on the plane can now be specified in two ways:

either (1) by its coordinates x, y referred to the old axes

or (2) by its coordinates ξ, η " " " new axes.

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If M_1 be any point on OX_1 the projections of OM_1 on the x and y axes are $OM_1\cos A$, $OM_1\sin A$.

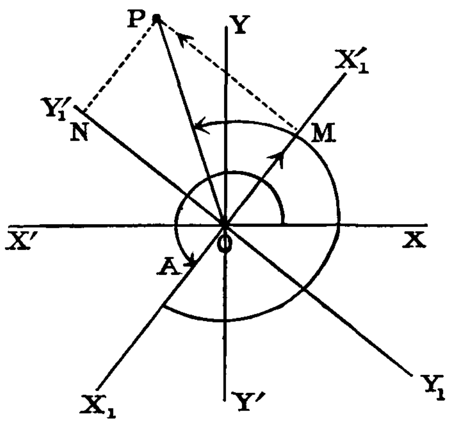
If M_1' be any point on OX_1' the projections of OM_1' on the x and y axes are $-OM_1'\cos A$, $-OM_1'\sin A$.

Thus we may make the following general statement :—

If M be any point on the line X_1OX_1' specified by the coordinate ξ, the projections of OM on the x and y axes are $\xi\cos A$, $\xi\sin A$.

Again if N be any point on the line Y_1OY_1' specified by the coordinate η, the projections of ON on the x and y axes are

$$\eta\cos(A+90°), \quad \eta\sin(A+90°).$$



Formulae for the sine and cosine of $(A+B)$.

Let the revolving line of length r start from OX and trace out an angle A. It will then coincide with OX_1. Let it further revolve through an angle B and take up the position OP indicated in the diagram, the coordinates of P referred to the new axes being ξ, η. The line OP now makes an angle $(A+B)$ with OX.

If we consider the diagram X_1OX_1', Y_1OY_1' we have an angle B traced out by the revolving line starting from the initial position OX_1. Thus

$$\cos B = \frac{\xi}{r}, \quad \sin B = \frac{\eta}{r},$$

by the definition of the sine and cosine of an angle.

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Now draw PM, PN perpendiculars to X_1OX_1', Y_1OY_1'.

The journey from O to P can be performed in two ways:

(1) along OP,

(2) along OM and then along MP.

By the theorem on projections we have

$$\begin{aligned} \text{projection of OP} &= \text{projection of OM} + \text{projection of MP} \\ &= \text{projection of OM} + \text{projection of ON}. \end{aligned}$$

Take these on the x-axis:

$$\begin{aligned} r\cos(A+B) &= \xi\cos A + \eta\cos(A+90^\circ) \\ &= r\cos B\cos A + r\sin B\cos(A+90^\circ); \end{aligned}$$

$$\therefore \quad \cos(A+B) = \cos A\cos B - \sin A\sin B,$$

$$\text{since } \cos(A+90^\circ) = -\sin A.$$

Again take projections on the y-axis:

$$\begin{aligned} r\sin(A+B) &= \xi\sin A + \eta\sin(A+90^\circ) \\ &= r\cos B\sin A + r\sin B\sin(A+90^\circ); \end{aligned}$$

$$\therefore \quad \sin(A+B) = \sin A\cos B + \cos A\sin B,$$

$$\text{since } \sin(A+90^\circ) = +\cos A.$$

To find formulae for the sine and cosine of $(A-C)$.

The position OP of the revolving line may be attained in the following way:

First by a revolution through an angle $+A$ and then a further revolution through an angle $-C$. Hence by considering the coordinate diagram given by the axes X_1OX_1', Y_1OY_1' we have

$$\xi = r\cos(-C), \quad \eta = r\sin(-C) \quad \text{by definition.}$$

The angle which OP now makes with OX is $(A-C)$, and thus by the same theorems as before

$$\begin{aligned} r\cos(A-C) &= \xi\cos A + \eta\cos(A+90^\circ) \\ &= r\cos(-C)\cos A + r\sin(-C)\cos(A+90^\circ); \end{aligned}$$

$$\therefore \quad \cos(A-C) = \cos A\cos C + \sin A\sin C,$$

$$\text{since } \cos(-C) = +\cos C, \ \sin(-C) = -\sin C.$$

Again $r\sin(A-C) = \xi\sin A + \eta\sin(A+90^\circ)$;

$$\therefore \quad \sin(A-C) = \sin A\cos C - \cos A\sin C.$$

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