where $Q_n'(\mu) = \frac{d}{d\mu} Q_n(\mu)$, denoting $\cos\theta$ by $\mu$, and replacing the notation $Q_n(\theta)$ by $Q_n(\mu)$. The surface velocity thus vanishes, as is obvious from symmetry, where the axis of the ring cuts the spherical surface.

The expressions for the velocity potential in the form given above do not converge rapidly unless the distance of the ring from the centre of the sphere be considerable compared to the radius.

---

## On a problem in permutations.

By R. E. Allardice, M.A.

The problem to be considered may be stated as follows :—How many necklaces may be formed with $p$ pearls, $r$ rubies, and $d$ diamonds ?*

The peculiarity of this problem is that a general solution cannot be given in terms of $p$, $r$, and $d$ alone. The form of the solution depends on the nature of the numbers $p$, $r$, and $d$ ; and it is necessary in solving the problem to consider whether or not these numbers have a common measure, and how many of them are odd and how many even. All possible cases of the problem are not discussed in this paper ; but enough of them are considerod to illustrate the variety of forms that the solution may assume.

If we put $p+r+d=n$, the number of possible arrangements of the $n$ stones in a line is $n!/p!r!d!$ Hence the question is, how many of these arrangements will give the same necklace ; or, conversely, if we take any one form of the necklace, how many different arrangements of the stones we can get from it by breaking it at different parts and stretching it out straight. It is obvious that if the $n$ stones had been all different, the answer to the second of these questions would have been $2n$ ; in other words, with $n$ stones all different, we may form $n!/2n$ necklaces. The further question then naturally arises, In what cases, if any, are the $2n$ arrangements of the stones obtained from each form of the necklace all different when the stones are not all different ? Now these $2n$ arrangements comprise the $n$ that are obtained by a cyclical interchange of the stones, one at a time, together with the $n$ that are obtained by exactly reversing each of these $n$ arrangements.

---

* This problem was suggested to me by Professor Chrystal.

https://doi.org/10.1017/S0013091500030546 Published online by Cambridge University Press

The $2n$ arrangements may fail to be all different for one of the three following reasons :—

(1). Because an arrangement is not altered when it is reversed. In this case the arrangement must be symmetrical.

(2). Because an arrangement is reproduced after a cyclical interchange of a certain number of the stones. It may easily be seen that in this case the arrangement considered must be resolvable into a number of identical groups.

(3). Because an arrangement is reproduced when a number of the stones have been interchanged cyclically and the resulting arrangement reversed. In order that this may happen, the arrangement considered must consist of two symmetrical groups. These three cases may be conveniently referred to as the case of a single symmetrical group, the case of identical groups, and the case of two symmetrical groups, respectively.

A single symmetrical group will occur if not more than one of the numbers $p$, $r$, $d$, is odd ; and not otherwise.

Identical groups will occur if $p$, $r$, and $d$ have a common measure ; and not otherwise.

The case of the two symmetrical groups will occur if not more than two or the numbers $p$, $r$, $d$, are odd ; and not otherwise.

First Case.

Hence if $p$, $r$, and $d$ are all odd and have no common measure, the number of necklaces that may be formed is $n!/2n.p!r!d!$, but this formula will not hold good in any other case. [It may be pointed out that any two of the numbers may have a common measure and the formula will still hold ; and that a similar formula will apply to the case of stones of any number of different kinds, provided there be an odd number of each of three of the kinds.]

Second Case.

Suppose now that two of the numbers, $p$, $r$, $d$, are odd ; but that these numbers have still no factor in common.

Let $p = 2\pi + 1$, $r = 2\rho + 1$, $d = 2\delta$.

We may now have two symmetrical groups, a pearl occurring at the centre of one and a ruby at the centre of the other. It should be noted further that a cyclical interchange in the case of two symmetrical groups produces two symmetrical groups ; and that by means of

https://doi.org/10.1017/S0013091500030546 Published online by Cambridge University Press

a number of cyclical interchanges one of the groups may be reduced to a single stone. Hence, if we put $\nu = \pi + \rho + \delta$, the number of necklaces involving two symmetrical groups will be $\nu!/\pi!\rho!\delta!$. From each of these may be obtained $n$ linear arrangements of the stones; and thus there are altogether $n!/p!r!d! - n.\ \nu!/\pi!\rho!\delta!$ arrangements that do not involve symmetrical groups, giving $(n!/p!r!d! - n.\nu!/\pi!\rho!\delta!)/2n$ necklaces. Hence the whole number of necklaces is

$$\frac{1}{2n}\left\{\frac{n!}{p!r!d!} - \frac{n.\nu!}{\pi!\rho!\delta!}\right\} + \frac{\nu!}{\pi!\rho!\delta!} = \frac{(n-1)}{2.p!r!d!} + \frac{\nu!}{2.\pi!\rho!\delta!}$$

It may be well to illustrate this formula by actually writing out all the possible arrangements in a particular case.

Put $p = 1$, $r = 3$, $d = 2$; then $\pi = 0$, $\rho = 1$, $\delta = 1$; $n = 6$, $\nu = 2$.

The formula gives $\dfrac{5!}{2.1!3!2!} + \dfrac{2}{2.0!1!1!} = 6.$

The arrangements are the following, the two in the first column containing, and the others not containing, symmetrical groups:—

| | | |
|---|---|---|
| *prdddr* | *prrddd* | *pddrrd* |
| *pdrdrd* | *pddrdr* | *prddrd* |

THIRD CASE.

Next let only one of the numbers $p$, $r$, $d$, be odd; and suppose that these three numbers have still no common measure.

Let $p = 2\pi + 1$, $r = 2\rho$, $d = 2\delta$, $n = p + r + d$, $\nu = \pi + \rho + \delta$. We have now the case of a single symmetrical group, and also the case of two symmetrical groups to consider.

In the latter case one of the groups will contain an odd number of stones and will have a pearl as the centre one, while the other group will contain an even number. It may easily be seen that any such arrangement may be reduced by a number of cyclical interchanges to a single symmetrical group.

The number of necklaces involving symmetrical groups is $\nu!/\pi!\rho!\delta!$, giving $n.\nu!/\pi!\rho!\delta!$ different linear arrangements. Hence it may easily be seen that the whole number of different necklaces is

$$(n-1)!/2.p!r!d! + \nu!/2.\pi!\rho!\delta!$$

This formula is exactly the same as that obtained in the last case; but $\pi$ and $\nu$ have here slightly different meanings.

As an example, put $p = 1$, $r = 2$, $d = 2$; so that $\pi = 0$, $\rho = 1$, $\delta = 1$; $n = 5$, $\nu = 2$.

https://doi.org/10.1017/S0013091500030546 Published online by Cambridge University Press

The formula gives $4!/2.1!2!2! + 2!/2.0!1!1! = 4$.

The arrangements are the following, the two in the first column containing, and the other two not containing, symmetrical groups :—

| | |
|---|---|
| *rdpdr* | *rrpdd* |
| *drprd* | *drdrp* |

FOURTH CASE.

Suppose in the next case, that $p$, $r$, and $d$ are all even.

This case is an exception to the assumption made hitherto that $p$, $r$, and $d$ have no common measure. I shall consider only the case in which the other factors, when 2 is divided out, are all odd numbers, and have no common measure, that is,

$p = 2\pi$, $r = 2\rho$, $d = 2\delta$; where $\pi$, $\rho$ and $\delta$ are odd numbers and have no common measure.

We may now have (1) a single symmetrical group; (2) two symmetrical groups; (3) identical groups.

Under (2) we have to consider two cases, namely, ($\alpha$) that in which each of the symmetrical groups contains an odd number of stones, the central stones of the two groups being necessarily of the same kind; ($\beta$) that in which each of the symmetrical groups contain an even number of stones, an arrangement which is reducible to a single symmetrical group by means of cyclical interchanges.

(1). The number of necklaces involving single symmetrical groups is $\nu!/2.\pi!\rho!\delta!$, giving $n.\nu!/2.\pi!\rho!\delta!$ permutations.

(2). The number of necklaces involving two symmetrical groups of type ($\alpha$) is $\Sigma\{(\nu-1)!/2.(\pi-1)!\rho!\delta!\}$, the three terms corresponding to the three cases where the middle stones are 2 pearls, 2 rubies and 2 diamonds. The number of permutations that these give is

$$n.\Sigma\{(\nu-1)!/2.(\pi-1)!\rho!\delta!\} = n.(\nu-1)!\Sigma\pi/2.\pi!\rho!\delta! = n.\nu!/2.\pi!\rho!\delta!$$

(3). The number of *permutations* involving identical groups is $\nu!/\pi!\rho!\delta!$ giving $(\nu-1)!/2.\pi!\rho!\delta!$ necklaces.

Hence the whole number of permutations not involving any of these three cases is

$$\frac{n!}{p!r!d!} - \frac{n.\nu!}{\pi!\rho!\delta!} - \frac{\nu!}{\pi!\rho!\delta!};$$

and the whole number of necklaces is

$$\frac{1}{2n}\left\{\frac{n!}{p!r!d!} - \frac{n.\nu!}{\pi!\rho!\delta!} - \frac{\nu!}{\pi!\rho!\delta!}\right\} + \frac{\nu!}{\pi!\rho!\delta!} + \frac{(\nu-1)!}{2.\pi!\rho!\delta!}$$

$$= \frac{(n-1)!}{2.p!r!d!} + \frac{\nu!}{2.\pi!\rho!\delta!} + \frac{(\nu-1)!}{4.\pi!\rho!\delta!}$$

https://doi.org/10.1017/S0013091500030546 Published online by Cambridge University Press

As an example, put $p=2$, $r=2$, $d=2$; so that $\pi=1$, $\rho=1$, $\delta=1$, $n=6$, $\nu=3$.

The formula gives $5!/2.2.2.2+3!/2+2!/4=11$.

The arrangements are the following, those in (1) containing a single symmetrical group, those in (2) two symmetrical groups, those in (3) identical groups, and those in (4) having none of these peculiarities :—

| | | |
|---|---|---|
| (1) *dprrpd* | (2) *pdrprd* | (4) *prddpr* |
| *pdrrdp* | *drpdpr* | *rdpprd* |
| *prddrp* | *rpdrdp* | *pdrrpd* |
| | (3) *pdrpdr* | *pprrdd* |

FIFTH CASE.

Suppose next that $p$, $r$, and $d$ have a factor in common. In a full discussion of the problem a number of particular cases would require to be considered; but I shall limit myself to that in which $p$, $r$, and $d$ are all odd numbers and their G.C.M. is a prime number. In other words, I assume $p=\lambda\pi$, $r=\lambda\rho$, $d=\lambda\delta$, where $\lambda$, $\pi$, $\rho$, and $\delta$ are all odd numbers, $\lambda$ is a prime, and $\pi$, $\rho$, and $\delta$ have no factor in common.

A permutation may consist of $\lambda$ identical groups, each containing $\nu=\pi+\rho+\delta$ terms.

There will be $\nu!/\pi!\rho!\delta!$ such permutations, giving $(\nu!/\pi!\rho!\delta!)/2\nu$ necklaces.

Hence the whole number of necklaces is

$$\frac{1}{2n}\left\{\frac{n!}{p!r!d!}-\frac{\nu!}{\pi!\rho!\delta!}\right\}+\frac{(\nu-1)!}{2.\pi!\rho!\delta!}=\frac{(n-1)!}{2.p!r!d!}+\frac{(n-\nu).(\nu-1)!}{2n.\pi!\rho!\delta!}$$

$$=\frac{(n-1)!}{2.p!r!d!}+\frac{\lambda-1}{\lambda}\cdot\frac{(\nu-1)!}{2.\pi!\rho!\delta!}$$

As an example, put $p=r=d=3$; so that $\pi=\rho=\delta=1$, $n=9$, $\nu=3$. The formula gives $8!/2.3!3!3!+2/3=94$.

Only one of the necklaces contains identical groups, namely, *prdprdprd*.

Without writing out all the 94 arrangements, we may verify that this is the correct number by enumerating them in the following manner :—

We may set down the three pearls in a ring, and count the number of ways in which the other six stones may be arranged relatively

https://doi.org/10.1017/S0013091500030546 Published online by Cambridge University Press

to them. The number of stones to be put in each of the three spaces between the pairs of pearls is given in the following table, with the total number of corresponding necklaces in each case :—

| | | | | | |
|---|---|---|---|---|---|
| 0 | 0 | 6 | gives | 10 | necklaces, |
| 0 | 1 | 5 | ,, | 20 | ,, |
| 0 | 2 | 4 | ,, | 20 | ,, |
| 0 | 3 | 3 | ,, | 10 | ,, |
| 1 | 1 | 4 | ,, | 10 | ,, |
| 1 | 2 | 3 | ,, | 20 | ,, |
| 2 | 2 | 2 | ,, | 4 | ,, |
| | | | | 94 | |

---

*Sixth Meeting*, 21*th April* 1890.

---

R. E. Allardice, Esq., M.A., Vice-President, in the Chair.

## On a hydromechanical theorem.

By Dr A. C. Elliott.

Giffard's injector appeared more than thirty years ago. The first serious attempt to explain its action on dynamical principles was made by the late William Froude at the Oxford Meeting of the British Association in 1860. The history of mechanical science is almost everywhere deeply marked by Rankine; and it seems, just as it ought to be, that he should be found to have contributed not a little to the literature of this particular subject in a paper presented to the Royal Society of London in 1870. As serving to show how far the problem is still interesting, even from a high standpoint, attention may be directed to the exceedingly curious procedure of Professor Greenhill, where he deals cursorily with the matter at the page numbered 448 of his article on Hydromechanics in the *Encyclopædia Britannica.*

When first announced, the statement that the particles of a mere steam jet could, by the agency of this somewhat simple apparatus, force for themselves, in addition to a considerable quantity of more or less cold feed water, re-entrance into the identical boiler from whence they had escaped, seemed to involve an impossibility. But the mystery of that aforetime paradox would have been as nothing had it then been farther known what is now familiar—namely, that