SPANS OF TRANSLATES IN $L^p(G)$

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(received 6 August 1964)

1. Introduction and preliminaries

Throughout this paper, G denotes a Hausdorff locally compact Abelian group, X its character group, and $L^p(G)$ $(1 \leqq p \leqq \infty)$ the usual Lebesgue space formed relative to the Haar measure on G. If $f \in L^p(G)$, we denote by $T^p[f]$ the closure (or weak closure, if $p = \infty$) in $L^p(G)$ of the set linear combinations of translates of f.

Wiener's famous "closure of translations theorem" asserts that, if $f \in L^1(G)$, then $T^1[f] = L^1(G)$ if and only if $Z = \hat{f}^{-1}(0)$ is void, $\hat{f}$ denoting the Fourier transform of f. Wiener proved the result for $G = R$, the additive group of real numbers ([1], p. 98, Theorem 9); it has since been extended to general G (see, for example, [9], p. 162). Wiener also showed ([1], p. 100, Theorem 11) that, if $f \in L^2(G)$, then $T^2[f] = L^2(G)$ if and only if Z is locally null; this result also extends (and easily) to general G. If G is compact, the analogue of Wiener's theorems is true and easy to prove for $L^p(G)$, whatever the value of p ([2], Corollary 3.2.2). But, if G is noncompact, no such complete results are known for values of p other than 1 and 2. However, Segal ([2], Theorem 3.3). Pollard [3], Agnew [4], [5], and Edwards [6] have given partial results about $T^p(G)$ in case $f \in L^1(G) \cap L^p(G)$ and G is R or R^n; Segal ([2], Theorems 3.3 and 3.4) also gives partial results about $T^p[f]$ for general G, the assumption that f be integrable being replaced when $p > 2$ by the demand that f be the Fourier transform of some element of $L^{p'}(G)$ $(1/p+1/p' = 1)$. A unified treatment was given by Herz [15] (and, indirectly, [16] — the main concern of which is the uniform approximations by linear combinations of translates of bounded uniformly continuous functions). The writer pleads guilty to having overlooked [15] until the present paper had been completed and submitted for publication, at which time private correspondence with Professor Herz corrected the oversight.

In this paper we start almost *ab initio*. Sufficient conditions for $T^p[f]$ to exhaust $L^p(G)$ are obtained in Theorem (2.2) in a form slightly less demanding than in Herz's analogous Theorem 1. Partial converses appear in Theorems (2.5) and (6.2): these correspond roughly to Herz' Theorem 3.

These results include those of Segal, Agnew, and Pollard. The relationship with the results of Pollard are discussed in some detail in § 7: this is thought to be desirable because Pollard uses Abel summability for Fourier transforms, a technique which is not employed in our general treatment.

In § 3 we collect some results about the class of p-thin sets (our analogue of Herz's sets of type $U^{p'}$, $1/p+1/p' = 1$) and give an application in § 4. In § 5 we consider some connections between p-thinness for algebraic varieties and uniqueness theorems for associated partial differential equations, and use this to discuss some examples. Both here and in § 4, our examples amplify some of the remarks made in Herz [15]. The case $G = R^n$ is discussed further in §6.

We shall use systematically the generalised Fourier transform $\hat{\phi}$ of an arbitrary $\phi \in L^\infty(G)$, which transform exists as a pseudomeasure on X. Concerning pseudomeasures, see [7], Appendices II, III, and [8]. For our main Theorem (2.2) we shall require only the following facts:

(1.1) if $f \in L^1(G)$ and $\phi \in L^\infty(G)$, then $(f * \phi)^\wedge = \hat{f} \cdot \hat{\phi}$.

(1.2) Pseudomeasures can be localised, so that in particular one can define the support supp s of a pseudomeasure s to be the complement of the largest open subset of X on which s is zero. Then, if $f \in L^1(G)$ and $\phi \in L^\infty(G)$, the relation $\hat{f} \cdot \hat{\phi} = 0$ implies that sup $\hat{\phi} \subset \hat{f}^{-1}(0)$. The spectrum of ϕ can now be defined directly as the support supp $\hat{\phi}$ of $\hat{\phi}$.

(1.3) A pseudomeasure having a finite support $\{\xi_1, \cdots, \xi_n\} \subset X$ is a linear combination of Dirac measures placed at the points $\xi_1, \cdots, \xi_n$.

It should be noted that, although Theorem (2.2) could be stated so as to include the case $p = 1$ (*i.e.*, Wiener's theorem), our arguments do not really simplify the known proofs of the latter, inasmuch as the properties of pseudomeasures are based upon results about the ring structure of $L^1(G)$ which are of much the same depth as Wiener's theorem itself. Thus the emphasis is everywhere on the case in which $1 < p < \infty$ and G is non-compact.

2. The main theorem

We begin with a definition.

(2.1) Definition. A subset E of X is said to be p-thin if the relations

(2.1.1) $$\phi \in C_0(G) \cap L^{p'}(G), \text{ supp } \hat{\phi} \subset E$$

imply that

(2.1.2) $$\phi = 0.$$

In (2.1.1) it is understood that $C_0(G)$ denotes the space of continuous functions on G which tend to zero at infinity, whilst p' is defined by

$$1/p+1/p' = 1.$$

Some discussion of p-thin sets will be given in §§ 3 and 5.

Herz [15] uses, in place of our concept of p-thinness, the notion of type $U^{p'}$: a closed set $E \subset X$ is of type $U^{p'}$ if there exists no $\phi \neq 0$ which is bounded and continuous, belongs to $L^{p'}(G)$, and is such that $\operatorname{supp} \hat{\phi} \subset E$. His Theorem 1 is our Theorem (2.2) to follow, with "p-thin" replaced by "of type $U^{p'}$". It is evident that any set of type $U^{p'}$ is p-thin, so that Herz's Theorem 1 is implied by our Theorem (2.2). I do not know whether, when $p > 1$, there exist sets E which are p-thin but not of type $U^{p'}$.

(2.2) THEOREM. *Suppose that* $1 < p < \infty$, *that* $f \in L^1(G) \cap L^p(G)$, *and that* $Z = \hat{f}^{-1}(0)$ *is* p*-thin. Then* $T^p[f] = L^p(G)$.

PROOF. According to the Hahn-Banach theorem it suffices to show that if $g \in L^{p'}(G)$ satisfies

$$f * g = 0, \tag{2.2.1}$$

then $g = 0$ a.e. To this end, take any $k \in L^1(G) \cap L^p(G)$. Then (2.2.1) implies that

$$f * k * g = 0. \tag{2.2.2}$$

Here $\phi = k * g$ belongs to $C_0(G) \cap L^{p'}(G)$. Also, (2.2.2) yields via (1.1) the relation

$$\hat{f} \cdot \hat{\phi} = 0.$$

Using (1.2), this in turn leads to

$$\operatorname{supp} \hat{\phi} \subset Z.$$

Since Z is assumed to be p-thin, reference to (2.1) confirms that $\phi = k * g = 0$. This being the case for each $k \in L^1(G) \cap L^p(G)$, it follows easily that $g = 0$ a.e. The proof is complete.

A similar argument yields an analogous result for $p = \infty$, this time in an "if and only if" form, and without assuming that $f \in L^1(G)$.

(2.3) THEOREM. *Suppose that* $f \in L^\infty(G)$. *Then* $T^\infty[f] = L^\infty(G)$ *if and only if* $\operatorname{supp} \hat{f} = X$.

PROOF. The dual of $L^\infty(G)$ relative to its weak topology being $L^1(G)$, it has to be shown that

$$g \in L^1(G), \ f * g = 0 \tag{2.3.1}$$

implies $g = 0$ a.e., if and only if $\operatorname{supp} \hat{f} = X$. But, by (1.1), (2.3.1) is equivalent to the equation $\hat{g} \cdot \hat{f} = 0$. This implies $\hat{g} = 0$ (*i.e.*, $g = 0$ a.e.), if and only if $\operatorname{supp} \hat{f} = X$, as alleged.

(2.4) REMARK. There is an almost obvious extension of (2.2), giving a sufficient condition in order that a given family (f_i) of functions in $L^1(G) \cap L^p(G)$ be such that the vector subspace of $L^p(G)$ generated by the translates of all the f_i be dense in $L^p(G)$: the said sufficient condition is that $\bigcap \hat{f}_i^{-1}(0)$ be p-thin. There is a similar extension of (2.3).

We next consider a partial converse of (2.2); see also (6.2) for the case $G = R^n$.

(2.5) THEOREM. *Suppose that* $1 < p < \infty$, *that* $f \in L^1(G) \cap L^p(G)$, *and that* $T^p[f] = L^p(G)$. *Put* $Z = \hat{f}^{-1}(0)$. *Suppose that either*

(i) *the frontier* ∂Z *of* Z *relative to* X *is* p-*thin, or*

(ii) Z *is an* S-*set* ([9], p. 158).

Then Z *is* p-*thin.*

PROOF. The argument proceeds by contradiction. Suppose that Z were not p-thin. Then there exists a function $\phi \neq 0$ in $C_0(G) \cap L^{p'}(G)$ for which $\operatorname{supp} \hat{\phi} \subset Z$. It will suffice to show that in either case $f * \phi = 0$, *i.e.*, that $\hat{f} \cdot \hat{\phi} = 0$.

In case (ii), this follows from the known properties of S-sets. On the other hand, it is in any case evident that the relation $\hat{f} \cdot \hat{\phi} = 0$ holds on a neighbourhood of each point of Z' (complement in X) and on a neighbourhood of each point of the interior of Z. Hence, by the localisation principle for pseudomeasures, the support of $\hat{f} \cdot \hat{\phi}$ is contained in

$$Z \cap (\text{interior } Z)' = \partial Z.$$

Since $f * \phi \in C_0(G) \cap L^{p'}(G)$, (i) entails that $f * \phi = 0$ once more. The proof is complete.

(2.6) REMARKS. (i) As Herz remarks ([15], Theorem 2*), if $T^p[f] = L^p(G)$, then there exists no $\phi \neq 0$ which is both a Fourier-Stieltjes transform and a member of $L^{p'}(G)$ satisfying $\operatorname{supp} \hat{\phi} \subset Z = \hat{f}^{-1}(0)$. For, since $\hat{\phi}$ is now a bounded measure, the relation $\operatorname{supp} \hat{\phi} \subset Z$ entails that $\hat{f} \cdot \hat{\phi} = 0$ and so that $f * \phi = 0$; since $\phi \in L^{p'}(G)$ and $T^p[f] = L^p(G)$, this gives $\phi = 0$. Herein, instead of assuming that ϕ is a Fourier-Stieltjes transform, it is enough to assume that it is the weak limit in $L^\infty(G)$ of such transforms.

(ii) Herz ([15], Theorem 3) gives a different sort of partial converse of Theorem (2.2) in which $\hat{f}$ is further restricted; see also Theorem (6.2) and the Remarks which follow it.

3. Concerning p-thin sets

We shall collect a number of results which assist in showing that certain types of sets are p-thin, and thus assist in the application of (2.2).

(3.1) (i) Any subset of a p-thin set is p-thin.

(ii) A set E is p-thin if and only if every compact subset of E is p-thin.

(iii) A set E is p-thin if and only if, for each $\xi \in X$, there is a neighbourhood $U(\xi)$ of ξ such that $E \cap U(\xi)$ is p-thin.

(v) If E is p-thin, and if $q > p$, then E is q-thin.

PROOF. Statement (i) is trivial.

As for (ii) we observe first that, since supp $\hat{\phi}$ is always a closed set, (2.1) shows that E is p-thin if and only if every closed subset of E is p-thin. Next, assuming that E is closed, if ϕ be replaced in (2.1) by functions of the form $k * \phi$, where $k \in L^1(G)$ and supp $\hat{k}$ is compact, and if it be noted that ϕ is the uniform limit of such functions $k * \phi$, it appears that E is p-thin provided each compact subset of E is p-thin. The converse assertion is a trivial consequence of (i).

In proving (iii) we may, in view of (ii), assume that E is relatively compact in X. Then, if E satisfies the stated condition, we can find open sets U_m $(m = 1, 2, \cdots, n)$ which cover $\bar{E}$ and such that $E \cap U_m$ is p-thin for each m. By known properties of $L^1(G)$, functions $k_m \in L^1(G)$ may be chosen so that supp $\hat{k}_m \subset U_m$ and $\sum_{m=1}^{n} \hat{k}_m = 1$ on a neighbourhood of $\bar{E}$. Then, if ϕ is as in (2.1), we have

$$\phi = \sum_{m=1}^{n} (k_m * \phi).$$

On the other hand, $k_m * \phi \in C_0(G) \cap L^{p'}(G)$ and supp $(k_m * \phi)^\wedge \subset E \cap U_m$. Since $E \cap U_m$ is p-thin, $k_m * \phi = 0$ for each m, and so $\phi = 0$.

(iv) This statement is clear from the inclusion

$$C_0(G) \cap L^{q'}(G) \subset C_0(G) \cap L^{p'}(G),$$

valid whenever $q' < p'$, *i.e.*, whenever $q > p$.

(3.2) If G is noncompact and $p > 1$, each discrete subset of X is p-thin.

PROOF. According to (1.3), any finite subset of G is p-thin. The rest follows from (3.1 ii).

(3.3) (i) If $p \geqq 2$, any locally null E subset of X is p-thin.

(ii) If $1 \leqq p \leqq 2$, any p-thin subset E of X is locally null.

PROOF. (i) If $p \geqq 2$, then $p' \leqq 2$, so that if ϕ is as in (2.1), then the pseudomeasure $\hat{\phi}$ is defined by a function in $L^p(X)$. Since this same pseudomeasure has its support contained in E, the defining function must vanish l.a.e. outside E and therefore l.a.e. on X. But then $\phi = 0$, showing that E is p-thin.

(ii) Here we have $p' \geqq 2$. If E were not locally null, E would contain a compact set K having positive measure. If ϕ is the inverse Fourier transform of the characteristic function of K, then $\phi \in C_0(G) \cap L^2(G) \subset C_0(G) \cap L^{p'}(G)$ and satisfies $\phi(0) = \int_K d\xi > 0$. Thus E is not p-thin.

(3.4) If G is noncompact and $p > 1$, and if E is a compact subset of X which supports no true pseudomeasures, then E is p-thin. (It may be shown without difficulty that these hypotheses are satisfied whenever E is both a Helson set and an S-set).

PROOF. If ϕ is as in (2.1), then $\hat{\phi}$ is a bounded measure with support contained in E. Moreover, as may be shown without much difficulty, the fact that E supports no true pseudomeasures entails that E is a Helson set. The conclusion $\phi = 0$ now follows from [9], Theorem 5.6.10, p. 119.

For G the discrete additive group of integers, examples of such sets E are given in [10].

(3.5) Suppose that G is noncompact and $p > 1$. Let E be subset of X contained in an S-set S with the following property; if, for any complex number z of unit modulus, we define

$$A_z = \{x \in G : \xi(x) = z \text{ for all } \xi \in S\}$$

(so that A_1 is the annihilator in G of S), then the closed subgroup G_0 of G generated by

$$\bigcup \{A_z : |z| = 1\}$$

is noncompact. Then E is p-thin.

PROOF. Let ϕ be as in (2.1), and let $a \in A_z$ for some z. Since S is an S-set and $\operatorname{supp} \hat{\phi} \subset S$, ϕ is the strict (and hence the pointwise) limit of trigonometric polynomials formed from elements of S. It follows at once that $\phi(x+a) = z \cdot \phi(x)$ identically in $x \in G$. Consequently $|\phi(x+a)| = |\phi(x)|$ for all $x \in G$ and all $a \in G_0$. Since $\phi \in C_0(G)$ and G_0 is noncompact, it follows that $\phi = 0$.

(3.6) It is convenient to list here a few categories of S-sets; for the following results, see [9], pp. 161, 169–172.

(i) If E is closed and ∂E contains no nonvoid perfect sets, then E is an S-set. Any C-set is an S-set.

(ii) Any finite set is a C-set. If ∂E is a C-set, so too is E.

(iii) A finite union of C-sets is a C-set.

(iv) Any closed subgroup of X is a C-set.

(v) Any translate of an S-set [resp. a C-set] is an S-set [resp. a C-set].

(vi) If E is a closed semigroup in X such that 0 belongs to the closure of the interior of E, then E is an S-set.

(vii) If $G = R^n = X$, any closed rectilinear simplex, any vector subspace, any closed halfspace, any closed polyhedral set, and any star-shaped body is a C-set.

(3.7) (i) Suppose that $E_1 \subset E$ are subsets of X, that E_1 is a p-thin C-set, and that $E \cap U'$ is p-thin for every open set $U \supset E_1$. Then E is p-thin.

(ii) If E_1 and E_2 are p-thin subsets of X, E_1 being a C-set, then $E = E_1 \cup E_2$ is p-thin.

(iii) If $E_1, \cdots, E_n$ are p-thin C-sets, then so too is $E = E_1 \cup \cdots \cup E_n$.

Proof. Statement (ii) follows directly from (i) since, if the hypotheses of (ii) are fulfilled, $E \cap U' \subset E_2$ for every $U \supset E_1$. Statement (iii) follows from (ii) by induction, in view of (3.6.iii). Thus all depends on proving (i), which we shall effect in two steps.

(a) Denote by $A(X)$ the set of all functions u on X of the form

$$u(\xi) = \int_G v(x)\overline{\xi(x)}dx \equiv \hat{v}(\xi),$$

v ranging over $L^1(G)$. $A(X)$ is made into a Banach space under the norm $\|u\|_A = \|v\|_1$. The dual of $A(X)$ is precisely the space $P(X)$ of pseudomeasures on X. We aim to show that, under the hypotheses of (i), every pseudomeasure s on X is the weak limit in $P(X)$ of pseudomeasures of the form

$$\mu + \hat{g} \cdot s, \tag{3.7.1}$$

where μ is a bounded Radon measure on X satisfying $\operatorname{supp} \mu \subset E_1$ and $g \in L^1(G)$ is such that $\operatorname{supp} \hat{g} \subset U'$ for some neighbourhood U of E_1, U possibly depending on g. In order to do this, we have to show that any $u \in A(X)$, orthogonal to all pseudomeasures of the form (3.7.1), is orthogonal to s.

Now, if u is orthogonal to all pseudomeasures of the form (3.7.1), it appears first (by taking $g = 0$) that u vanishes on E_1. Since E_1 is a C-set, this entails ([9], p. 169) that u is the limit in $A(X)$ of functions $\hat{g} \cdot u$, where the variable function g is as specified in (3.7.1). But then

$$s(u) = \lim s(\hat{g} \cdot u) = \lim \hat{g} \cdot s(u) = \lim 0 = 0,$$

since by hypothesis u is orthogonal to all pseudomeasures of the form (3.7.1). This establishes the possibility of the said approximation.

(b) Suppose now that ϕ is as in (2.1), and that the hypotheses of (i) are satisfied. By (a) we can write

$$\hat{\phi} = \lim (\mu_i + \hat{g}_i \cdot \hat{\phi}) \tag{3.7.2}$$

weakly in $P(X)$, the μ_i being bounded Radon measures on X satisfying $\operatorname{supp} \mu_i \subset E_i$, and $g_i \in L^1(G)$ being such that $\operatorname{supp} \hat{g}_i \subset U_i'$ for some neighbourhood U_i of E_1. Now $\hat{g}_i \cdot \hat{\phi}$ is the transform of $g_i * \phi$, which (like ϕ) belongs to $C_0(G) \cap L^{p'}(G)$. Since also $\operatorname{supp} \hat{g}_i \cdot \hat{\phi} \subset E \cap U_i'$, and since $E \cap U_i'$ is p-thin by hypothesis, it follows that $g_i * \phi = 0$. Thus (3.7.2) reads simply

$$\hat{\phi} = \lim \mu_i$$

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weakly in $P(X)$, which shows that supp $\hat{\phi} \subset E_1$. So, since E_1 is p-thin, $\phi = 0$. This completes the proof of (i).

(3.8) If G is noncompact and $p > 1$, and if E is a subset of X whose derived set E_1 is a p-thin C-set, then E is p-thin.

PROOF. If U is any neighbourhood of E_1, $E \cap U'$ is discrete. It suffices now to apply (3.7.i).

(3.9) Let (E_i) be a locally finite, disjoint family of closed p-thin sets. Then $E = \bigcup E_i$ is p-thin.

PROOF. In view of (3.1), it suffices to show that the union, E, of two disjoint compact p-thin sets, E_1 and E_2, is p-thin.

Now E_1 and E_2 possess disjoint neighbourhoods U_1 and U_2. Choose f_k $(k = 1, 2)$ from $L^1(G)$ such that $\hat{f}_k = 1$ on a neighbourhood of E_k and supp $\hat{f}_k \subset U_k$. If ϕ is as in (2.1) we shall have $\phi = f_1 * \phi + f_2 * \phi$, since $\hat{f}_1 + \hat{f}_2 = 1$ on a neighbourhood of $E \supset$ supp $\hat{\phi}$. Then $f_k * \phi \in C_0(G) \cap L^{p'}(G)$ and supp $(f_k * \phi)\hat{} \subset U_k \cap E = E_k$. Since E_k is p-thin, so $f_k * \phi = 0$ and therefore $\phi = 0$. Thus E is p-thin.

(3.10) Both (3.7) and (3.9) prompt the question: Is it always true that the union of two p-thin sets is again p-thin? An affirmative answer, for the special case in which $G = R^n$ and the sets concerned are closed, is given in (6.2).

Some more specialised examples of p-thin sets are given in § 5.

(3.11) Herz ([15], Theorem 4) gives two conditions, each of which is sufficient to ensure (when $p \leq 2$) that a closed set $E \subset R^n$ is of type $U^{p'}$ (and therefore certainly p-thin), namely:

(i) the (Haar) measure of the set of points at distance below h from K is $o[h^{n(2/p-1)}]$ as $h \to 0$, K being any compact subset of E;

(ii) the Hausdorff dimension of E is inferior to $2n(p-1)/p$.

4. An application

We discuss an application of (2.2) which in a sense extends the result of Segal, and presents at the same time a multidimensional generalisation of Agnew's results.

(4.1) Suppose that G_k $(k = 1, 2, \cdots, n)$ are noncompact groups, the character group of G_k being denoted by X_k. Put $G = G_1 \times \cdots \times G_n$, whose character group is (isomorphic to) $X = X_1 \times \cdots \times X_n$.

Let $f_k \in L^1(G_k) \cap L^p(G_k)$ be such that

$$Z_k = \hat{f}^{k-1}(0) \text{ is discrete.} \tag{4.1.1}$$

Let $f \in L^1(G) \cap L^p(G)$ be defined by

$$f(x_1, \cdots, x_n) = f_1(x_1) \cdots f_n(x_n). \tag{4.1.2}$$

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(4.2) THEOREM. *Assume that the hypotheses of* (4.1) *are fulfilled, and that* $1 < p < \infty$. *Then* $T^p[f] = L^p(G)$.

PROOF. From (4.1.2) it follows that

$$\hat{f}(\xi_1, \cdots, \xi_n) = \hat{f}_1(\xi_1) \cdots \hat{f}_n(\xi_n),$$

whence it appears that

(4.2.1) $$Z \equiv \hat{f}^{-1}(0) = (Z_1 \times X_2 \times \cdots \times X_n) \cdots (X_1 \times \cdots \times X_{n-1} \times Z_n).$$

We must show that Z is a p-thin subset of X.

For each k, let K_k be a compact subset of X_k. If $K = K_1 \times \cdots \times K_n$, then (4.2.1) shows that

(4.2.2) $$K \cap Z \subset Q_1 \cup \cdots \cup Q_n,$$

where

$$Q_1 = (K_1 \cap Z_1) \times X_2 \times \cdots \times X_n$$

and the remaining Q_k are similarly defined. Since Z_1 is discrete, $K_1 \cap Z_1$ is finite. Therefore Q_1 is a finite union of sets of the form

$$\{\alpha_1\} \times X_2 \times \cdots \times X_n,$$

where $\alpha_1 \in X_1$. Each of these latter sets is a translate of $\{0\} \times X_2 \times \cdots \times X_n = P_1$, say. The set P_1 is a C-set, by (3.6.iv), and its annihilator in G is $G_1 \times \{0\} \times \cdots \times \{0\}$, which is noncompact. By (3.5), therefore, P_1 is p-thin. That Q_1 is a p-thin C-set now follows from (3.6.v) and (3.7.iii). Similarly, each Q_k is a p-thin C-set. Applying (3.7.iii) again, (4.2.2) shows that $K \cap Z$ is p-thin. Since the compact sets K here considered form a base for the compact subsets of X, it follows from (3.1) that Z is p-thin.

The proof is completed by appeal to (2.2).

(4.3) REMARK. If, in (4.2), one or more of the G_k are compact, the theorem will remain valid provided the corresponding sets Z_k are void.

(4.4) COROLLARY. *Suppose that* $1 < p < \infty$ *and that* f *is a non-null function on* R^n *of the form*

$$f(x) = f_1(x_1) \cdots f_n(x_n),$$

where for each $k = 1, 2, \cdots, n$, $f_k \in L^p(R)$ *and vanishes a.e. outside some compact subset of* R. *Then* $T^p[f] = L^p(R^n)$.

PROOF. In this case, $\hat{f}_k^{-1}(0)$ is a discrete subset of R (identified with its own character group in the usual way), since $\hat{f}_k$ is an entire function which does not vanish identically.

(4.5) Notwithstanding Corollary (4.4), when $n > 1$ it is *not* the case that any non-null $f \in C_c(R^n)$ has the property that $T^p[f] = L^p(R^n)$ for every p satisfying $1 < p < \infty$. (The corresponding assertion with $n = 1$ is excluded by Corollary (4.4), of course.) A simple counterexample follows.

In general we identify the character group of R^n with R^n itself, the character function being

$$\xi(x) = \exp\,(-2\pi i \sum_{k=1}^{n} \xi_i x_i).$$

In R^n, let S denote the unit hypersphere, s the surface measure on S, and $|S| = \int_S ds$. The function ϕ on R^n defined by

$$\phi(x) = |S|^{-1} \int_S \exp 2\pi i (x_1\xi_1 + \cdots + x_n\xi_n) ds(\xi)$$

is expressible as a nonzero constant multiple of $r^{-\frac{1}{2}n+\frac{1}{2}} J_{\frac{1}{2}n-\frac{1}{2}}(2\pi r)$, where $r = (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}$ and J_ν denotes the ν-th order Bessel function. Well-known properties of J_ν show that $\phi \in C_0(R^n) \cap L^{p'}(R^n)$ provided $n > 1$ and $p' > 2n/(n-1)$, *i.e.*, provided $n > 1$ and $p < 2n/(n+1)$. Since $\hat{\phi}$ is a measure supported by S, it follows that S is not p-thin for any p satisfying $1 \leqq p < 2n/n+1$.

Now suppose that $f \in C_c(R^n)$ is of the form

$$f = u + (4\pi^2)^{-1} \Delta u,$$

where $u \not\equiv 0$ belongs to $C_c(R^n)$, and where Δ denotes the Laplacian. Then, if $\rho = (\xi_1^2 + \cdots + \xi_n^2)^{\frac{1}{2}}$,

$$\hat{f} = (1-\rho^2)\hat{u}$$

vanishes on S, and $f \not\equiv 0$. It follows, since $\hat{\phi}$ is a measure supported by S, that $f * \phi = 0$. Since $\phi \not\equiv 0$, this last equation shows that $T^p[f] \neq L^p(R^n)$ for $1 \leqq p < 2n/(n+1)$.

For this same f, Theorem (2.2) and (3.3.i) combine to show that $T^p[f] = L^p(R^n)$ whenever $p \geqq 2$. The truth of the relation $T^p[f] = L^p(R^n)$ remains undecided for values of p satisfying $2n/(n+1) \leqq p < 2$. See also (5.6) and (6.3).

Herz ([15], final paragraph) remarks that "consideration of a few Bessel functions" will show that if $p < 2n/(n+1)$ there exist non-null functions $f \in L^p(R^n)$ with a compact support such that $T^p[f] \neq L^p(R^n)$. In (5.4) *infra* we see in detail how Bessel functions appear in a related connection.

5. Algebraic varieties and p-thin sets

(5.1) Throughout this section we take $G = R^n$, identified with its own character group as in (4.5). In this case, as is easily verified, the pseudomeasure $\hat{\phi}$ can be identified with the distributional Fourier transform of ϕ.

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We shall consider, in respect of p-thinness, sets $V \subset R^n$ which are (not necessarily irreducible) algebraic varieties in R^n. Thus V will be defined by a system of equations

$$P_i(\xi) \equiv P_i(\xi_1, \cdots, \xi_n) = 0 \qquad (i \in I), \tag{5.1.1}$$

each P_i being a polynomial over the complex field in n indeterminates. (The polynomial ring being Noetherian, it is always possible to define V by a system (5.1.1) in which the index set I is finite, but we do not need to assume this here.)

For each polynomial P we denote by $P(D)$ the linear partial differential operator

$$P[(2\pi i)^{-1}\partial/\partial x_1, \cdots, (2\pi i)^{-1}\partial/\partial x_n].$$

It is a convenient piece of notation to denote by $F^p(R^n)$ the set of functions ϕ on R^n which, together with each of their partial derivatives, belong to $C_0(R^n) \cap L^{p'}(R^n)$, and which are such that supp $\hat{\phi}$ is compact. Each $\phi \in F^p(R^n)$ is necessarily analytic on R^n (and even extendible into an entire-analytic function of n complex variables).

The following simple result will be needed.

Lemma. *For $n = 1, 2, \cdots$ and $1 \leqq p \leqq \infty$, define*

$$m_{n,p} = \begin{cases} 0 & \text{if } \ p \geqq 2, \\ 2[(2-p)n/4p]+2 & \text{if } \ n \geqq 2 \text{ and } 1 \leqq p < 2, \\ 1 & \text{if } \ n = 1 \text{ and } 1 \leqq p < 2, \end{cases} \tag{5.1.2}$$

where the square brackets on this occasion denotes the integral part. If $\phi \in F^p(R^n)$, then $\hat{\phi}$ is a distribution of order at most $m_{n,p}$.

Proof. If $p \geqq 2$, $\hat{\phi}$ is a function. If $1 \leqq p < 2$ and $n \geqq 2$, Hölder's inequality shows that, if $m = m_{n,p}$ and $\phi \in F^p(R^n)$, then $\phi = (1+r^2)^{\frac{1}{2}m} f$, where $f \in L^2(R^n)$ and $r = (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}$. Consequently,

$$\hat{\phi} = (1-\Delta/4\pi^2)^{\frac{1}{2}m}\hat{f},$$

where Δ denotes the n-dimensional Laplacian and $\hat{f} \in L^2(R^n)$. Similar estimates apply when $n = 1$.

We can now relate the property of p-thinness of an algebraic variety V to a uniqueness property of the corresponding system of partial differential equations.

(5.2) Theorem. (i) *Suppose that V, defined by* (5.1.1), *is p-thin, and that (m_i) is any family of nonnegative integers. Then the system*

$$\phi \in C_0(R^n) \cap L^{p'}(R^n), \quad P_i(D)^{m_i}\phi = 0 \quad (i \in I) \tag{5.2.1}$$

has only the trivial solution $\phi \equiv 0$.

(ii) *Let $m = m_{n,p}$ be defined by* (5.1.2) *and suppose that the system of partial differential equations*

$$(5.2.2) \qquad \phi \in F^p(R^n), \quad P_i(D)^{m+1}\phi = 0 \quad (i \in I)$$

has only the trivial solution $\phi \equiv 0$.

Then V, defined by (5.1.1), *is p-thin.*

PROOF. (i) If ϕ satisfies (5.2.1), then, on taking Fourier transforms, it is seen that

$$P_i(\xi)^{m_i}\hat{\phi} = 0 \quad (i \in I).$$

This system of equations entails that $\operatorname{supp}\hat{\phi} \subset P_i^{-1}(0)$ for $i \in I$, and hence that $\operatorname{supp}\hat{\phi} \subset V$. Since V is p-thin, it follows that $\phi \equiv 0$.

(ii) Suppose that $\phi \in C_0(R^n) \cap L^{p'}(R^n)$ and $\operatorname{supp}\hat{\phi} \subset V$: it must be shown that $\phi \equiv 0$. By considering in place of ϕ functions of the type $\phi * h$, where h is the inverse Fourier transform of an element of $C_c^\infty(R^n)$, we may assume from the outset that $\phi \in F^p(R^n)$. Now, since $\operatorname{supp}\hat{\phi}$ is a subset of V, the lemma in (5.1) combines with a known theorem ([14], pp. 97–98, Théorème XXXIII) to show that $P_i^{m+1} \cdot \hat{\phi} = 0$, *i.e.*, that $P_i(D)^{m+1}\phi = 0$, for each $i \in I$. Thus ϕ is a solution of the system (5.2.2) and is therefore trivial.

(5.3) COROLLARY. *Let V be an algebraic variety in R^n defined by an equation*

$$(5.3.1) \qquad P(\xi) = 0,$$

P being a polynomial. In order that V be p-thin, it is

(i) *necessary that the implication*

$$(5.3.2) \qquad \phi \in C_0(R^n) \cap L^{p'}(R^n), \quad P(D)\phi = 0 \Rightarrow \phi = 0$$

be valid, and

(ii) *sufficient that the implication*

$$(5.3.2) \qquad \phi \in F^p(R^n), \quad P(D)\phi = 0 \Rightarrow \phi = 0$$

be valid.

PROOF. (i) The necessity of the validity of (5.3.2) follows at once from (5.2.i).

(ii) The sufficiency of the validity of (5.3.2) follows from (5.2.ii), if one remarks that $F^p(R^n)$ is stable under partial differentiations and hence under the operator $P(D)$.

(5.4) As an application of Corollary (5.3), we will show that if $n > 1$ an $(n-1)$-dimensional hypersphere S in R^n is p-thin if and only if $p \geqq 2n/(n+1)$.

Indeed, the arguments in (4.5) show that S is not p-thin if $p < 2n/(n+1)$. Turning to the converse, we start from the associated differential equation, which in this case takes the form

$$\Delta\phi + c^2\phi = 0, \tag{5.4.1}$$

where $c > 0$. Suppose that ϕ is a solution of (5.4.1) which belongs to $C_0(R^n) \cap L^{p'}(R^n)$. We aim to show that, if $p \geqq 2n/(n+1)$, then $\phi = 0$. By replacing ϕ by any translate thereof, it will suffice to show that $\phi(0) = 0$. To this end, we denote by S_r the hypersphere in R^n with centre 0 and radius r, and write s_r for the surface measure on S_r. Then ([12], p. 289) one has

$$\Gamma(\tfrac{1}{2}n)(cr)^{-\frac{1}{2}n+1}J_{\frac{1}{2}n-1}(cr)\phi(0) = |S_r|^{-1}\int ds_r, \tag{5.4.2}$$

where $|S_r| = \int ds_r = \text{const.}\, r^{n-1}$. By Hölder's inequality,

$$\begin{aligned}\int |\phi| ds &\leqq \left(\int |\phi|^{p'} ds_r\right)^{1/p'} \left(\int ds_r\right)^{1/p} \\ &= |S_r|^{1/p} \cdot M(r)^{1/p'},\end{aligned} \tag{5.4.3}$$

where

$$M(r) = \int |\phi|^{p'} ds_r.$$

Now

$$||\phi||_{p'}^{p'} = \int_0^\infty dr \int |\phi|^{p'} ds_r = \int_0^\infty M(r)dr < \infty.$$

Also, as $r \to \infty$,

$$J_{\frac{1}{2}n-1}(cr) \sim (2/\pi cr)^{\frac{1}{2}} \cos\left[cr - (\tfrac{1}{2}n-1)\pi/2 - \pi/4\right]. \tag{5.4.4}$$

The cosine factor here is bounded away from zero on each of an infinite sequence of disjoint congruent intervals. Since $\int_0^\infty M(r)dr < \infty$, it follows that a sequence $r_i \to \infty$ may be chosen from these intervals such that $r_i M(r_i) \to 0$. From (5.4.2), (5.4.3), and (5.4.4) it then appears that

$$|\phi(0)| \leqq \text{const.}\, r^{\frac{1}{2}n-1} \cdot |S_r|^{-1} \cdot |S_r|^{1/p} \cdot M(r)^{1/p'}/J_{\frac{1}{2}n-1}(cr).$$

Taking $r = r_i$, this yields

$$|\phi(0)| \leqq \text{const.}\, r_i^{(n-1)(\frac{1}{2}-1/p')-1/p} \cdot [r_i M(r_i)]^{1/p'}.$$

Letting $i \to \infty$, this gives $\phi(0) = 0$, provided that

$$(n-1)(\tfrac{1}{2}-1/p')-1/p' \leqq 0,$$

i.e., provided that $p \geqq 2n/(n+1)$.

(5.5) The result in (5.4) for hyperspheres naturally extends to images of hyperspheres under vector space isomorphisms of R^n. We note also that results given by Littman [13] show that sufficiently smooth $(n-1)$-dimensional surfaces in R^n which have everywhere positive Gaussian curvature fail to be p-thin for $1 \leqq p < 2n/(n+1)$.

(5.6) Example. Consider a function f on R^n which is a function of $r = (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}$ only, say $f(x) = F(r)$, where

$$\int_0^\infty |F(r)| r^{n-1} dr < \infty, \quad \int_0^\infty |F(r)|^p r^{n-1} dr < \infty.$$

The Fourier transform of f is then of the form $\hat{f}(\xi) = G(\rho)$, where $\rho = (\xi_1^2 + \cdots + \xi_n^2)^{\frac{1}{2}}$ and

$$G(\rho) = 2\pi\rho^{-\frac{1}{2}n+1} \int_0^\infty r^{\frac{1}{2}n} J_{\frac{1}{2}n-1}(2\pi\rho r) F(r) dr.$$

If we assume that $G(\rho)$ is zero for a set of $\rho \geqq 0$ which is discrete, and that $p \geqq 2n/(n+1)$, then (2.2), (3.9), and (5.4) combine to show that $T^p[f] = L^p(R^n)$. The stated condition on the zeros of G is certainly satisfied if f is nonnull and has a compact support.

6. Further results for $G = R^n$

The principal result of this section, Theorem (6.2), gives for $G = R^n$ another partial converse of Theorem (2.2) in which f itself (rather than merely the set $\hat{f}^{-1}(0)$) is further restricted. At the same time, it provides an almost complete answer (for $G = R^n$) to the question raised in (3.10).

The proof of (6.2) uses a lemma, which is valid for general G.

(6.1) Lemma. *Suppose that* $f_i \in L^1(G) \cap L^p(G)$ *and* $T^p[f_i] = L^p(G)$ *for* $i = 1, 2$. *Then* $f = f_1 * f_2 \in L^1(G) \cap L^p(G)$ *and* $T^p[f] = L^p(G)$.

Proof. It is simple to verify that, if $h \in L^p(G)$ and $T^p[h] = L^p(G)$, then to each $\varepsilon > 0$ and each $g \in L^p(G)$ there corresponds a function k which is continuous and has a compact support such that $\|h * k - g\|_p < \varepsilon$. (Notice that each translate of h is the limit in $L^p(G)$ of functions $h * k$ with k as specified.) This being so, we first choose k_1 so that $\|f_1 * k_1 - g\|_p < \frac{1}{2}\varepsilon$. Then, since $k_1 \in L^p(G)$, we may choose k_2 so that $\|f_2 * k_2 - k_1\|_p < \frac{1}{2}\varepsilon \cdot \|f_1\|_1^{-1}$. Combining these inequalities, it is seen that $\|f * k_2 - g\|_p < \varepsilon$. Finally, $f * k_2$ is the limit in $L^p(G)$ of linear combinations of translates of f. Thus $g \in T^p[f]$ and the lemma follows.

Let now $m = m_{n,p}$ be defined as in (5.1.2), and let us denote by $K^p(R^n)$ the set of $f \in L^1(R^n) \cap L^p(R^n)$ such that $\hat{f} \in C^m(R^n)$. Obviously, $K^p(R^n)$ is a convolution algebra containing the Schwartz space $\mathscr{S}(R^n)$. It is simple to show that any closed subset E of R^n is the zero-set $\hat{f}^{-1}(0)$ for some $f \in \mathscr{S}(R^n)$. In fact, let U_r be the set of points of R^n at distance

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less than r^{-1} from E. By Urysohn's lemma, there exists a continuous function $F_r : R^n \to [0, 1]$ which vanishes on E and takes the value 1 on U'_r. By regularisation, we may assume that $F_r \in C^\infty(R^n)$ and that each mixed partial derivative $D^p F_r$ is bounded. Let

$$c_r = r^{-2} [\mathrm{Sup}_{|p| \leqq r} ||D^p F_r||_\infty]^{-1},$$

so that $||D^p(c_r F_r)||_\infty \leqq r^{-2}$ for $r \geqq |p|$. It follows then that

$$F = \sum_{r=1}^{\infty} c_r F_r \in C^\infty(R^n)$$

and that $D^p F$ is bounded for each p. The function $\xi \to e^{-|\xi|^2} F(\xi)$ belongs to $\mathscr{S}(R^n)$ and so can be expressed as $\hat{f}$ for some $f \in \mathscr{S}(R^n)$. Evidently, $F^{-1}(0) = E$, which confirms the claim made above.

We can now state and prove the main result of this section.

(6.2) Theorem. *(a) Let E be a closed subset of R^n. In order that E be p-thin it is*

(i) *sufficient that*

$$f \in \mathscr{S}(R^n),\ \hat{f}^{-1}(0) \subset E \Rightarrow T^p[f] = L^p(R^n), \tag{6.2.1}$$

and

(ii) *necessary that*

$$f \in K^p(R^n),\ \hat{f}^{-1}(0) \subset E \Rightarrow T^p[f] = L^p(R^n). \tag{6.2.2}$$

(b) The validity of either implication, (6.2.1) *or* (6.2.2), *is thus necessary and sufficient that E be p-thin.*

(c) The union of two p-thin closed subsets of R^n is p-thin.

Proof. (a) Suppose that the implication (6.2.1) is valid. As we have shown, E can be written as $\hat{f}^{-1}(0)$ for some $f \in \mathscr{S}(R^n)$. If E were not p-thin, we could choose $\phi \in C_0(R^n) \cap L^{p'}(R^n)$ such that supp $\hat{\phi} \subset E$ and $\phi \neq 0$. Let $g = f * \cdots * f$, with $m+1$ factors. Then $g \in \mathscr{S}(R^n)$ and $\hat{g}$ and its partial derivatives of orders at most m all vanish on E. So ([14], pp. 97—98, Théorème XXXIII again) $\hat{g} \cdot \hat{\phi} = 0$, *i.e.*, $g * \phi = 0$. Since $\phi \neq 0$, this shows that $T^p[g] \neq L^p(R^n)$ and so, by (6.1), that $T^p[f] \neq L^p(R^n)$. This establishes the sufficiency of (6.2.1).

The necessity of (6.2.2) is a special case of (2.2).

(b) This follows at once from (a) and the obvious implication (6.2.2) $\Rightarrow$ (6.2.1).

(c) Suppose E_i $(i = 1, 2)$ are closed p-thin subsets of R^n and that $E = E_1 \cup E_2$. Write $E_i = \hat{f}_i^{-1}(0)$ with $f_i \in \mathscr{S}(R^n)$. Put $f = f_1 * f_2$, which belongs to $\mathscr{S}(R^n)$. By (b), $T^p[f_i] = L^p(R^n)$ for $i = 1, 2$ and so, by Lemma (6.1), $T^p[f] = L^p(R^n)$. Since $\hat{f}^{-1}(0) = E$, another application of (b) entails that E is p-thin.

REMARKS. Part (a) of Theorem (6.2) as akin to Theorem 3 of Herz [15], inasmuch as both constitute partial converses of our Theorem (2.2) and his Theorem 1, respectively. On the other hand, Herz's Theorem 3 corresponds to a considerably stronger form of the implication (6.2.2), differentiability properties of f being replaced by Lipschitz conditions on $\hat{f}$. As is implicit in [15] and [16], it is possible to show that if s is any pseudo-measure on X, and if $f \in L^1(G)$ is such that $\hat{f}$ satisfies a Lipschitz condition of order $\alpha > 0$ and vanishes on supp s, then $\hat{f}^m \cdot s = 0$ holds for all sufficiently large integers m. We here interpret the Lipschitz condition on $\hat{f}$ as meaning that, for some base (U_i) of relatively compact neighbourhoods of 0 in X,

$$|\hat{f}(\xi')-\hat{f}(\xi)| \leqq \text{const. } [\text{meas } U_i]^\alpha$$

for $\xi'-\xi \in U_i$.

More precisely and more generally: if $f \in L^1(G) \cap L^p(G)$ $(1 \leqq p \leqq \infty)$ and $\phi \in L^{p'}_c(G)$, then $f * \phi = 0$ provided $\hat{f} = 0$ on supp $\hat{\phi}$ and

$$\hat{f}(\xi) = O\left([\text{meas } U_i]^{1/p-\frac{1}{2}}\right)$$

for $\xi \in K+U_i$, K being any compact subset of supp $\hat{\phi}$. (The Lipschitz condition becomes void, and can be dropped entirely, if $p > 2$.) The case $p = 1$ is an extension of a result of Pollard [17] for the case $G = R$. Compare Herz [16], Lemma 4.4.

(6.3) We collect here a few remarks bearing upon a problem first raised by Herz ([15], final paragraph).

Consider again the case in which $f \in L^p(R^n)$ is nonnull and vanishes a.e. outside some compact subset of R^n. The following facts have already emerged:

(a) If $n = 1$ and $p > 1$, or if n is arbitrary and $p \geqq 2$, then $T^p[f] = L^p(R)$ (see Theorem (2.2));

(b) If $p > 1$ and n is arbitrary, and if f has the special form described in (4.4), then $T^p[f] = L^p(R^n)$; and likewise if $n > 1$ and $p \geqq 2n/(n+1)$ (see (5.6));

(c) if $n > 1$ and $1 \leqq p < 2n/(n+1)$, then $T^p[f]$ is in general a proper subspace of $L^p(R^n)$ (see (4.5)).

Concentrating on the case $n > 1$, it is natural to ask whether there exist values of p (necessarily greater than or equal to $2n/(n+1)$) such that $T^p[f] = L^p(R^n)$ for all f of the type considered. Now Theorem (6.2) shows that it is equivalent to ask whether there exist values of p $(\geqq 2n/(n+1))$ such that $\hat{f}^{-1}(0)$ is p-thin for each f of the type considered. Furthermore, by the Paley-Wiener-Schwartz theorem, it is the same thing to ask whether there exist such values of p such that $F^{-1}(0)$ is p-thin for all functions $F \neq 0$ on R^n which are extendible into entire functions of exponential type of n complex variables. In view of (3.1.iii) and the Weierstrass Vorbereitungsatz, this is reduced to determining whether a locus, defined in a neighbourhood of the origin, by an equation of the form

$$\xi_n^s + A_{s-1}(\xi_1, \cdots, \xi_{n-1})\xi_n^{s-1} + \cdots + A_0(\xi_1, \cdots, \xi_{n-1}) = 0,$$

where s is a positive integer and the A_j are analytic and vanish at the origin, is p-thin for $p \geqq 2n/(n+1)$.

Whilst (3.3.i) implies an affirmative answer for $p \geqq 2$, the problem is open for $2n/(n+1) \leqq p < 2$.

7. A comparison

In this section we suppose that $G = R$, the additive group of real numbers. In Pollard's version of (2.2), the condition on $Z = \hat{f}^{-1}(0)$, which corresponds to our demand that Z be p-thin, reads as follows: the relations

$$g \in L^{p'}(R), \ \lim_{\sigma \downarrow 0} \int e^{-\sigma|x|-2\pi i \xi x} g(x)dx = 0 \qquad (\xi \in Z') \tag{7.1}$$

shall imply that

$$g = 0 \text{ a.e.} \tag{7.2}$$

If we write $g_\sigma(x) = e^{-\sigma|x|}g(x)$, then $g_\sigma \in L^1(R)$ for $\sigma > 0$ and (7.1) reads

$$\lim_{\sigma \downarrow 0} \hat{g}_\sigma(\xi) = 0 \qquad (\xi \in Z') \tag{7.3}$$

We aim to show that this condition is in fact equivalent to the requirement that Z be p-thin.

Suppose first that (7.3) holds, and let $\hat{g}$ denote the Fourier-Schwartz transform of g. From (7.3) it follows (compare the discussion in [11]) that $\lim_{\sigma \downarrow 0} \hat{g}_\sigma = 0$ locally uniformly on Z'. Since $g_\sigma \to g$ in the Schwartz space $\mathscr{S}'(R)$, it follows at once that $\hat{g} = 0$ on Z', *i.e.*, that $\operatorname{supp} \hat{g} \subset Z$.

Conversely, if $g \in L^{p'}(R)$ is such that $\operatorname{supp} \hat{g} \subset Z$, and if

$$K_\sigma(\xi) = 2\sigma/(\sigma^2 + 4\pi^2\xi^2)$$

denotes the Fourier transform of $e^{-\sigma|x|}$, then

$$\hat{g}_\sigma = K_\sigma * \hat{g}. \tag{7.4}$$

This formula holds indeed in the pointwise sense, as one may verify most easily by observing that $\hat{g}$ is distributionally of the form $u + dv/d\xi$, where $u, v \in L^2(R)$. This special form of $\hat{g}$ combines with (7.4) to show also that $\hat{g}_\sigma \to 0$ pointwise on Z', which is (7.3).

Thus Pollard's condition signifies exactly that if $g \in L^{p'}(R)$ and $\operatorname{supp} \hat{g} \subset Z$, then $g = 0$ a.e. That this is equivalent to saying that Z is

p-thin in the sense of (2.1), follows by considering functions ϕ of the form $k * g$ with (say) k continuous and having a compact support. Each such function ϕ will belong to $C_0(R) \cap L^{p'}(R)$, and $\text{supp}\,\hat{\phi} \subset \text{supp}\,\hat{g} \subset Z$.

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https://doi.org/10.1017/S1446788700026781 Published online by Cambridge University Press