J. Austral. Math. Soc. **20** (Series B) (1978), 370–374

ASYMPTOTIC BEHAVIOUR OF A CLASS OF DISCONTINUOUS DIFFERENCE EQUATIONS

M. E. FISHER and P. E. KLOEDEN

(Received 6 March 1978)

(Revised 24 April 1978)

Abstract

Sufficient conditions for an equilibrium point to be an attractor or a global attractor are derived for a class of first-order difference equations which need not be continuous at the equilibrium point. These conditions involve Lyapunov-like functions which need not be continuous and are applied to the logistic equation with a piecewise continuous control.

1. Introduction

Lyapunov functions provide an effective tool for the analysis of stability and asymptotic properties of equilibria points of difference equations [1, 3]. To date their use has been restricted to difference equations described by continuous functions. Discontinuities in the describing functions, however, arise quite naturally in a control context due to switches in admissible controls. If an equilibrium point is a point of discontinuity it can never be stable, but it may possess desirable asymptotic properties such as being an attractor or a global attractor.

In this note sufficient conditions involving Lyapunov-like functions are derived for an equilibrium point to be an attractor or a global attractor. These are valid for a class of first-order autonomous difference equations on a normed linear space which admit discontinuities of a certain kind at an equilibrium point. Significantly the Lyapunov-like functions need not be continuous, even at the equilibrium point. Moreover, the sufficient conditions include the usual ones for asymptotic and global asymptotic stability as a special case. A simple example of the logistic equation with a piecewise continuous control is given to illustrate the use of the above conditions.

2. Lyapunov-like sufficient conditions for attractors

Consider a first-order autonomous difference equation

$$x_{n+1} = f(x_n) \tag{1}$$

described by a function $f\colon X \to X$, where X is a normed linear space, for which there exists a subset $X_1 \subset X$, a point $x^* \in X_1$, and a constant $\zeta > 0$ such that

$$f(x^*) = x^*, \tag{2}$$

$$f|_{X_1} \text{ is continuous at } x^*, \tag{3}$$

$$f(X_1 \cup S_\zeta(x^*)) \subset X_1. \tag{4}$$

Here $S_\zeta(x^*)$ is the open ball in X of radius ζ and centre x^* defined by

$$S_\zeta(x^*) = \{x \in X;\ \|x - x^*\| < \zeta\}.$$

Such difference equations include many of the common discrete-time density dependent population models with non-overlapping generations, with or without spatial distribution [4]. They also include controlled versions of such models where switches in the controls introduce discontinuities into the describing functions [1, 2]. An example is given in Section 3 where such a switch introduces a discontinuity at an equilibrium point $x^* \in \partial X_1$.

An equilibrium point x^* of a difference equation (1) is said [3] to be

(i) *stable* if for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that $x_n \in S_\varepsilon(x^*)$ for $n = 1, 2, 3, \ldots$ whenever $x_0 \in S_\delta(x^*)$;

(ii) an *attractor* if there exists a $\delta_0 > 0$ such that $x_n \to x^*$ as $n \to \infty$ whenever $x_0 \in S_{\delta_0}(x^*)$;

(iii) *asymptotically stable* if it is stable and an attractor;

(iv) a *global attractor* if $x_n \to x^*$ for all $x_0 \in X$;

(v) *globally asymptotically stable* if it is stable and a global attractor.

Clearly an equilibrium point x^* can never be stable when the function f has a discontinuity at x^*, though it may be an attractor or a global attractor. The following theorems give sufficient conditions involving Lyapunov-like functions for an equilibrium point x^* to be an attractor or a global attractor. They include the usual Lyapunov sufficient conditions for asymptotic or global asymptotic stability [3] when x^* is an interior point of the subset X_1, in which case f is continuous at x^*.

THEOREM 1 (*Sufficient conditions for an attractor*). *Suppose that* f *satisfies conditions* (2), (3) *and* (4) *and that there exists a function*

$$V\colon X_1 \to \mathbf{R}^+$$

with

$$a(\|x - x^*\|) \leqslant V(x) \leqslant b(\|x - x^*\|) \tag{5}$$

and

$$V(f(x)) - V(x) \leqslant -c(\|x - x^*\|) \tag{6}$$

for all $x \in X_1$, *where* a, b *and* c *are continuous, strictly increasing real valued functions of a real variable with* $a(0) = b(0) = c(0) = 0$.

Then x^* *is an attractor for difference equation* (1). *If in addition* $x^* \in \operatorname{int} X_1$, *then* x^* *is asymptotically stable for difference equation* (1).

Proof. Conditions (5) and (6) imply the global asymptotic stability of x^* for the restriction of difference equation (1) to the subset X_1. The proof of this is exactly the same as for the usual Lyapunov sufficient conditions for global asymptotic stability [3], which, in this case, really only requires the functions V and f to be continuous in the relative topology on X_1 at the equilibrium point x^*. Hence $x_n \to x^*$ as $n \to \infty$ for all $x_0 \in X_1$.

Let $\delta_0 = \zeta$. Then by condition (4) $x_1 = f(x_0) \in X_1$ for all $x_0 \in S_{\delta_0}(x^*)$. Hence $x_n \to x^*$ as $n \to \infty$ for all $x_0 \in S_{\delta_0}(x^*)$, that is x^* is an attractor for difference equation (1).

If $x^* \in \operatorname{int} X_1$, then the stability of x^* for the restriction of (1) to X_1 implies the stability of x^* for (1) on X since all small neighbourhoods of x^* in the relative topology on X_1 are also neighbourhoods of x^* in X. Hence x^* is both stable and an attractor, that is, x^* is asymptotically stable for (1).

Theorem 2 (*Sufficient conditions for a global attractor*). *Suppose that* f *satisfies conditions* (2), (3) *and* (4) *and that there exists a function*

$$V: X \to \mathbf{R}^+$$

with

$$a(\|x - x^*\|) \leqslant V(x) \leqslant \chi_{X_1}(x) \,.\, b_1(\|x - x^*\|) + \chi_{X - X_1}(x) \,.\, b_2(\|x - x^*\|) \tag{7}$$

and

$$V(f(x)) - V(x) \leqslant -c(\|x - x^*\|) \tag{8}$$

for all $x \in X$, *where* a, b_1, b_2 *and* c *are continuous, strictly increasing real valued functions of a real variable with* $a(0) = b_1(0) = c(0) = 0 \leqslant b_2(0)$ *and* χ_{X_1} *and* $\chi_{X - X_1}$ *are the characteristic functions of subsets* X_1 *and* $X - X_1$, *respectively.*

Then x^* *is a global attractor for difference equation* (1). *If in addition* $x^* \in \operatorname{int} X_1$, *then* x^* *is globally asymptotically stable for difference equation* (1).

Proof. From Theorem 1 it follows that $x_n \to x^*$ as $n \to \infty$ for all $x_0 \in X_1 \cup S_\zeta(x^*)$. Also, for each $x_0 \in X - X_1 \cup S_\zeta(x^*)$, there exists an integer $n_0 = n_0(x_0)$ such that $x_{n_0} \in S_\zeta(x^*)$ and hence such that $x_n \to x^*$ as $n \to \infty$. For, if this were not the case, then $x_n \in X - X_1$ and $\zeta \leqslant \|x_n - x^*\|$ for $n = 0, 1, 2, \ldots$, and so by conditions (7) and (8)

$$\begin{aligned} 0 < a(\zeta) &\leqslant a(\|x_n - x^*\|) \\ &\leqslant V(x_n) \\ &\leqslant V(x_0) - nc(\zeta) \\ &\leqslant b_2(\|x_0 - x^*\|) - nc(\zeta) \\ &< 0 \end{aligned}$$

for all $n > b_2(\|x_0 - x^*\|)/c(\zeta)$, which is absurd.

Hence x^* is a global attractor for difference equation (1).

If, in addition $x^* \in \operatorname{int} X_1$, then x^* is also stable and hence globally asymptotically stable for (1) on the whole space X for the same reasons as in Theorem 1.

The Lyapunov-like function in Theorem 1 is defined only on the subset X_1 of which the equilibrium point may be a boundary point. In Theorem 2 it is defined on the whole space X, but need not be continuous at the equilibrium point since $b_2(0) \geqslant 0$. As less is demanded of these Lyapunov-like functions, they should in specific problems be easier to find than are the usual Lyapunov functions.

3. An example

To illustrate the application of the above theorems consider the controlled logistic equation

$$x_{n+1} = ax_n(1 - x_n) + u(x_n) \tag{9}$$

on $X = [\alpha, \beta] \subset \mathbf{R}$ with control $u\colon X \to \mathbf{R}$ defined by

$$u(x) = \begin{cases} -c & \text{if } 1 - a^{-1} < x < \gamma, \\ 0 & \text{elsewhere}, \end{cases} \tag{10}$$

where $1 < a < 2$, $0 < \alpha < 1 - a^{-1} < \beta < 1$ with $\alpha \leqslant a\beta(1-\beta)$, $1 - a^{-1} < \gamma < \beta$ with $a\gamma(1-\gamma) \leqslant 1 - a^{-1}$ and $0 < c < a\gamma(1-\gamma) - \alpha$.

This gives a difference equation (1) described by a function $f\colon X \to X$ defined by

$$f(x) = \begin{cases} ax(1-x) - c & \text{for } 1 - a^{-1} < x < \gamma, \\ ax(1-x) & \text{elsewhere} \end{cases}$$

with $x^* = 1 - a^{-1}$, $X_1 = [\alpha, x^*]$ and $\zeta = \gamma - x^*$.

Also the discontinuous Lyapunov-like function $V\colon X \to \mathbf{R}^+$ defined by

$$V(x) = \begin{cases} x^* - x & \text{for } x \in X_1, \\ a + \alpha a & \text{for } x \in X - X_1 \end{cases}$$

satisfies conditions (7) and (8) of Theorem 2 with $a(r) = b_1(r) = r$, $c(r) = \alpha ar$ and $b_2(r) = a + \alpha a + r$. Hence the equilibrium point x^* is a global attractor.

In a biological harvesting context, such as whaling, a control (10) corresponds to harvesting at a constant rate c when the population is larger, but not too much larger, than the equilibrium population and no harvesting otherwise. Theorem 2 says that the population always tends to the equilibrium population with such a harvesting policy.

References

[1] M. E. Fisher, B. S. Goh and T. L. Vincent, "Some stability conditions for discrete time single species models", to appear in *Bull. Math. Biology*.

[2] B. S. Goh, G. Leitmann and T. L. Vincent, "Optimal control of a predator–prey system", *Mathematical Biosciences* 19 (1974), 263–286.

[3] J. P. La Salle, *The stability of dynamical systems*, Regional Conference Series in Applied Mathematics, S.I.A.M., Philadelphia (1976).

[4] E. C. Pielou, *An introduction to mathematical ecology* (Wiley–Interscience, New York, 1969).

Department of Mathematics
University of W.A.
Nedlands
Western Australia, 6009
and
School of Mathematical and Physical Sciences
Murdoch University
Murdoch
Western Australia, 6153

https://doi.org/10.1017/S0334270000001727 Published online by Cambridge University Press