# MINKOWSKI'S THEOREM WITH CURVATURE LIMITATIONS (I)

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1. The well-known theorem of Minkowski, [1], [2], states that:

(M) a plane convex region, symmetric about the origin O, includes a lattice point other than O if its area is greater than 4.

By a lattice point we shall understand a point in the plane, both of whose coordinates are rational integers. In connection with (M) a critical region is defined to be a convex symmetric region of area 4, which includes no lattice point other than O. One such region is the open square $S = \{(x,y) \mid |x| < 1, |y| < 1\}$, an infinite set of critical regions is formed by the parallelograms bounded by the lines $y = x+1$, $y = x-1$, $y-1 = k(x-1)$, $y+1 = k(x+1)$, $0 \leq -k < \infty$. Finally, there is a critical hexagon H, bounded by the following six lines: $y-1 = -(x-1)$, $y+1 = -(x+1)$, $y-1 = mx$, $y+1 = mx$, $y = (1/m)(x-1)$, $y = (1/m)(x+1)$, where $m = \tan \pi/12$. All the vertex angles of H are equal to $2\pi/3$.

Since all critical regions are convex polygons it is natural to expect that the constant 4 in (M) ought to be diminished if the boundary of the convex symmetric region is required to curve more strongly than a straight line. It is the aim of this article to find the corresponding critical regions and the constants which replace 4 in (M) under suitable curvature restrictions on the region.

It should be emphasized that the topology of various regions does not enter into the picture. It will be clear in each case whether the point-set used is to be taken as open or closed.

The following notation is used. Any region, unless otherwise specified, is assumed to be convex and symmetric about the fixed origin O. A p.l.p. (proper lattice point) is a lattice

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point other than O. Capital letters, other than A, B and O, will denote regions. A(X) and B(X) will stand for the area and the boundary of X, respectively. Small letters will denote points or numbers, and small Greek letters will denote curves or angles. For any set Z, $Z'$ will be its image in O.

2. Let R be a region and let $p \in B(R)$. A circle $\gamma$ is said to be a supporting circle of R at p if it is tangent there to a supporting line of R, and if B(R), and therefore R as well, lies inside or on $\gamma$. R is said to be an r-region if at each point of B(R) there exists a supporting circle of radius $\leq r$, while for at least one point there is no supporting circle of smaller radius.

It follows from this definition that if B(R) consists of a finite number of arcs along each of which there exists a continuously varying radius of curvature (continuity is one-sided at the corners), then R is an r-region, $0 < r < \infty$. However, it is easy to give an example of an r-region R, such that curvature fails to exist at an uncountable subset of B(R). We give the following example based on the properties of the Cantor set. Let $\alpha$ be the unit circle about O. On the upper semi-circle let $\mathfrak{A}$ be the set which becomes the ordinary ternary Cantor set based on $[0, \pi]$ when $\alpha$ is unrolled onto a line. Let r be real, $1 < r < \infty$. Replace the arc of $\alpha$, corresponding to the first excluded middle third, by an arc of a circle of radius r located so that the result is a convex curve. Repeat the same on each middle third of the succeeding stages. Finally, carry out the same procedure for the lower semi-circle. It is clear that one obtains in this way a convex curve B(K) which bounds an r-region K, while along B(K) curvature fails to exist at all points of $\mathfrak{A}$ and $\mathfrak{A}'$.

An r-region is said to be maximal if it contains no p.l.p. and if no other r-region containing no p.l.p. has bigger area. We assume henceforth that $r > 1$, since otherwise the problem is trivial. Strictly speaking, it would be necessary to obtain a proof that there exists a maximal r-region. Such a proof can be constructed, as in the isoperimetric problem, by the use of Blaschke's selection principle, [3], but it is of no interest here.

3. LEMMA 1. Let K be a maximal r-region. Then B(K) consists of a finite number of circular arcs of radius r.

Suppose that this is false. Then there are on B(K) two points $p_1$ and $p_2$, bounding the (shorter) arc $\alpha$, such that the following properties hold. First, $\alpha$ is neither an arc of radius r nor a union of a finite number of such arcs. Then, letting $\tau_1$ and $\tau_2$ be any two supporting lines at $p_1$ and $p_2$, the non-convex region N bounded by $\alpha$, $\tau_1$ and $\tau_2$, contains no p.l.p. Now replace $\tau_1$ and $\tau_2$ by the corresponding supporting circles of radius r at $p_1$ and $p_2$. Let M be the non-convex region bounded by $\alpha$ and by these two circles. Finally let $L = K \cup M \cup M'$. Then L is an r-region containing no p.l.p. and $A(L) > A(K)$, which is a contradiction. We note further that since $r > 1$, there are at least two arcs.

In view of this lemma we shall use the self-explanatory terms 'corner' and 'interior point' for the points on the boundary B(K) of a maximal r-region K.

LEMMA 2. If K is a maximal r-region, then each arc of B(K) contains an interior p.l.p.

Suppose that this is false. Then there is an arc $\beta$ in B(K), with corners $q_1$ and $q_2$, and without interior p.l.p's. Continue the arcs, or arc, adjoining $\beta$ at $q_1$ and $q_2$, beyond $q_1$ and $q_2$, and repeat the same with $\beta'$. Now let $\beta$ be translated by a parallel motion through a small distance a away from the origin. Let the same operation be repeated on $\beta'$. We obtain thus a new r-region L which includes no p.l.p's if a is small enough. Since $A(L) > A(K)$, the desired contradiction is proved and the lemma follows.

LEMMA 3. Let K be a maximal r-region and let $\alpha$ be one of the arcs of B(K). If $\alpha$ contains exactly one interior p.l.p. p then p is the centre of $\alpha$.

Let $q_1$ and $q_2$ be the corners of $\alpha$ and suppose that p is nearer to $q_1$ than to $q_2$. Let $\alpha$ and the neighbouring arcs (or arc) be continued beyond $q_1$ and $q_2$. Rotate the extended $\alpha$ through a small angle $\theta$ about p so as to leave $q_1$ outside the new region. Repeat the same operation on $\alpha'$. Suppose that in the process the corners $q_1$ and $q_2$ are moved to $q_3$ and $q_4$ respectively. By the hypothesis $\widehat{pq_1} < \widehat{pq_2}$ and $\theta$ is small; therefore $\widehat{pq_3} \cong \widehat{pq_1}$ and $\widehat{pq_4} \cong \widehat{pq_2}$. Further for $\theta$ sufficiently small, $\widehat{pq_1} < \widehat{pq_4}$ and $\widehat{pq_3} < \widehat{pq_2}$. Consider now the two circularly bounded non-convex regions $pq_1q_3$ and $pq_2q_4$. Let the first one be reflected in the line through p which bisects the angle at p between the tangents to $\widehat{pq_1}$ and $\widehat{pq_4}$. Under this reflexion the image of $q_3$ is on the interior of $\widehat{pq_2}$ and that of $q_1$ on the interior of $\widehat{pq_4}$. It follows that the first region has smaller area than the second one. That is, more area is gained than lost in the process, and the maximality of K is contradicted, which completes the proof.

LEMMA 4. Let K be a maximal r-region and let $\alpha$ be one of the arcs of B(K). Let $q_1$, $q_2$ and c be the corners and the centre of $\alpha$ respectively. If $\alpha$ contains exactly two interior p.l.p.'s $p_1$ and $p_2$, then $p_1$ lies between $q_1$ and c (or possibly at c), and $p_2$ between c and $q_2$. Also, $\widehat{p_1p_2} < \widehat{cq_1}$.

Suppose that $p_1$ and $p_2$ are both between $q_1$ and c. Rotate $\alpha$ as in the previous lemma, about $p_2$ - the maximality of K is contradicted as before. Suppose now that $\widehat{p_1p_2} > \widehat{cq_1}$. Rotate $\alpha$ as before, but this time twice: once about $p_1$ so as to leave $q_1$ outside, and once about $p_2$ so as to leave $q_2$ outside. Now $\alpha$ is replaced by the union of the two rotated arcs and the maximality of K is again contradicted since the new region has larger area.

LEMMA 5. Let K be a maximal r-region and let $\alpha$, $\beta$, $\gamma$, $\delta$ be four arcs lying in this order on B(K). Let $\lambda$, $\mu$, $\nu$ be the angles between $\alpha$ and $\beta$, $\beta$ and $\gamma$, $\gamma$ and $\delta$. Suppose that $\beta$ and $\gamma$, corners included, contain one p.l.p. each. Then $\lambda + \mu + \nu \geqslant 2\pi$.

Let $q_1$ be the corner between $\alpha$ and $\beta$, $q_2$ that between $\beta$ and $\gamma$, and $q_3$ that between $\gamma$ and $\delta$. Let $p_1$ and $p_2$ be the p.l.p.'s on $\beta$ and $\gamma$ respectively. Rotate $\beta$ through a small angle about $p_1$ and $\gamma$ through a small angle about $p_2$, extending the two arcs as necessary. The sense of rotations is such as to leave $q_1$ and $q_3$ outside the new region. Suppose that the process moves $q_1$, $q_2$ and $q_3$ to $q_4$, $q_5$ and $q_6$ respectively. Let $\rho$ be the angle at $q_2$ between the straight segment $q_2q_5$ and the tangent at $q_2$ to $\beta$; similarly let $\sigma$ be the angle between $q_2q_5$ and the tangent at $q_2$ to $\gamma$. Therefore

(1) $$\rho + \sigma + \mu = 2\pi.$$

Let the same operation be repeated at $\alpha'$, $\beta'$, $\gamma'$, $\delta'$.

The total change of area of K during the process is

(2) $$\Delta A = 2[A(p_1q_2q_5) + A(p_2q_2q_5) - A(p_1q_1q_4) - A(p_2q_3q_6)],$$

where each summand is the area of the circularly bounded non-convex region with indicated corners. Assume now that $\lambda, \nu > \pi/2$; it will be seen that the proof holds a fortiori if either $\lambda$ or $\nu$ is $\leqslant \pi/2$.

By Lemma 3, $p_1$ is the centre of $\beta$ and $p_2$ that of $\gamma$. Performing some elementary calculations we get

$$(3)\ \Delta A = 2a_1^2[\tan(\rho - \pi/2) - \tan(\lambda - \pi/2)] + 2a_2^2[\tan(\sigma - \pi/2) - \tan(\nu - \pi/2)] + e$$

where $a_1$ and $a_2$ are the distances from $q_1$ and $q_3$ to $\widehat{q_4 q_5}$ and $\widehat{q_5 q_6}$ respectively, and the error term e is much smaller than the other two summands if $q_5$ is sufficiently close to $q_2$, i.e. if the angles of rotation of $\beta$ and $\gamma$ are small enough. Now

$$(4) \qquad \Delta A = 2a_1^2 \sin(\rho - \lambda)\sec(\rho - \pi/2)\sec(\lambda - \pi/2) + 2a_2^2 \sin(\sigma - \nu)\sec(\sigma - \pi/2)\sec(\nu - \pi/2) + e;$$

by the assumptions on various angles the four secants in (4) are positive. Comparing (1) and (4) we see that $q_5$ can be selected so as to make the two sines in (4) positive, provided that $\lambda + \mu + \nu < 2\pi$. If the latter holds then the total change $\Delta A$ of area in the perturbation process is positive, and the maximality of K is contradicted.

LEMMA 6. Let K be a maximal r-region and let n be the number of p.l.p.'s on B(K). Then n=4 or n=6.

In the first place, n is a positive integer and, by the symmetry of K, n is even. Let the n p.l.p.'s be, in order, $p_i = (x_i, y_i)$, $i = 1, 2, \ldots, n$. It is clear than the $p_i$'s are vertices of a convex polygon P with O in its interior. Also, by the determinant formula for the area of a triangle

$$A(P) = \tfrac{1}{2} \sum_{i=1}^{n} \left| x_i y_{i+1} - y_i x_{i+1} \right| ,$$

where $x_{n+1} = x_1$ and $y_{n+1} = y_1$. Each summand is a non-negative integer and since O is inside P, each summand is in fact positive. Therefore

$$A(P) \geqslant n/2 .$$

Suppose now that $n \geqslant 8$. Then $A(K) > A(P) \geqslant 4$, and so by (M) K includes a p.l.p. which is a contradiction. It remains to show that $n > 2$. Suppose, to the contrary, that n=2. Then, by Lemmas 2 and 3, B(K) consists of two arcs $\alpha$ and $\alpha'$, and each has a p.l.p. at its centre. Let the latter be p and p'. Then it is easy to show that K contains also the circle about O passing



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through p and p′. Since this circle contains two other p.l.p.'s (or more) it follows that either B(K) has at least four p.l.p.'s, or K includes a p.l.p. in its interior. Since either case leads to a contradiction the lemma is proved.

It has been proved by now that if K is a maximal r-region then B(K) consists of m arcs of radius r and contains n p.l.p.'s. Further, m and n are even positive integers, $m \leq n$, and n=4 or n=6. Therefore there are five possibilities for K: $T_{2,4}, T_{2,6}, T_{4,4}, T_{4,6}, T_{6,6}$. Here $T_{m,n}$ is the type of region bounded by m arcs and containing exactly n p.l.p.'s on its boundary.

It turns out that the maximal r-region K is completely determined as follows. K can be of the type $T_{2,4}$ only if $r < (5/2)^{1/2}$; the four p.l.p.'s are then $(0, \pm 1)$ and $(\pm 1, 0)$, and the two arcs have their centres on the line $x = -y$. K can be of the type $T_{2,6}$ only if $r = (5/2)^{1/2}$; the region is then as above, with the addition of p.l.p.'s (1, 1) and (-1, -1). K can be of the type $T_{4,4}$ only if the four arcs have their centres on the coordinate axes and the four p.l.p.'s are $(0, \pm 1)$ and $(\pm 1, 0)$. K can be of the type $T_{4,6}$ only if $r > (5/2)^{1/2}$; the four arcs have then their centres on the lines $x = \pm y$ and the six p.l.p.'s are $(0, \pm 1)$, $(\pm 1, 0)$ (1, 1) and (-1, -1). Finally K can be of the type $T_{6,6}$ only if $r > (5/2)^{1/2}$ and the six p.l.p.'s are as above; the centres of the arcs are here as follows: two on the line x=y and the other four in the second and fourth quadrants, situated symmetrically with respect to the line $x = -y$. It is to be understood that the above determination holds only up to reflexions in the coordinate axes and their two diagonals.

Now, the complete determination is reduced to the computation of different areas. For a fixed r the critical constant replacing 4 in (M) is, of course, the largest area of the admissible maximal r-region.

This programme will be carried out in the second part of this article. In the remainder of this part we consider extensions to the n-dimensional space $E_n$.

4. Before proceeding to the general case we remark that in $E_3$ the lemmas 1 and 6 possess the following extensions:

Let K be a maximal r-region in $E_3$. Then B(K) consists of a finite number of regions, each region being a part of the



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sphere of radius r, bounded by a finite number of other such spheres. Each such region contains at least one interior p.l.p.

Let K be a maximal r-region in $E_3$ and let n be the number of p.l.p.'s on B(K): Then $n \leq 24$.

In $E_n$ the constant 4 of (M) is replaced by $2^n$. All regions will be assumed, as before, to be convex and symmetric about the origin O. The definition of an r-region is the same as before, supporting hyperplanes and n-spheres replacing supporting lines and circles. The definition of a maximal r-region is clear. As before, we assume that $r > 1$.

THEOREM 1. Let K be a maximal r-region in $E_n$. Then K is the intersection of f(n) solid n-spheres of radius r, where f(n) is an even integer and $f(n) < 2^n n(n-2)! + n - 2/(n-1)$. Moreover, each n-spherical part of B(K) contains an interior p.l.p.

The proof follows exactly the proofs of Lemmas 1, 2, and 6 and the only important difference is the determination of the bound on the number f(n). We enumerate the p.l.p.'s on B(K): $p_1, p_2, \ldots, p_{g(n)}$. It is clear that $f(n) \leq g(n)$. The points $p_1, \ldots, p_{g(n)}$ are vertices of a convex polytope P whose volume V(P) satisfies $V(P) < V(K)$. By the use of the generalized Euler-Poincaré formula it can be shown that B(P) can be partitioned into $h(n) = (n-1)g(n) - n^2 + n + 2$ n-simplices, such that all the vertices of each simplex are among the points $p_i$, and no two simplices overlap except on a lower-dimensional simplex. If the n vertices of each simplex are joined by straight segments to O, we obtain a decomposition of the polytope P into h(n) (n+1)-simplices whose vertices have all their coordinates integral. Then, following the proof of Lemma 6, and making use of the determinant formula for the volume of a simplex, we show that

$$1/n!\ h(n) < 2^n,$$

since otherwise K includes a p.l.p. in its interior. The above inequality leads at once to the estimate of f(n) in the theorem.

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