Bull. Austral. Math. Soc. 30D45, 30D50
Vol. 68 (2003) [405–421]

n-TH DERIVATIVE CHARACTERISATIONS, MEAN GROWTH OF DERIVATIVES AND $F(p,q,s)$

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Various n-th derivative characterisations involving different kinds of oscillations of $F(p,q,s)$ functions are established, and the mean growth of derivatives of $F(p,q,s)$ functions is considered. Moreover, inclusion relations between certain analytic function spaces are discussed.

1. Introduction

Integral characterisations involving n-th derivatives for functions in the weighted Bergman spaces were essentially proved by Flett [6] in 1972. In 1989 Stroethoff [11] proved similar results for Bloch functions and little Bloch functions. Further, in 1998 Aulaskari, Nowak and Zhao [2] obtained n-th derivative characterisations for functions in the spaces Q_p on $Q_{p,0}$. Recently these results were generalised for the general family of function spaces $F(p,q,s)$ and $F_0(p,q,s)$ by Rättyä [10]: for an analytic function f on the unit disc of the complex plane the conditions $f \in F(p,q,s)$ (respectively $F_0(p,q,s)$) and $f^{(n)} \in F(p,np+q,s)$ (respectively, $F_0(p,np+q,s)$) are equivalent, provided that the spaces are not trivial and $1 < p < \infty$.

The present paper is organised as follows. We begin by briefly recalling the basic notations of function spaces, as well as some elementary inequalities needed later on in Section 2. In Section 3, we first note that the condition $1 < p < \infty$ in the result above can be removed, see Theorem 3.2, and then apply it in order to obtain various n-th derivative characterisations, involving different kind of oscillations, for $F(p,q,s)$ functions, see Theorems 3.4, 3.5 and 3.7, generalising resent results by Yoneda [14]. In Section 4 we briefly discuss inclusion relations between the closely related families of function spaces $F(p,q,s)$ and $F^n(p,q,s)$, and, as an consequence, answer partially a question posed in [14]. Section 5 is devoted to the study of the mean growth of derivatives of $F(p,q,s)$ functions, and the results therein generalise and/or improve results in [1] and [3].

Received 15th April, 2003
The author wishes to thank Dr. J. Heittokangas and Dr. R. Korhonen for several valuable comments and suggestions.

2. Notations and prerequisites

Let D denote the unit disc of the complex plane. For $a \in D$, let the Möbius transformation $\varphi_a : D \to D$ be defined by

$$\varphi_a(z) = \frac{a - z}{1 - \overline{a}z},$$

and let the Green's function of D with logarithmic singularity at a be

$$g(z,a) = \log\left|\frac{1-\overline{a}z}{z-a}\right| = \log\frac{1}{|\varphi_a(z)|}.$$

Simple calculations show that $\varphi_a^{-1} = \varphi_a$ and

$$1 - |\varphi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\overline{a}z|^2} = (1-|z|^2)|\varphi_a'(z)|.$$

For $a \in D$ and $0 < r < 1$, the pseudo-hyperbolic disc is defined by

$$D(a,r) = \left\{z \in D : |\varphi_a(z)| < r\right\}.$$

The pseudo-hyperbolic disc $D(a,r)$ is an Euclidean disc centred at $(1-r^2)a/(1-|a|^2r^2)$ with radius $(1-|a|^2)r/(1-|a|^2r^2)$, see [**7**, p. 3]. Let $d\sigma_z$ denote the Euclidean area element, and let $|K|$ denote the Euclidean area of K. Obviously

$$|D(a,r)| = \pi r^2 \frac{(1-|a|^2)^2}{(1-|a|^2r^2)^2}. \tag{2.1}$$

The constants, which might vary from one occurence to another, are denoted by C.

For $0 < p < \infty$, $-2 < q < \infty$ and $0 \leqslant s < \infty$, $F(p,q,s)$ and $F_0(p,q,s)$ are defined as the sets of all analytic functions f on D for which

$$\|f\|_{p,q,s}^p = \sup_{a\in D} \iint_D |f'(z)|^p (1-|z|^2)^q g^s(z,a)\, d\sigma_z < \infty$$

and

$$\lim_{|a|\to 1} \iint_D |f'(z)|^p (1-|z|^2)^q g^s(z,a)\, d\sigma_z = 0, \quad 0 < s < \infty,$$

respectively. For convenience, we also define $F_0(p,q,0) = F(p,q,0)$. For $1 \leqslant p < \infty$, $F(p,q,s)$ is a Banach space with respect to the norm $\|f\|_{p,q,s} + |f(0)|$, and so is $F_0(p,q,s)$ as a closed subspace of $F(p,q,s)$, see [**13**, Theorem 2.10 and Proposition 2.15]. The spaces $F(p,q,s)$, introduced by R. Zhao in [**13**], are known as the *general family of function spaces*. The importance of these spaces stems from the fact that for appropriate parameter values p, q and s they coincide with several classical function spaces. For instance, it is well known that $F(2,1,0)$ is just the Hardy space H^2, the proof being based on the Parseval's relation, $F(p,p,0)$ is the Bergman space L^p_a, $F(2,0,s) = Q_s$ and $F(2,0,1) = BMOA$, the space of analytic functions with bounded mean oscillation. For other relations, the reader is invited to see [**13**].

A positive measure μ on D is said to be a *bounded s-Carleson measure*, if

$$\sup_I \frac{\mu(S(I))}{|I|^s} < \infty, \quad 0 < s < \infty,$$

where $|I|$ denotes the arc length of a subarc I of the boundary ∂D, $S(I) = \{z \in D : z/|z| \in I, 1 - |z| \leqslant |I|/2\pi\}$ is the Carleson box based on I and the supremum is taken over all subarcs I of ∂D. Moreover, if

$$\lim_{|I|\to 0} \frac{\mu(S(I))}{|I|^s} = 0, \quad 0 < s < \infty,$$

then μ is a *compact s-Carleson measure.* For $s = 1$ we have the standard definitions of bounded and compact Carleson measures.

Easy calculations show that for $w \in D(z,r)$ and $a \in D$,

$$\frac{1-r}{1+r}(1-|z|^2) \leqslant 1 - |w|^2 \leqslant \frac{1+r}{1-r}(1-|z|^2), \tag{2.2}$$

$$\frac{1}{1+r}(1-|z|^2) \leqslant |1 - \overline{w}z| \leqslant \frac{1}{1-r}(1-|z|^2) \tag{2.3}$$

and

$$\left(\frac{1-r}{1+r}\right)^3 \left(1 - |\varphi_a(z)|^2\right) \leqslant \left(1 - |\varphi_a(w)|^2\right) \leqslant \left(\frac{1+r}{1-r}\right)^3 \left(1 - |\varphi_a(z)|^2\right). \tag{2.4}$$

By (2.1),

$$\frac{\sqrt{\pi}r}{1+r^2}(1-|z|^2) \leqslant \left|D(z,r)\right|^{1/2} \leqslant \frac{\sqrt{\pi}r}{1-r^2}(1-|z|^2). \tag{2.5}$$

If $R = (2r)/(1+r^2) \in (r,1)$ and $w \in D(z,r)$, then $D(w,r) \subset D(z,R)$ and

$$\frac{1}{|D(w,r)|^{1/2}} \leqslant \frac{(1+r^2)^2}{(1-r)^3(1+r)} \frac{1}{\left|D(z,R)\right|^{1/2}}. \tag{2.6}$$

Moreover, it is evident that

$$\chi_{D(z,r)}(w) = \chi_{D(w,r)}(z). \tag{2.7}$$

Formulas (2.2)–(2.7) will be used frequently in sequel.

For an analytic function f on D and $0 < r < 1$,

$$\sup_{w \in D(z,r)} \left|f(z) - f(w)\right|$$

https://doi.org/10.1017/S0004972700037813 Published online by Cambridge University Press

and

$$\frac{1}{|D(z,r)|}\iint_{D(z,r)}\left|\widehat{f}_r(z)-f(w)\right|d\sigma_z\,,$$

where

$$\widehat{f}_r(z)=\frac{1}{|D(z,r)|}\iint_{D(z,r)}f(w)\,d\sigma_z\,,$$

are called the *oscillation* and the *mean oscillation* of f at z in the Bergman metric, respectively. To characterise $F(p,q,s)$ functions, we use various different oscillations and mean oscillations similar to these two, see Theorems 3.4, 3.5 and 3.7.

3. On n-th derivative characterisations for $F(p,q,s)$

We begin with an auxiliary lemma which is essentially due to Flett.

Lemma 3.1. *Let f be an analytic function on D and let $0 < p < \infty$ and $-1 < q < \infty$. Then there exist two positive constants C_1 and C_2, depending only on p and q, such that*

$$C_1\Big(\left|f(0)\right|^p+\iint_D\left|f'(z)\right|^p\big(1-|z|^2\big)^{p+q}\,d\sigma_z\Big)\leqslant\iint_D\left|f(z)\right|^p\big(1-|z|^2\big)^q\,d\sigma_z$$
$$\leqslant C_2\Big(\left|f(0)\right|^p+\iint_D\left|f'(z)\right|^p\big(1-|z|^2\big)^{p+q}\,d\sigma_z\Big).$$

Proof: By [**6**, Theorems 6 and 7],

$$C_1\iint_D\left|f'(z)\right|^p|z|^{p-1}\big(1-|z|\big)^{p+q}\,d\sigma_z\leqslant\int_0^1\int_0^{2\pi}\left|f(re^{i\theta})\right|^p(1-r)^q\,d\theta\,dr$$
$$\leqslant C_2\bigg(\left|f(0)\right|^p+\iint_D\left|f'(z)\right|^p|z|^{p-1}\big(1-|z|^2\big)^{p+q}\,d\sigma_z\bigg),$$

from which [**8**, Lemma 4.6] and [**10**, Lemma 5.3.1] with elementary estimates yield the assertion. ∎

For the proof of Lemma 3.1 the reader is also invited to see [**2**, Lemma 1] (the case $p=2$), [**15**, p. 58] (the case $1\leqslant p<\infty$ and $q=0$) and [**10**, Lemma 4.1.4] (the case $1<p<\infty$).

In function space language, Lemma 3.1 states that for an analytic function on D, the conditions $f\in A_q^p$ and $f^{(n)}\in A^p_{np+q}$, $n\in\mathbb{N}$, where $A_q^p=F(p,p+q,0)$ stands for the weighted Bergman space, are equivalent.

Applying Lemma 3.1 to the function $(f\circ\varphi_w)(rz)$, $0<r<1$, we obtain

$$\iint_{D(w,r)}\left|f(z)\right|^p d\sigma_z\leqslant C\left(\iint_{D(w,r)}\left|f'(z)\right|^p\big(1-|z|^2\big)^p\,d\sigma_z+\left|f(w)\right|^p\right), \tag{3.1}$$

where the positive constant C depends only on p and r.

The following two theorems fulfill the gap $0 < p \leqslant 1$ in [**10**, Theorems 4.2.1-4.2.4, 4.3.3 and 4.3.4].

Theorem 3.2. *Let f be an analytic function on D and let $0 < p < \infty$, $-2 < q < \infty$ and $0 \leqslant s < \infty$. Let $n \in \mathbb{N}$ and $q + s > -1$ or $n = 0$ and $q + s - p > -1$. Then the following conditions are equivalent:*

(1) $f \in F(p,q,s)$,

(2) $\sup_{a\in D} \iint_D |f^{(n)}(z)|^p (1-|z|^2)^{np-p+q} \left(1 - |\varphi_a(z)|^2\right)^s d\sigma_z < \infty$,

(3) $\sup_{a\in D} \iint_D |f^{(n)}(z)|^p (1-|z|^2)^{np-p+q} g^s(z,a)\, d\sigma_z < \infty$,

(4) $d\mu(z) = |f^{(n)}(z)|^p (1-|z|^2)^{np-p+q+s}\, d\sigma_z$ *is a bounded s-Carleson measure.*

Theorem 3.3. *Let f be an analytic function on D and let $0 < p < \infty$, $-2 < q < \infty$ and $0 < s < \infty$. Let $n \in \mathbb{N}$ and $q + s > -1$ or $n = 0$ and $q + s - p > -1$. Then the following conditions are equivalent:*

(1) $f \in F_0(p,q,s)$,

(2) $\lim_{|a|\to 1} \iint_D |f^{(n)}(z)|^p (1-|z|^2)^{np-p+q} \left(1 - |\varphi_a(z)|^2\right)^s d\sigma_z = 0$,

(3) $\lim_{|a|\to 1} \iint_D |f^{(n)}(z)|^p (1-|z|^2)^{np-p+q} g^s(z,a)\, d\sigma_z = 0$,

(4) $d\mu(z) = |f^{(n)}(z)|^p (1-|z|^2)^{np-p+q+s}\, d\sigma_z$ *is a compact s-Carleson measure.*

Theorems 3.2 and 3.3 can be proved as [**10**, Theorems 4.2.1-4.2.4, 4.3.3 and 4.3.4] by applying Lemma 3.1. We note that as an immediate consequence of the above two theorems, [**10**, Theorems 4.4.2 and 4.4.3] hold also for $0 < p \leqslant 1$.

We now use Lemma 3.1 and Theorem 3.2 to prove n-th derivative characterisations, involving the oscillation of $f^{(n)}$ at z in the Bergman metric and similar oscillations, for $F(p,q,s)$ functions.

Theorem 3.4. *Let f be an analytic function on D and let $0 < p < \infty$, $-2 < q < \infty$, $0 \leqslant s < \infty$ and $0 < r < 1$. Let $n \in \mathbb{N}$ and $q + s > -1$ or $n = 0$ and $q + s - p > -1$. Let $\alpha + \beta = np - p + q$ and $s_1 + s_2 = s$. Then the following conditions are equivalent:*

(1) $f \in F(p,q,s)$,

(2)
$$\sup_{a\in D} \iint_D \left(\sup_{w\in D(z,r)} |f^{(n)}(w) - f^{(n)}(z)| (1-|z|^2)^{\alpha/p} (1-|w|^2)^{\beta/p} \times \left(1 - |\varphi_a(z)|^2\right)^{s_1/p} \left(1 - |\varphi_a(w)|^2\right)^{s_2/p} \right)^p d\sigma_z < \infty,$$

(3)
$$\sup_{a\in D} \iint_D \left(\frac{1}{|D(z,r)|} \iint_{D(z,r)} |f^{(n)}(w) - f^{(n)}(z)|^p (1-|z|^2)^{\alpha} (1-|w|^2)^{\beta} \times \left(1 - |\varphi_a(z)|^2\right)^{s_1} \left(1 - |\varphi_a(w)|^2\right)^{s_2} d\sigma_w \right) d\sigma_z < \infty,$$

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$$(4)\quad \sup_{a\in D}\iint_D\left(\frac{1}{|D(z,r)|}\iint_{D(z,r)}\left|f^{(n)}(w)-f^{(n)}(z)\right|\left(1-|z|^2\right)^{\alpha/p}\left(1-|w|^2\right)^{\beta/p}\right.$$
$$\left.\times\left(1-|\varphi_a(z)|^2\right)^{s_1/p}\left(1-|\varphi_a(w)|^2\right)^{s_2/p}d\sigma_w\right)^p d\sigma_z<\infty.$$

PROOF: We will prove the implications (4) $\Rightarrow$ (1) $\Rightarrow$ (3) $\Rightarrow$ (2) while (2) $\Rightarrow$ (4) is trivial.

(4) $\Rightarrow$ (1). An easy calculation shows that, for an analytic function g on D,

$$g'(0)=\frac{2}{r^4\pi}\iint_{D(0,r)}\overline{w}g(w)\,d\sigma_w. \tag{3.2}$$

Applying this to the function $g(w)=(f^{(n)}\circ\varphi_z)(w)-f^{(n)}(z)$ we obtain

$$\left|f^{(n+1)}(z)\right|\left(1-|z|^2\right)\leqslant C\iint_{D(0,r)}\left|(f^{(n)}\circ\varphi_z)(w)-f^{(n)}(z)\right|d\sigma_w, \tag{3.3}$$

where the positive constant C depends only on r. By (3.3) and the elementary inequalities (2.2)–(2.5),

$$\iint_D\left|f^{(n+1)}(z)\right|^p\left(1-|z|^2\right)^{np+q}\left(1-|\varphi_a(z)|^2\right)^s d\sigma_z$$
$$\leqslant C\iint_D\left(\iint_{D(0,r)}\left|(f^{(n)}\circ\varphi_z)(w)-f^{(n)}(z)\right|d\sigma_w\right)^p$$
$$\times\left(1-|z|^2\right)^{np-p+q}\left(1-|\varphi_a(z)|^2\right)^s d\sigma_z$$
$$\leqslant C\iint_D\left(\frac{1}{|D(z,r)|}\iint_{D(z,r)}\left|f^{(n)}(u)-f^{(n)}(z)\right|\left(1-|z|^2\right)^{\alpha/p}\left(1-|u|^2\right)^{\beta/p}\right.$$
$$\left.\times\left(1-|\varphi_a(z)|^2\right)^{s_1/p}\left(1-|\varphi_a(u)|^2\right)^{s_2/p}d\sigma_u\right)^p d\sigma_z,$$

and it follows by Theorem 3.2 that $f\in F(p,q,s)$.

(1) $\Rightarrow$ (3). By the inequalities (2.2) and (2.4), and by applying (3.1) to the function $f^{(n)}(w)-f^{(n)}(z)$, we obtain

$$\iint_D\left(\frac{1}{|D(z,r)|}\iint_{D(z,r)}\left|f^{(n)}(w)-f^{(n)}(z)\right|^p\left(1-|z|^2\right)^\alpha\left(1-|w|^2\right)^\beta\right.$$
$$\left.\times\left(1-|\varphi_a(z)|^2\right)^{s_1}\left(1-|\varphi_a(w)|^2\right)^{s_2}d\sigma_w\right)d\sigma_z$$
$$\leqslant C\iint_D\left(\frac{1}{|D(z,r)|}\iint_{D(z,r)}\left|f^{(n+1)}(w)\right|^p\left(1-|w|^2\right)^p d\sigma_w\right)\left(1-|z|^2\right)^{np-p+q}$$
$$\times\left(1-|\varphi_a(z)|^2\right)^s d\sigma_z.$$

Moreover, by (2.2)–(2.5), (2.7) and Fubini's Theorem,

$$\iint_D\left(\iint_{D(z,r)}\left|f^{(n+1)}(w)\right|^p\left(1-|w|^2\right)^p d\sigma_w\right)\left(1-|z|^2\right)^{np-p+q}$$

$$
\begin{aligned}
&\times\left(1-\left|\varphi_a(z)\right|^2\right)^s d\sigma_z \\
&\leqslant C\iint_D\left(\iint_D \chi_{D(z,r)}(w)\left|f^{(n+1)}(w)\right|^p\left(1-|w|^2\right)^{np+q-2}\left(1-\left|\varphi_a(w)\right|^2\right)^s d\sigma_w\right) d\sigma_z \\
&= C\iint_D\left(\iint_D \chi_{D(w,r)}(z)\, d\sigma_z\right)\left|f^{(n+1)}(w)\right|^p\left(1-|w|^2\right)^{np+q-2}\left(1-\left|\varphi_a(w)\right|^2\right)^s d\sigma_w \\
&\leqslant C\iint_D\left|f^{(n+1)}(w)\right|^p\left(1-|w|^2\right)^{np+q}\left(1-\left|\varphi_a(w)\right|^2\right)^s d\sigma_u,
\end{aligned}
$$

and the implication follows by Theorem 3.2.

(3) $\Rightarrow$ (2). By the inequalities (2.2) and (2.4), and by the subharmonicity of $\left|f^{(n)}(w)-f^{(n)}(z)\right|^p$,

$$
\begin{aligned}
&\sup_{w\in D(z,r)}\left|f^{(n)}(w)-f^{(n)}(z)\right|\left(1-|z|^2\right)^{\alpha/p}\left(1-|w|^2\right)^{\beta/p} \\
&\qquad\times\left(1-\left|\varphi_a(z)\right|^2\right)^{s_1/p}\left(1-\left|\varphi_a(w)\right|^2\right)^{s_2/p} \\
&\leqslant C\sup_{w\in D(z,r)}\left(\left|f^{(n)}(w)-f^{(n)}(z)\right|^p\right)^{1/p}\left(1-|z|^2\right)^{(np-p+q)/p}\left(1-\left|\varphi_a(z)\right|^2\right)^{s/p} \\
&\leqslant C\sup_{w\in D(z,r)}\left(\frac{1}{|D(w,r)|}\iint_{D(w,r)}\left|f^{(n)}(u)-f^{(n)}(z)\right|^p\right. \\
&\qquad\left.\times\left(1-|z|^2\right)^{np-p+q}\left(1-\left|\varphi_a(z)\right|^2\right)^s d\sigma_u\right)^{1/p}.
\end{aligned}
$$

Setting $R=\left((2r)/(1+r^2)\right)\in(r,1)$, we have $D(w,r)\subset D(z,R)$ for $w\in D(z,r)$, and it follows by (2.6) that the last expression above is dominated by a constant times

$$
\begin{aligned}
&\left(\frac{1}{|D(z,R)|}\iint_{D(z,R)}\left|f^{(n)}(w)-f^{(n)}(z)\right|^p\right. \\
&\qquad\left.\times\left(1-|z|^2\right)^{\alpha}\left(1-|w|^2\right)^{\beta}\left(1-\left|\varphi_a(z)\right|^2\right)^{s_1}\left(1-\left|\varphi_a(w)\right|^2\right)^{s_2} d\sigma_w\right)^{1/p},
\end{aligned}
$$

and the implication follows. ∎

Next we prove equivalent conditions involving oscillations derived from the mean

$$
\widehat{f}_r(z)=\frac{1}{|D(z,r)|}\iint_{D(z,r)} f(w)\, d\sigma_z.
$$

THEOREM 3.5. *Let f be an analytic function on D and let $0<p<\infty$, $-2<q<\infty$, $0\leqslant s<\infty$ and $0<r<1$. Let $n\in\mathbb{N}$ and $q+s>-1$ or $n=0$ and $q+s-p>-1$. Let $\alpha+\beta=np-p+q$ and $s_1+s_2=s$. Then the following conditions are equivalent:*

(1) $f\in F(p,q,s)$,

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$$\text{(2)}\quad \sup_{a\in D}\iint_D \Bigl(\sup_{w\in D(z,r)}|\widehat{f^{(n)}}_r(z)-f^{(n)}(w)|(1-|z|^2)^{\alpha/p}(1-|w|^2)^{\beta/p} \times \Bigl(1-|\varphi_a(z)|^2\Bigr)^{s_1/p}\Bigl(1-|\varphi_a(w)|^2\Bigr)^{s_2/p}\Bigr)^p\, d\sigma_z<\infty,$$

$$\text{(3)}\quad \sup_{a\in D}\iint_D \Bigl(\frac{1}{|D(z,r)|}\iint_{D(z,r)}|\widehat{f^{(n)}}_r(z)-f^{(n)}(w)|^p(1-|z|^2)^{\alpha}(1-|w|^2)^{\beta} \times \Bigl(1-|\varphi_a(z)|^2\Bigr)^{s_1}|(1-|\varphi_a(w)|^2\Bigr)^{s_2}\,d\sigma_w\Bigr)\, d\sigma_z<\infty,$$

$$\text{(4)}\quad \sup_{a\in D}\iint_D \Bigl(\frac{1}{|D(z,r)|}\iint_{D(z,r)}|\widehat{f^{(n)}}_r(z)-f^{(n)}(w)|(1-|z|^2)^{\alpha/p}(1-|w|^2)^{\beta/p} \times \Bigl(1-|\varphi_a(z)|^2\Bigr)^{s_1/p}\Bigl(1-|\varphi_a(w)|^2\Bigr)^{s_2/p}\,d\sigma_w\Bigr)^p\, d\sigma_z<\infty.$$

PROOF: We will only prove the implication (1) $\Rightarrow$ (3), since the other implications ((3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (4) $\Rightarrow$ (1)) can be proved as the corresponding parts of Theorem 3.4.

(1) $\Rightarrow$ (3). Following the proof of the corresponding part of Theorem 3.4 we deduce

$$\begin{aligned}
&\iint_D \Bigl(\frac{1}{|D(z,r)|}\iint_{D(z,r)}|f^{(n)}(w)-\widehat{f^{(n)}}_r(z)|^p(1-|z|^2)^{\alpha}(1-|w|^2)^{\beta}\\
&\qquad\qquad \times\Bigl(1-|\varphi_a(z)|^2\Bigr)^{s_1}\Bigl(1-|\varphi_a(w)|^2\Bigr)^{s_2}\,d\sigma_w\Bigr)\,d\sigma_z\\
&\leqslant C\iint_D \Bigl(\frac{1}{|D(z,r)|}\iint_{D(z,r)}|f^{(n+1)}(w)|^p(1-|w|^2)^p\,d\sigma_w\Bigr)(1-|z|^2)^{np-p+q}\\
&\qquad\qquad \times\Bigl(1-|\varphi_a(z)|^2\Bigr)^{s}\,d\sigma_z\\
&\qquad\qquad +C\iint_D|f^{(n)}(z)-\widehat{f^{(n)}}_r(z)|^p(1-|z|^2)^{np-p+q}\Bigl(1-|\varphi_a(z)|^2\Bigr)^{s}\,d\sigma_z.
\end{aligned}$$

The first term can be treated in the usual manner, and the second term is easily seen to be dominated by a constant times the integral expression in the condition (4) of Theorem 3.4. $\square$

In order to prove equivalent conditions involving oscillations derived from

$$\left|\frac{f^{(n-1)}(z)-f^{(n-1)}(w)}{z-w}\right|,$$

we need two auxiliary lemmas. We quote [**10**, Lemma 5.2.1] as

LEMMA A. *Let u be a subharmonic function on D and let $-2<q<\infty$, $0\leqslant s<\infty$, $0<r<1$ and $a\in D$. Then, for every $w\in D$, there exists a positive constant C, depending only on q, r and s, such that*

$$u(w)(1-|w|^2)^{q+2}\Bigl(1-|\varphi_a(w)|^2\Bigr)^s\leqslant C\iint_{D(w,r)}u(z)(1-|z|^2)^q\Bigl(1-|\varphi_a(z)|^2\Bigr)^s\,d\sigma_z.$$

Lemma 3.6. *Let f be an analytic function on D and let $0 < p < \infty$ and $-1 < q < \infty$. Then there exists a positive constant C, depending only on q, such that*

$$\iint_D \left|\frac{f(z)-f(0)}{z}\right|^p (1-|z|^2)^q \, d\sigma_z \leqslant C \iint_D |f'(z)|^p (1-|z|^2)^{p+q}\, d\sigma_z .$$

In the view of Lemma 3.1, Lemma 3.6 follows by the proof [**10**, Lemma 5.3.2]. For a slightly different proof in the case $p \geqslant 1$, see [**12**, Lemma 6].

As an immediate application of Lemma 3.1, we see that

$$\iint_{D(0,r)} \left|\frac{f(z)-f(0)}{z}\right|^p d\sigma_z \leqslant C \iint_{D(0,r)} |f'(z)|^p (1-|z|^2)^p \, d\sigma_z , \tag{3.4}$$

where C is a positive constant, see also [**14**, Lemma 3.2].

Theorem 3.7. *Let f be an analytic function on D and let $0 < p < \infty$, $-2 < q < \infty$, $0 \leqslant s < \infty$ and $0 < r < 1$. Let $n \in \mathbb{N}$ and $q + s > -1$ or $n = 0$ and $q + s - p > -1$. Let $\alpha + \beta = np - p + q$ and $s_1 + s_2 = s$. Then the following conditions are equivalent:*

(1) $f \in F(p,q,s)$,

(2) $$\sup_{a\in D} \iint_D \left(\sup_{w\in D(z,r)} \left|\frac{f^{(n-1)}(z)-f^{(n-1)}(w)}{z-w}\right| (1-|z|^2)^{\alpha/p} (1-|w|^2)^{\beta/p} \times \left(1-|\varphi_a(z)|^2\right)^{s_1/p} \left(1-|\varphi_a(w)|^2\right)^{s_2/p} \right)^p d\sigma_z < \infty,$$

(3) $$\sup_{a\in D} \iint_D \left(\frac{1}{|D(z,r)|} \iint_{D(z,r)} \left|\frac{f^{(n-1)}(z)-f^{(n-1)}(w)}{z-w}\right|^p (1-|z|^2)^{\alpha} (1-|w|^2)^{\beta} \times \left(1-|\varphi_a(z)|^2\right)^{s_1} \left(1-|\varphi_a(w)|^2\right)^{s_2} d\sigma_w \right) d\sigma_z < \infty,$$

(4) $$\sup_{a\in D} \iint_D \left(\frac{1}{|D(z,r)|} \iint_{D(z,r)} \left|\frac{f^{(n-1)}(z)-f^{(n-1)}(w)}{z-w}\right| (1-|z|^2)^{\alpha/p} (1-|w|^2)^{\beta/p} \times \left(1-|\varphi_a(z)|^2\right)^{s_1/p} \left(1-|\varphi_a(w)|^2\right)^{s_2/p} d\sigma_w \right)^p d\sigma_z < \infty.$$

Proof: We will prove the implications (4) $\Rightarrow$ (1) $\Rightarrow$ (3) while (3) $\Rightarrow$ (2) $\Rightarrow$ (4) can be proved as the corresponding parts of Theorem 3.4.

(4) $\Rightarrow$ (1). By Lemma 3,

$$\begin{aligned} &\iint_D \left(\frac{1}{|D(z,r)|} \iint_{D(z,r)} \left|\frac{f^{(n-1)}(z)-f^{(n-1)}(w)}{z-w}\right| (1-|z|^2)^{\alpha/p} (1-|w|^2)^{\beta/p} \right. \\ &\qquad\qquad \left. \times \left(1-|\varphi_a(z)|^2\right)^{s_1/p} \left(1-|\varphi_a(w)|^2\right)^{s_2/p} d\sigma_w \right)^p d\sigma_z \\ &\geqslant C \iint_D |f^{(n)}(z)|^p (1-|z|^2)^{\alpha+\beta} \left(1-|\varphi_a(z)|^2\right)^{s_1+s_2} d\sigma_z , \end{aligned}$$

from which the implication follows by Theorem 3.2 in view of the inequalities (2.2) and (2.4).

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(1) $\Rightarrow$ (3). By the inequalities (2.2)–(2.5),

$$\iint_D \left(\frac{1}{|D(z,r)|} \iint_{D(z,r)} \left| \frac{f^{(n-1)}(z) - f^{(n-1)}(w)}{z-w} \right|^p (1-|z|^2)^\alpha (1-|w|^2)^\beta \right.$$
$$\left. \times \left(1-|\varphi_a(z)|^2\right)^{s_1} \left(1-|\varphi_a(w)|^2\right)^{s_2} d\sigma_w \right) d\sigma_z$$
$$\leqslant C \iint_D (1-|z|^2)^{np+q-2p} \left(1-|\varphi_a(z)|^2\right)^s$$
$$\times \left(\iint_{D(0,r)} \left| \frac{(f^{(n-1)} \circ \varphi_z)(u) - (f^{(n-1)} \circ \varphi_z)(0)}{u} \right|^p d\sigma_u \right) d\sigma_z .$$

Using the inequality (3.4) we see that

$$\iint_D (1-|z|^2)^{np+q-2p} \left(1-|\varphi_a(z)|^2\right)^s$$
$$\times \left(\iint_{D(0,r)} \left| \frac{(f^{(n-1)} \circ \varphi_z)(u) - (f^{(n-1)} \circ \varphi_z)(0)}{u} \right|^p d\sigma_u \right) d\sigma_z$$
$$\leqslant C \iint_D (1-|z|^2)^{np+q-2p} \left(1-|\varphi_a(z)|^2\right)^s$$
$$\times \left(\iint_{D(z,r)} \left|f^{(n)}(w)\right|^p (1-|w|^2)^p \left|\varphi_z'(w)\right|^2 d\sigma_w \right) d\sigma_z ,$$

from which the implication follows by (2.7) and Fubini's Theorem. $\square$

REMARK. (1) Analogous results to Theorems 3.4, 3.5 and 3.7 hold for the spaces $F_0(p,q,s)$ by the same proofs.

(2) Theorems 3.4, 3.5 and 3.7 and their $F_0(p,q,s)$ counterparts generalise [**14**, Theorems 3.3 and 3.5] to the spaces $F(p,q,s)$ and $F_0(p,q,s)$, respectively.

4. On inclusion relations between $F(p,q,s)$ and $F^n(p,q,s)$

Let us first recall the definition of the spaces $F^n(p,q,s)$ and $F_0^n(p,q,s)$ from [**10**]. For $0 < p < \infty$, $-2 < q < \infty$, $0 \leqslant s < \infty$ and $n \in \mathbb{N}$, $F^n(p,q,s)$ and $F_0^n(p,q,s)$ are defined as the sets of all analytic functions f on D for which

$$\sup_{a \in D} \iint_D \iint_D \left| \frac{f^{(n-1)}(z) - f^{(n-1)}(w)}{z-w} \right|^p (1-|z|^2)^{np-p+q/2-1} (1-|w|^2)^{(np-p+q)/2-1}$$
$$\times \left(1-|\varphi_a(z)|^2\right)^{s/2} \left(1-|\varphi_a(w)|^2\right)^{s/2} d\sigma_z \, d\sigma_w < \infty ,$$

and

$$\lim_{|a| \to 1} \iint_D \iint_D \left| \frac{f^{(n-1)}(z) - f^{(n-1)}(w)}{z-w} \right|^p (1-|z|^2)^{(np-p+q)/2-1} (1-|w|^2)^{(np-p+q)/2-1}$$
$$\times \left(1-|\varphi_a(z)|^2\right)^{s/2} \left(1-|\varphi_a(w)|^2\right)^{s/2} d\sigma_z \, d\sigma_w = 0, \quad 0 < s < \infty ,$$

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respectively. For convenience, we also define $F_0^n(p,q,0) = F^n(p,q,0)$.

To consider inclusion relations between the spaces $F(p,q,s)$ and $F^n(p,q,s)$ (respectively, $F_0(p,q,s)$ and $F_0^n(p,q,s)$), it suffices to consider the case $n=1$, since by the definition, $f \in F^n(p,q,s)$ (respectively, $f \in F_0^n(p,q,s)$) if and only if $f^{(n-1)} \in F^1(p, np-p+q, s)$ (respectively, $f^{(n-1)} \in F_0^1(p, np-p+q, s)$), if f is not a polynomial of degree less than or equal to $n-1$.

Probably the first paper containing results involving these kind of double integrals is due Stroethoff. He proved in [**12**] that $F^1(p, p-2, 0) = F(p, p-2, 0) = B_p$, where B_p stands for the classical Besov space. Yoneda considered more general situation, and showed that $F^n(p, p-2, 0) \subset B_p$, if $p \geqslant 1$ and $np > 2$. In view of Lemma 3 and Lemma 3.6, we see, by observing the proofs of [**10**, Theorems 5.2.2 and 5.2.3], that the latter result can be generalised.

THEOREM 4.1. *Let f be an analytic function in D and let $0 < p < \infty$, $-2 < q < \infty$, $0 \leqslant s < \infty$ and $n \in \mathbb{N}$ with $np - p + q + s > 0$. Then*

$$F^n(p,q,s) \subset F(p,q,s)$$

and

$$F_0^n(p,q,s) \subset F_0(p,q,s).$$

We note that Theorem 4.1 is also an immediate consequence of Theorem 3.7.

Further, it is known that $F^1(p, p-2, s) = F(p, p-2, s)$, $F_0^1(p, p-2, s) = F_0(p, p-2, s)$ and $F^1(p, \alpha p - 2, 0) = F(p, \alpha p - 2, 0)$ for $1 < p < \infty$, $0 \leqslant s < \infty$ and $1 < \alpha < 2$ with $\alpha p > 2$, see [**10**, Theorems 5.3.4 and 5.3.6]. These results can be generalised as well.

THEOREM 4.2. *Let f be an analytic function in D and let $0 < p < \infty$, $-2 < q < \infty$ and $0 \leqslant s < \infty$. If*

$$\max\{s-2, p-s-1, -s\} < q < 2p - s - 2\,,$$

then $F(p,q,s) = F^1(p,q,s)$ and $F_0(p,q,s) = F_0^1(p,q,s)$.

PROOF: By a change of variable,

$$\begin{aligned}
B(p,q,s,a;f) &:= \iint_D \iint_D \left|\frac{f(z)-f(w)}{z-w}\right|^p (1-|z|^2)^{(q/2)-1}(1-|w|^2)^{(q/2)-1} \\
&\qquad \times \left(1-|\varphi_a(z)|^2\right)^{s/2}\left(1-|\varphi_a(w)|^2\right)^{s/2} d\sigma_z\, d\sigma_w \\
&= \iint_D \left(\iint_D \left| \frac{(f\circ\varphi_w)(u) - (f\circ\varphi_w(0)}{u(1-\overline{w}u)^{(q-p-2)/p}\left(1-\overline{\varphi_a(w)}u\right)^{s/p}} \right|^p (1-|u|^2)^{(q+s)/2-1}\, d\sigma_u \right) \\
&\qquad \times (1-|w|^2)^{q-p}\left(1-|\varphi_a(w)|^2\right)^s d\sigma_w\,,
\end{aligned}$$

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from which an application of Lemma 3.6 yields

$$B(p,q,s,a;f) \leqslant C\Big(\iint_D |f'(z)|^p (1-|z|^2)^q \Big(1-|\varphi_a(z)|^2\Big)^s C(p,q,s,z,a)\, d\sigma_z \\ + \iint_D |f(z)|^p (1-|z|^2)^{q-p} \Big(1-|\varphi_a(z)|^2\Big)^s C(p,q,s,z,a)\, d\sigma_z \Big),$$

where

$$C(p,q,s,z,a) \\ = \frac{\big(1-|z|^2\big)^{(p-q-2)/2}}{\big(1-|\varphi_a(z)|^2\big)^{s/2}} \iint_D (1-|w|^2)^{(q-p-2)/2} \Big(1-|\varphi_w(z)|^2\Big)^{p/2} \Big(1-|\varphi_a(w)|^2\Big)^{s/2} d\sigma_w .$$

However,

$$C(p,q,s,z,a) = \big(1-|z|^2\big)^{(2p-q-s-2)/2} |1-\bar{a}z|^s \iint_D \frac{\big(1-|w|^2\big)^{(q+s-2)/2}}{|1-\bar{z}w|^p |1-\bar{a}w|^s}\, d\sigma_w ,$$

and therefore, by [**5**, Lemma 2.5],

$$\sup_{a,z\in D} C(p,q,s,z,a) < \infty .$$

The assertions now follow by Theorems 3.2, 3.3 and 4.1. ∎

An immediate, yet false, see [**10**, Theorem 5.3.10], conjecture is that the spaces $F(p,q,s)$ and $F^n(p,q,s)$ coincide for all parameter values p, q, s and n. The following theorem shows that if n is sufficiently large, then $F^n(p,q,s)$ is strictly included in $F(p,q,s)$.

Theorem 4.3. *Let $0 < p < \infty$, $-2 < q < \infty$, $0 \leqslant s < \infty$ and $n \in \mathbb{N}$ with $np - p + q + s > 0$. If*

$$n > 3 + \frac{s-q-2}{p}, \tag{4.1}$$

then $F^n(p,q,s) \subsetneq F(p,q,s)$.

Proof: By Theorem 4.1 it suffices to show that the inclusion is strict. Let f be an analytic function on D such that

$$f'(z) = (1-z)^{-\beta},$$

where $\beta < (q+2)/p$, $\beta \in \mathbb{Q}$. Let us first prove that $f \in F(p,q,s)$. If $s = 0$, an application of [**4**, p. 65 Lemma] shows that $f \in F_0(p,q,0)$. If $s > 1$, $F(p,q,s) = \mathcal{B}^{(q+2)/p}$ and obviously $f \in F(p,q,s)$ (in fact $f \in F_0(p,q,s) = \mathcal{B}_0^{(q+2)/p}$). If $0 < s \leqslant 1$, consider $f_b(z) = f(bz)$, $0 < b < 1$. A simple calculation with an application of [**5**, Lemma 2.5] and Lebesgue's dominated convergence theorem shows that $f_1 = f \in F(p,q,s)$.

To see that $f \notin F^n(p,q,s)$, we may follow the reasoning in [**10**, Theorem 5.3.10]. Since β is rational, there exists an $r(\beta,n)$ and a disc $D(\beta,n) \subset D$, depending only on β and n, such that

$$|1-z|^{\beta+n-2}\left|\frac{f^{(n-1)}(z)-f^{(n-1)}(w)}{z-w}\right| > 0$$

for $(z,w) \in \left(\overline{D\setminus D(0,1-r(\beta,n))} \times \overline{D(\beta,n)}\right)$. It follows that

$$\begin{aligned}\sup_{a\in D}\iint_D\iint_D&\left|\frac{f^{(n-1)}(z)-f^{(n-1)}(w)}{z-w}\right|^p(1-|z|^2)^{((np-p+q)/2)-1}(1-|w|^2)^{((np-p+q)/2)-1}\\&\qquad\times(1-|\varphi_a(z)|^2)^{s/2}\left(1-|\varphi_a(w)|^2\right)^{s/2}d\sigma_z\,d\sigma_w\\\geqslant C&\iint_{D\setminus D(0,1-r(\beta,n))}\frac{1}{|1-z|^{p(\beta+n-2)}}(1-|z|^2)^{((np-p+q+s)/2)-1}\,d\sigma_z\\\geqslant C&\int_{1-r(\beta,n)}^1(1-r^2)^{((np-p+q+s)/2)-1}\left(\int_0^{2\pi}\frac{d\theta}{(1-2r\cos\theta+r^2)^{(p(\beta+n-2))/2}}\right)dr\,.\end{aligned}$$

However,

$$\int_0^{2\pi}\frac{d\theta}{(1-2r\cos\theta+r^2)^{(p(\beta+n-2))/2}} \geqslant C\frac{1}{(1-r)^{((p(\beta+n-2))/2)-1}},$$

which can be seen by the change of variable $\theta=(1-r)t$, see [**10**, Theorem 5.3.10], and hence

$$\begin{aligned}\sup_{a\in D}\iint_D\iint_D&\left|\frac{f^{(n-1)}(z)-f^{(n-1)}(w)}{z-w}\right|^p(1-|z|^2)^{((np-p+q)/2)-1}(1-|w|^2)^{((np-p+q)/2)-1}\\&\qquad\times\left(1-|\varphi_a(z)|^2\right)^{s/2}\left(1-|\varphi_a(w)|^2\right)^{s/2}d\sigma_z\,d\sigma_w\\\geqslant C&\int_{1-r(\beta,n)}^1(1-r^2)^{((np-p+q+s)/2)-1-p(\beta+n-2)+1}\,dr\,.\end{aligned}$$

The last integral diverges, if

$$\beta \geqslant \frac{3}{2}-\frac{n}{2}+\frac{((q+s)/2)+1}{p}.$$

Combining this condition with the inequality $\beta < (q+2)/p$ we get (4.1). ∎

REMARK If $(q+2)/p \in \mathbb{Q}$ and $0 < s < \infty$, we may take $\beta=(q+2)/p$ in the proof above, and then the slightly weaker condition

$$n \geqslant 3+\frac{s-q-2}{p}$$

implies $F^n(p,q,s) \subsetneq F(p,q,s)$. This, with $q=p-2$ and $1<s<\infty$, reduces to [**10**, Theorem 5.3.10].

As an immediate consequence of Theorem 4.3, we have

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Corollary 4.4. *Let $0 < p < \infty$ and $n \in \mathbb{N}$ with $np - 2 > 0$. If $n > 2$, then $F^n(p, p-2, 0) \subsetneq B_p = F(p, p-2, 0)$.*

Corollary 4.4 answers partially the question in [**14**, pp. 416 and 446], but it still remains open whether $F^2(p, p-2, 0)$ equals to B_p for $p > 1$ or not.

5. Mean growth of derivatives of $F(p, q, s)$ functions

A careful reader observes that the mean growth of $F(p, q, s)$ functions is quite well studied by Flett in terms of generalised Hardy-Littlewood integral means

$$M(p, q, \alpha; f) = \int_0^1 M_p^q(r, f)(1-r)^{\alpha q - 1}\, dr,$$

where $M_p(r, f)$ are the standard L^p means of the restriction of f to the circle of radius r centred at the origin, in his work [**6**] on multiplier transformations which may be regarded as fractional derivatives or integrals. Because of this fact, we settle to prove the following result and some of its consequences.

Theorem 5.1. *Let f be an analytic function on D and let $0 < p < \infty$, $-2 < q < \infty$ and $0 \leqslant s \leqslant 1$. Let φ be an increasing function of r on the interval $(0, 1)$ such that $|f'(z)| \leqslant \varphi(r)$ for all $z = re^{i\theta} \in D$. If*

$$\int_0^1 \varphi(r)^p (1-r^2)^{q+s}\, dr < \infty, \tag{5.1}$$

then $f \in F_0(p, q, s)$.

Proof: Let $0 \leqslant s \leqslant 1$. Then

$$\begin{aligned} &\iint_D |f'(z)|^p (1-|z|^2)^q \left(1 - |\varphi_a(z)|^2\right)^s d\sigma_z \\ &\quad = \int_0^1 \int_0^{2\pi} \left|f'(re^{i\theta})\right|^p (1-r^2)^{q+s} \frac{\left(1-|a|^2\right)^s}{|1-\bar{a}re^{i\theta}|^{2s}} r\, d\theta\, dr \\ &\quad \leqslant \int_0^1 \varphi(r)^p (1-r^2)^{q+s} \left(\left(1-|a|^2\right)^s \int_0^{2\pi} \frac{d\theta}{|1-\bar{a}re^{i\theta}|^{2s}}\right) dr, \end{aligned} \tag{5.2}$$

and since

$$\int_0^{2\pi} \frac{d\theta}{|1-\bar{a}re^{i\theta}|^{2s}} \leqslant \begin{cases} C, & \text{if } 0 \leqslant s < \frac{1}{2} \\ C \log \dfrac{1}{1-|a|r}, & \text{if } s = \frac{1}{2} \\ C \dfrac{1}{\left(1-|a|r\right)^{2s-1}}, & \text{if } \frac{1}{2} < s \leqslant 1 \end{cases},$$

we conclude, by the assumption (5.1), that the last integral in (5.2) converges for all $a \in D$. Hence the assertion follows for $s = 0$. For $0 < s \leqslant 1$, the integrand tends to zero, as $|a| \to 1$, for all $r \in (0,1)$, and we conclude, by Lebesgue's dominated convergence theorem, that

$$\int_0^1 \varphi(r)^p (1-r^2)^{q+s} \Big((1-|a|^2)^s \int_0^{2\pi} \frac{d\theta}{|1-\bar{a}re^{i\theta}|^{2s}} \Big) \, dr \to 0 \,,$$

as $|a| \to 1$. Thus, by [**13**, Theorem 2.5], $f \in F_0(p,q,s)$. ∎

By the proof above it is evident that if the condition (5.1) is satisfied, then the function f lies actually in $F_0(p, q+s-1, 1)$ which is strictly included in $F_0(p,q,s)$, if $0 \leqslant s < 1$. Hereafter this observation will be ignored for simplicity.

The case $p=2$, $q=0$ and $s=1$ of Theorem 5.1 (that is, $f \in VMOA$, the space of analytic functions with vanishing mean oscillation) has been proved in a different way by Danikas [**3**, p. 25].

At this point it is clear that combining Theorems 3.2 and 5.1, we may deduce a similar result involving n-th derivatives.

THEOREM 5.2. *Let f be an analytic function on D and let $0 < p < \infty$, $-2 < q < \infty$ and $0 \leqslant s \leqslant 1$. Let $n \in \mathbb{N}$ and $q+s > -1$ or $n = 0$ and $q+s-p > -1$. Let φ be an increasing function of r on the interval $(0,1)$ such that $|f^{(n)}(z)| \leqslant \varphi(r)$ for all $z = re^{i\theta} \in D$. If*

$$\int_0^1 \varphi(r)^p (1-r^2)^{np-p+q+s} \, dr < \infty \,, \tag{5.3}$$

then $f \in F_0(p,q,s)$.

Theorem 5.2 (as well as Theorem 5.1) can be proved in the following alternative way by applying s-Carleson measures. Let $d\mu(z) = |f^{(n)}(z)|^p (1-|z|^2)^{np-p+q+s} \, d\sigma_z$. By (5.3),

$$\frac{1}{|I|^s} \iint_{S(I)} d\mu(z) \leqslant C|I|^{1-s} \int_{1-|I|}^1 \varphi(r)^p (1-r^2)^{np-p+q+s} \, dr < \infty \,, \tag{5.4}$$

and by letting $|I| \to 0$, it follows that $d\mu(z)$ is a compact s-Carleson measure, and hence $f \in F_0(p,q,s)$ by Theorem 3.3.

Defining

$$M_\infty(r,f) = \max_{0 \leqslant \theta \leqslant 2\pi} |f(re^{i\theta})| \,,$$

we conclude the following immediate corollary of Theorem 5.2.

COROLLARY 5.3. *Let f be an analytic function on D and let $0 < p < \infty$, $-2 < q < \infty$ and $0 \leqslant s \leqslant 1$. Let $n \in \mathbb{N}$ and $q+s > -1$ or $n = 0$ and $q+s-p > -1$. If*

$$\int_0^1 M_\infty(r, f^{(n)})^p (1-r^2)^{np-p+q+s} \, dr < \infty \,, \tag{5.5}$$

then $f \in F_0(p,q,s)$.

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Corollary 5.3 generalises and improves [**1**, Theorem 3.2]. We complete the section by using the standard arguments to show that the condition (5.5) is also necessary for the containment in $F_0(p,q,s)$, if the function f has Hadamard gaps, that is, if the power series representation $f(z)=\sum_{k=0}^{\infty}a_kz^{n_k}$ satisfies $n_{k+1}/n_k\geqslant\lambda>1$, for all $k\geqslant 0$. To do this we need the following lemma by Zygmund, see [**16**, p. 215] and [**9**, p. 314].

Lemma 1. *Let $0<p<\infty$, $0<\alpha<\infty$, $k\in\mathbb{N}\cup\{0\}$, $0\leqslant a_k<\infty$, $I_k=\{j:2^k\leqslant j<2^{k+1},\ j\in\mathbb{N}\}$, $t_k=\sum_{j\in I_k}a_j$, and let $f(x)=\sum_{k=0}^{\infty}a_kx^k$. Then there exists a positive constant C, depending only on p and α, such that*

$$\frac{1}{C}\sum_{k=0}^{\infty}2^{-k\alpha}t_k^p\leqslant\int_0^1(1-x)^{\alpha-1}f(x)^p\,dx\leqslant C\sum_{k=0}^{\infty}2^{-k\alpha}t_k^p.$$

Theorem 5.4. *Let $0<p<\infty$, $-2<q<\infty$ and $0\leqslant s\leqslant 1$. Let $n\in\mathbb{N}$ and $q+s>-1$ or $n=0$ and $q+s-p>-1$. If the analytic function $f(z)=\sum_{k=0}^{\infty}a_kz^{n_k}$ has Hadamard gaps, then $f\in F_0(p,q,s)$ if and only if (5.5) is satisfied.*

Proof: In view of Corollary 5.3, it suffices to show that if $f\in F_0(p,q,s)$ has Hadamard, then (5.5) is satisfied. For simplicity, we will prove this only in the case $n=1$, the general case can be proved in a similar manner. We note that Lemma 1 holds also if $f(x)=\sum_{k=0}^{\infty}a_kx^{k-1}$, and therefore

$$\begin{aligned}\int_0^1 M_\infty(r,f')^p(1-r^2)^{q+s}\,dr&\leqslant 2^{q+s}\int_0^1\Big(\sum_{k=1}^{\infty}n_k|a_{n_k}|r^{n_k-1}\Big)^p(1-r)^{q+s}\,dr\\&\leqslant 2^{q+s}C\sum_{k=1}^{\infty}2^{-k(q+s+1)}\Big(\sum_{n_j\in I_k}n_j|a_{n_j}|\Big)^p\\&\leqslant 2^{2q+2s+1}C\sum_{k=1}^{\infty}\Big(\sum_{n_j\in I_k}n_j^{1-((q+s+1)/p)}|a_{n_j}|\Big)^p.\end{aligned}$$

Since the power series representation of f has at most $\log_\lambda 2+1$ terms $a_jz^{n_j}$ such that $n_j\in I_k$, we deduce

$$\int_0^1 M_\infty(r,f')^p(1-r^2)^{q+s}\,dr\leqslant 2^{2q+2s+1}C(\log_\lambda 2+1)^p\sum_{k=1}^{\infty}n_k^{p-q-s-1}|a_{n_k}|^p,$$

from which (5.5), with $n=1$, follows by [**13**, Theorem 5.5]. ∎

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