Bull. Aust. Math. Soc. **109** (2024), 26–36
doi:10.1017/S0004972723000229

A CONJECTURE OF MERCA ON CONGRUENCES MODULO POWERS OF 2 FOR PARTITIONS INTO DISTINCT PARTS

JULIA Q. D. DU and DAZHAO TANG

(Received 17 January 2023; accepted 13 February 2023; first published online 27 March 2023)

Abstract

Let $Q(n)$ denote the number of partitions of n into distinct parts. Merca ['Ramanujan-type congruences modulo 4 for partitions into distinct parts', *An. Şt. Univ. Ovidius Constanţa* **30**(3) (2022), 185–199] derived some congruences modulo 4 and 8 for $Q(n)$ and posed a conjecture on congruences modulo powers of 2 enjoyed by $Q(n)$. We present an approach which can be used to prove a family of internal congruence relations modulo powers of 2 concerning $Q(n)$. As an immediate consequence, we not only prove Merca's conjecture, but also derive many internal congruences modulo powers of 2 satisfied by $Q(n)$. Moreover, we establish an infinite family of congruence relations modulo 4 for $Q(n)$.

2020 *Mathematics subject classification*: primary 11P83; secondary 05A17.

Keywords and phrases: congruences, internal congruences, partitions, distinct parts, modular forms.

1. Introduction

A partition π of a positive integer n is a finite weakly decreasing sequence of positive integers $\pi_1 \geq \pi_2 \geq \cdots \geq \pi_r$ such that $\sum_{i=1}^{r} \pi_i = n$. The π_i are called the parts of the partition π. Let $p(n)$ denote the number of partitions of n with the convention that $p(0) = 1$. The generating function of $p(n)$, derived by Euler, is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty},$$

The first author was partially supported by the National Natural Science Foundation of China (No. 12201177), the Natural Science Foundation of Hebei Province (No. A2021205018), the Science and Technology Research Project of Colleges and Universities of Hebei Province (No. BJK2023092), the Doctor Foundation of Hebei Normal University (No. L2021B02), the Program for Foreign Experts of Hebei Province and the Program for 100 Foreign Experts Plan of Hebei Province. The second author was partially supported by the National Natural Science Foundation of China (No. 12201093), the Natural Science Foundation Project of Chongqing CSTB (No. CSTB2022NSCQ–MSX0387), the Science and Technology Research Program of Chongqing Municipal Education Commission (No. KJQN202200509) and the Doctoral Start-up Research Foundation (No. 21XLB038) of Chongqing Normal University.

where, here and throughout this paper, we always assume that q is a complex number such that $|q| < 1$ and adopt the customary notation:

$$(a;q)_\infty = \prod_{j=0}^{\infty}(1 - aq^j).$$

In 1919, Ramanujan discovered three celebrated congruences for the partition function $p(n)$ (see [4]), which were later confirmed by Atkin [2] and Watson [15]: for any $n \geq 0$ and $\alpha \geq 1$,

$$p(5^\alpha n + \delta_{5,\alpha}) \equiv 0 \pmod{5^\alpha}, \tag{1.1}$$

$$p(7^\alpha n + \delta_{7,\alpha}) \equiv 0 \pmod{7^{\lfloor \alpha/2 \rfloor + 1}}, \tag{1.2}$$

$$p(11^\alpha n + \delta_{11,\alpha}) \equiv 0 \pmod{11^\alpha}, \tag{1.3}$$

where $\delta_{p,\alpha}$ is the least positive integer satisfying $24\delta_{p,\alpha} \equiv 1 \pmod{p^\alpha}$ with $p \in \{5, 7, 11\}$. Since then, congruence properties for various partition functions have been a hot topic in the theory of partitions and have motivated a large amount of research.

Another ingredient of the theory of partitions is the study of partition identities. In 1748, Euler [7] proved the most well-known partition theorem which states that there are as many partitions of n into distinct parts as into odd parts. In terms of the generating function,

$$\sum_{n=0}^{\infty} Q(n)q^n = (-q;q)_\infty = \frac{1}{(q;q^2)_\infty} = \frac{(q^2;q^2)_\infty}{(q;q)_\infty}, \tag{1.4}$$

where $Q(n)$ denotes the number of partitions of n into distinct parts. According to Euler's pentagonal number theorem [1, page 17, (1.4.11)],

$$(q;q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2},$$

we find that almost all values of $Q(n)$ are even, that is,

$$\lim_{X\to\infty} \frac{\#\{0 \leq n \leq X : Q(n) \equiv 0 \pmod 2\}}{X} = 1. \tag{1.5}$$

Indeed, $Q(n)$ is odd if and only if n is a generalised pentagonal number. Motivated by (1.1)–(1.5), many scholars subsequently investigated congruence properties and arithmetic density properties of $Q(n)$. For instance, in 1997, Gordon and Ono [8] proved the striking result that for any positive integer m, $Q(n)$ is divisible by 2^m for almost all nonnegative integers n, that is,

$$\lim_{X\to\infty} \frac{\#\{0 \leq n \leq X : Q(n) \equiv 0 \pmod{2^m}\}}{X} = 1. \tag{1.6}$$

The identity (1.6) is a powerful result on the arithmetic properties of $Q(n)$. However, it is not a constructive result and the theory of modular forms used in the proof of (1.6)

TABLE 1. A table of values of c_p.

p	11	13	17	19	23	31	37	41	43	47	59
c_p	3	5	15	27	89	1	45	231	131	305	51
p	61	67	71	79	83	89	103	107	109	113	
c_p	21	107	5769	1	27	23	1	3	37	367	

cannot be applied to derive the explicit congruences enjoyed by $Q(n)$. Therefore, it is still of interest to find explicit congruences for $Q(n)$.

In a recent paper, Merca [9] derived some congruences modulo 4 and 8 for $Q(n)$ by using Smoot's Mathematica implementation [13] of Radu's algorithm [12] on Ramanujan–Kolberg identities for partition functions. At the end of his paper, Merca posed the following conjecture on congruences modulo powers of 2 for $Q(n)$.

CONJECTURE 1.1 (Merca [9], Conjecture). Let $(p, k) \in S$. For any $n \not\equiv 0 \pmod{p}$,

$$Q\Big(pn + \frac{p^2-1}{24}\Big) \equiv 0 \pmod{2^k},$$

where

$$\begin{aligned} S \in \{&(11,5),(13,6),(17,8),(19,9),(23,11),(31,3),(37,6),\\ &(41,8),(43,9),(47,11),(59,6),(61,6),(67,10),(71,13),\\ &(79,3),(83,5),(89,9),(103,3),(107,6),(109,6),(113,9)\}. \end{aligned} \tag{1.7}$$

In this paper, we prove the following result.

THEOREM 1.2. *Let S be defined as in (1.7). Then for any $(p, k) \in S$,*

$$\sum_{n=0}^{\infty} Q\Big(pn + \frac{p^2-1}{24}\Big)q^n \equiv c_p \sum_{n=0}^{\infty} Q(n)q^{pn} \pmod{2^k}, \tag{1.8}$$

where c_p is given in Table 1.

As an immediate consequence of (1.8), we obtain the following congruences and internal congruences enjoyed by $Q(n)$, which confirms Conjecture 1.1.

COROLLARY 1.3. *Let S be defined as in (1.7). Then for any $(p, k) \in S$ and $1 \le i \le p-1$,*

$$Q\Big(p^2 n + \frac{(24i+p)p-1}{24}\Big) \equiv 0 \pmod{2^k}.$$

Moreover, for any $n \ge 0$,

$$Q\Big(p^2 n + \frac{p^2-1}{24}\Big) \equiv c_p Q(n) \pmod{2^k},$$

where c_p is given in Table 1.

https://doi.org/10.1017/S0004972723000229 Published online by Cambridge University Press

The following theorem shows that there are an infinite family of congruence relations of the form (1.8) satisfied by $Q(n)$.

THEOREM 1.4. *Let $p \geq 5$ be a prime number. If $(\frac{-24}{p}) = -1$, then*

$$\sum_{n=0}^{\infty} Q\Big(pn + \frac{p^2-1}{24}\Big)q^n \equiv (-1)^{(\pm p-1)/6} \sum_{n=0}^{\infty} Q(n)q^{pn} \pmod 4, \tag{1.9}$$

where $(\frac{\cdot}{p})$ is the Legendre symbol and

$$(\pm p - 1)/6 = \begin{cases} (p-1)/6 & \text{if } p \equiv 1 \pmod 6, \\ (-p-1)/6 & \text{if } p \equiv 5 \pmod 6). \end{cases} \tag{1.10}$$

The rest of this paper is organised as follows. In Section 2, we collect some notation and terminology on modular forms. The proof of Theorem 1.2 is presented in Section 3 and that of Theorem 1.4 in Section 4. Finally, we pose a conjecture on congruence relations for $Q(n)$ modulo 4 which strengthens both (1.9) and a result of Merca.

2. Preliminaries

We first recall some terminology from the theory of modular forms. The full modular group is given by

$$\Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, \text{ and } ad - bc = 1 \right\},$$

and for a positive integer N, the congruence subgroup $\Gamma_0(N)$ is defined by

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \pmod N \right\}.$$

Let γ be the matrix $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})$ from now on. Then γ acts on $\tau \in \mathbb{C}$ by the linear fractional transformation

$$\gamma\tau = \frac{a\tau + b}{c\tau + d} \quad \text{and} \quad \gamma\infty = \lim_{\tau \to \infty} \gamma\tau.$$

Let N, k be positive integers and $\mathbb{H} = \{\tau \in \mathbb{C} : \operatorname{Im}(\tau) > 0\}$. A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is called a modular function of weight k for $\Gamma_0(N)$ if it satisfies the following two conditions:

(1) for all $\gamma \in \Gamma_0(N)$, $f(\gamma\tau) = (c\tau + d)^k f(\tau)$;
(2) for any $\gamma \in \Gamma$, $(c\tau + d)^{-k} f(\gamma\tau)$ has a Fourier expansion of the form

$$(c\tau + d)^{-k} f(\gamma\tau) = \sum_{n=n_\gamma}^{\infty} a(n) q_{w_\gamma}^n,$$

where $a(n_\gamma) \neq 0$, $q_{w_\gamma} = e^{2\pi i\tau/w_\gamma}$ and $w_\gamma = N/\gcd(c^2, N)$.

https://doi.org/10.1017/S0004972723000229 Published online by Cambridge University Press

In particular, if $n_\gamma \geq 0$ for all $\gamma \in \Gamma$, then we call f a modular form of weight k for $\Gamma_0(N)$. A modular function with weight 0 for $\Gamma_0(N)$ is referred to as a modular function for $\Gamma_0(N)$. For a modular function $f(\tau)$ of weight k with respect to $\Gamma_0(N)$, the order of $f(\tau)$ at the cusp $a/c \in \mathbb{Q} \cup \{\infty\}$ is defined by

$$\mathrm{ord}_{a/c}(f) = n_\gamma$$

for some $\gamma \in \Gamma$ such that $\gamma\infty = a/c$; $\mathrm{ord}_{a/c}(f)$ is well defined (see [6, page 72]).

Radu [12] developed the Ramanujan–Kolberg algorithm to derive the Ramanujan–Kolberg identities on a class of partition functions defined in terms of eta-quotients using modular functions for $\Gamma_0(N)$ (see [11]). Smoot [13] developed a Mathematica package `RaduRK` to implement Radu's algorithm.

Let the partition function $a(n)$ be defined by

$$\sum_{n=0}^{\infty} a(n)q^n = \prod_{\delta|M} (q^\delta, q^\delta)_\infty^{r_\delta}, \tag{2.1}$$

where M, δ are positive integers and r_δ are integers. For any $m \geq 1$ and $0 \leq t \leq m-1$, Radu [12] defined

$$g_{m,t}(\tau) = q^{(t+\ell)/m} \sum_{n=0}^{\infty} a(mn+t)q^n,$$

where

$$\ell = \frac{1}{24} \sum_{\delta|M} \delta r_\delta,$$

and gave a criterion for a function involving $g_{m,t}(\tau)$ to be a modular function with respect to $\Gamma_0(N)$, where N satisfies the following conditions, with $\kappa = \gcd(1-m^2, 24)$:

(1) for every prime p, $p \mid m$ implies $p \mid N$;
(2) for every δ dividing M with $r_\delta \neq 0$, $\delta \mid M$ implies $\delta \mid mN$;
(3) $\kappa m N^2 \sum_{\delta|M} r_\delta/\delta \equiv 0 \pmod{24}$;
(4) $\kappa N \sum_{\delta|M} r_\delta \equiv 0 \pmod 8$;
(5) $24m/\gcd(\kappa(-24t - \sum_{\delta|M} \delta r_\delta), 24m) \mid N$;
(6) if $2 \mid m$, then $\kappa N \equiv 0 \pmod 4$ and $8 \mid Ns$, or $2 \mid s$ and $8 \mid N(1-j)$, where $\prod_{\delta|M} \delta^{|r_\delta|} = 2^s j$ and $j, s \in \mathbb{Z}$ with j odd.

Given a positive integer n and an integer x, we denote by $[x]_n$ the residue class of x modulo n. Let

$$\mathbb{Z}_n^* = \{[x]_n \in \mathbb{Z}_n : \gcd(x, n) = 1\} \quad \text{and} \quad \mathbb{S}_n = \{y^2 : y \in \mathbb{Z}_n^*\}.$$

Define the set

$$P_m(t) = \left\{ \left[ts + \frac{s-1}{24} \sum_{\delta|M} \delta r_\delta \right]_m : s \in \mathbb{S}_{24m} \right\}.$$

https://doi.org/10.1017/S0004972723000229 Published online by Cambridge University Press

Recall that the Dedekind eta-function $\eta(\tau)$ is defined by

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n),$$

where $q = e^{2\pi i\tau}$ and $\tau \in \mathbb{H}$.

Theorem 2.1 [12, Theorem 45]. *For a partition function $a(n)$ defined as in (2.1), and integers $m \geq 1, 0 \leq t \leq m-1$, suppose that N is a positive integer satisfying the conditions (1)–(6). Let*

$$F(\tau) = \prod_{\delta|N}\eta^{s_\delta}(\delta\tau)\prod_{t'\in P_m(t)} g_{m,t'}(\tau),$$

where s_δ are integers. Then $F(\tau)$ is a modular function for $\Gamma_0(N)$ if and only if the s_δ satisfy the following conditions:

(1) $|P_m(t)|\sum_{\delta|M} r_\delta + \sum_{\delta|N} s_\delta = 0$;
(2) $\sum_{t'\in P_m(t)}(1-m^2)(24t' + \sum_{\delta|M}\delta r_\delta)/m + |P_m(t)|m\sum_{\delta|M}\delta r_\delta + \sum_{\delta|N}\delta s_\delta \equiv 0 \pmod{24}$;
(3) $|P_m(t)|mN\sum_{\delta|M} r_\delta/\delta + \sum_{\delta|N}(N/\delta)s_\delta \equiv 0 \pmod{24}$;
(4) $(\prod_{\delta|M}(m\delta)^{|r_\delta|})^{|P_m(t)|}\prod_{\delta|N}\delta^{|s_\delta|}$ *is a square.*

Radu [12, Theorem 47] also gave lower bounds for the orders of $F(\tau)$ at cusps of $\Gamma_0(N)$.

Theorem 2.2. *For a partition function $a(n)$ defined as in (2.1) and integers $m \geq 1$, $0 \leq t \leq m-1$, let*

$$F(\tau) = \prod_{\delta|N}\eta^{s_\delta}(\delta\tau)\prod_{t'\in P_m(t)} g_{m,t'}(\tau)$$

be a modular function for $\Gamma_0(N)$, where s_δ are integers and N satisfies the conditions (1)–(6). Let $\{s_1, s_2, \ldots, s_\epsilon\}$ be a complete set of inequivalent cusps of $\Gamma_0(N)$ and, for $1 \leq i \leq \epsilon$, let $\gamma_i \in \Gamma$ be such that $\gamma_i\infty = s_i$. Then

$$\mathrm{ord}_{s_i}(F(\tau)) \geq \frac{N}{\gcd(c^2, N)}(|P_m(t)|p(\gamma_i) + p^*(\gamma_i)),$$

where

$$p(\gamma_i) = \min_{\lambda\in\{0,1,\ldots,m-1\}}\frac{1}{24}\sum_{\delta|M} r_\delta\frac{\gcd^2(\delta(a+\kappa\lambda c), mc)}{\delta m}$$

and

$$p^*(\gamma_i) = \frac{1}{24}\sum_{\delta|N} s_\delta\frac{\gcd^2(\delta, c)}{\delta}.$$

The following theorem of Sturm [14, Theorem 1] plays an important role in proving congruences using the theory of modular forms.

https://doi.org/10.1017/S0004972723000229 Published online by Cambridge University Press

THEOREM 2.3. *Let k be an integer and $g(\tau) = \sum_{n=0}^{\infty} c(n)q^n$ a modular form of weight k for $\Gamma_0(N)$. For any given positive integer u, if $c(n) \equiv 0 \pmod{u}$ holds for all $n \leq (kN/12)\prod_{p|N,\, p \text{ prime}}(1 + 1/p)$, then $c(n) \equiv 0 \pmod{u}$ holds for any $n \geq 0$.*

3. Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. The following lemma plays a vital role in the proof of Theorem 1.2.

LEMMA 3.1. *Let p be a prime with $p \geq 5$ and define $k_1 = \lceil (p^2 - 1)/48p \rceil$ and $k_2 = \lceil (p^2 - 1)/48p^2 \rceil$. Then for any constant c,*

$$\frac{\eta^{24k_1}(\tau)\eta^{16k_2}(2p\tau)}{\eta^{8k_2}(p\tau)}\left(q^{p/24}\frac{\eta(p\tau)}{\eta(2p\tau)}\sum_{n=0}^{\infty} Q\left(pn + \frac{p^2-1}{24}\right)q^n - c\right)$$

is a modular form of weight $12k_1 + 4k_2$ for $\Gamma_0(2p)$.

PROOF. Recall that the generating function of $Q(n)$ is

$$\sum_{n=0}^{\infty} Q(n)q^n = \frac{(q^2;q^2)_\infty}{(q;q)_\infty}.$$

Taking $M = 2$, $(r_1, r_2) = (-1, 1)$, $m = p$, $t = (p^2 - 1)/24$ in Theorem 2.1, one can find that $N = 2p$ satisfies the conditions (1)–(6), and for $(s_1, s_2, s_p, s_{2p}) = (0, 0, 1, -1)$,

$$F(\tau) = q^{p/24}\frac{\eta(p\tau)}{\eta(2p\tau)}\sum_{n=0}^{\infty} Q\left(pn + \frac{p^2-1}{24}\right)q^n$$

is a modular function for $\Gamma_0(2p)$.

By Theorem 2.2, we derive lower bounds for the orders of $F(\tau)$ at the cusps of $\Gamma_0(2p)$:

$$\mathrm{ord}_0(F(\tau)) \geq -\frac{p^2-1}{24}, \quad \mathrm{ord}_{1/2}(F(\tau)) \geq -\frac{1}{24p},$$
$$\mathrm{ord}_{1/p}(F(\tau)) \geq \frac{2p^2-1}{24p}, \quad \mathrm{ord}_\infty(F(\tau)) \geq -\frac{p^2-1}{24p},$$

which implies that

$$\mathrm{ord}_0(F(\tau) - c) \geq -\frac{p^2-1}{24}, \quad \mathrm{ord}_{1/2}(F(\tau) - c) \geq 0,$$
$$\mathrm{ord}_{1/p}(F(\tau) - c) \geq 0, \quad \mathrm{ord}_\infty(F(\tau) - c) \geq -\frac{p^2-1}{24p}.$$

By [10, Theorems 1.64 and 1.65], one easily shows

$$F_1(\tau) = \eta^{24}(\tau) \quad \text{and} \quad F_2(\tau) = \frac{\eta^{16}(2p\tau)}{\eta^8(p\tau)}$$

https://doi.org/10.1017/S0004972723000229 Published online by Cambridge University Press

TABLE 2. A table of values of l_p.

p	11	13	17	19	23	31	37	41	43	47	59
l_p	48	56	72	80	96	128	152	168	176	192	420
p	61	67	71	79	83	89	103	107	109	113	
l_p	434	476	504	560	588	630	1040	1080	1100	1140	

are modular forms with weight 12 and 4 for $\Gamma_0(2p)$, respectively, and the orders at the cusps of $\Gamma_0(2p)$ are

$$\begin{aligned}\mathrm{ord}_0(F_1(\tau)) = 2p,\quad \mathrm{ord}_{1/2}(F_1(\tau)) = p,\quad \mathrm{ord}_{1/p}(F_1(\tau)) = 2,\quad \mathrm{ord}_\infty(F_1(\tau)) = 1,\\ \mathrm{ord}_0(F_2(\tau)) = 0,\quad \mathrm{ord}_{1/2}(F_2(\tau)) = 1,\quad \mathrm{ord}_{1/p}(F_2(\tau)) = 0,\quad \mathrm{ord}_\infty(F_2(\tau)) = p.\end{aligned}$$

Therefore, the orders of $F_1^{k_1}(\tau)F_2^{k_2}(\tau)F(\tau)$ at all cusps of $\Gamma_0(2p)$ are nonnegative, and so $F_1^{k_1}(\tau)F_2^{k_2}(\tau)F(\tau)$ is a modular form with weight $12k_2 + 4k_2$ for $\Gamma_0(2p)$. This completes the proof. □

PROOF OF THEOREM 1.2. Fix $k \geq 1$. By Lemma 3.1 and Sturm's theorem, to prove

$$\frac{(q^p;q^p)_\infty}{(q^{2p};q^{2p})_\infty}\sum_{n=0}^{\infty} Q\Big(pn + \frac{p^2-1}{24}\Big)q^n - c_p \equiv 0 \pmod{2^k},$$

we only need to check that the coefficients of the first $l_p = (p+1)(3k_1 + k_2)$ terms of the expansion of

$$\frac{\eta^{24k_1}(\tau)\eta^{16k_2}(2p\tau)}{\eta^{8k_2}(p\tau)}\Big(q^{p/24}\frac{\eta(p\tau)}{\eta(2p\tau)}\sum_{n=0}^{\infty} Q\Big(pn + \frac{p^2-1}{24}\Big)q^n - c_p\Big)$$

are congruent to 0 modulo 2^k. Here, k_1 and k_2 are defined in Lemma 3.1 and the corresponding l_p are displayed in Table 2. This information allows us to do the computations to complete the proof of Theorem 1.2. □

4. Proof of Theorem 1.4

In this section, we give a proof of Theorem 1.4. Before starting the proof, we need to introduce Ramanujan's theta function, given by

$$f(a,b) = \sum_{n=0}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2} = (-a,-b,ab;ab)_\infty,\quad |ab| < 1, \tag{4.1}$$

where the last identity in (4.1) is the celebrated Jacobi triple product [1, page 17, (1.4.8)]. Two important cases of $f(a,b)$ are

https://doi.org/10.1017/S0004972723000229 Published online by Cambridge University Press

$$\varphi(q) := f(q, q) = \sum_{n=0}^{\infty} q^{n^2} = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2 (q^4;q^4)_\infty^2}, \tag{4.2}$$

$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} q^{n(3n+1)/2} = (q;q)_\infty.$$

Replacing q by $-q$ in (4.2) yields

$$\varphi(-q) = \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty}.$$

The following p-dissections for $\varphi(-q)$ and $f(-q)$ play an important role in the proof of Theorem 1.4.

LEMMA 4.1. *Let $p \geq 5$ be a prime number. Then*

$$\varphi(-q) = \varphi(-q^{p^2}) + 2 \sum_{j=1}^{(p-1)/2} q^{j^2} f(-q^{p^2+2pj}, -q^{p^2-2pj}), \tag{4.3}$$

$$\begin{aligned} f(-q) = &\sum_{\substack{k=-(p-1)/2 \\ k \neq (\pm p-1)/6}}^{(p-1)/2} (-1)^k q^{k(3k+1)/2} f(-q^{(3p^2+(6k+1)p)/2}, -q^{(3p^2-(6k+1)p)/2}) \\ &+ (-1)^{(\pm p-1)/6} q^{(p^2-1)/24} f(-q^{p^2}), \end{aligned} \tag{4.4}$$

where $(\pm p - 1)/6$ is defined as in (1.10). Further, for $-(p-1)/2 \leq k \leq (p-1)/2$ and $k \neq (\pm p - 1)/6$,

$$\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod{p}.$$

PROOF. The identity (4.3) follows immediately from [3, page 49]. The identity (4.4) appears in [5, Theorem 2.2]. □

PROOF OF THEOREM 1.4. From (1.4), we find that

$$\sum_{n=0}^{\infty} Q(n) q^n = \frac{(q^2;q^2)_\infty}{(q;q)_\infty} = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty^4} \cdot \frac{(q;q)_\infty^3}{(q^2;q^2)_\infty} \equiv \varphi(-q) \cdot f(-q) \pmod{4}. \tag{4.5}$$

For a prime $p \geq 5$, $0 \leq j \leq (p-1)/2$, $-(p-1)/2 \leq k \leq (p-1)/2$, assume that

$$j^2 + \frac{3k^2+k}{2} \equiv \frac{p^2-1}{24} \pmod{p},$$

which implies that

$$24j^2 + (6k+1)^2 \equiv 0 \pmod{p}.$$

https://doi.org/10.1017/S0004972723000229 Published online by Cambridge University Press

Since $(\frac{-24}{p}) = -1$, we get $j = 0$ and $k = (\pm p - 1)/6$. Substituting (4.3) and (4.4) into (4.5), we find that

$$\sum_{n=0}^{\infty} Q\Big(pn + \frac{p^2-1}{24}\Big)q^n \equiv (-1)^{(\pm p-1)/6}\varphi(-q^p)f(-q^p)$$
$$\equiv (-1)^{(\pm p-1)/6}\sum_{n=0}^{\infty} Q(n)q^{pn} \pmod 4,$$

where we have used (4.5) in the last congruence. The congruence (1.9) follows. This completes the proof of Theorem 1.4. □

5. Concluding remarks

One can use Lemma 3.1 to establish congruence relations satisfied by $Q(n)$ similar to (1.8) for other primes p. For example,

$$\sum_{n=0}^{\infty} Q(127n+672)q^n \equiv \sum_{n=0}^{\infty} Q(n)q^{127n} \pmod{2^3},$$
$$\sum_{n=0}^{\infty} Q(131n+715)q^n \equiv 43\sum_{n=0}^{\infty} Q(n)q^{131n} \pmod{2^7},$$
$$\sum_{n=0}^{\infty} Q(137n+782)q^n \equiv 71\sum_{n=0}^{\infty} Q(n)q^{137n} \pmod{2^8},$$
$$\sum_{n=0}^{\infty} Q(139n+805)q^n \equiv 803\sum_{n=0}^{\infty} Q(n)q^{139n} \pmod{2^{10}}.$$

However, the corresponding bound l_p will become much larger as p increases.

Merca [9] proved the following infinite family of congruences modulo 4 for $Q(n)$.

THEOREM 5.1. *Let $p \geq 5$ be a prime number such that $p \not\equiv 1 \pmod{24}$. Then for any $n \not\equiv 0 \pmod p$,*

$$Q\Big(pn + \frac{p^2-1}{24}\Big) \equiv 0 \pmod 4. \tag{5.1}$$

The congruence (1.8) together with numerical evidence suggests the following conjecture, which contains (1.9) and (5.1) as special cases.

CONJECTURE 5.2. Let $p \geq 5$ be a prime number such that $p \not\equiv 1 \pmod{24}$. Then

$$\sum_{n=0}^{\infty} Q\Big(pn + \frac{p^2-1}{24}\Big)q^n \equiv c_p\sum_{n=0}^{\infty} Q(n)q^{pn} \pmod 4,$$

where $c_p = -1$ or 1.

https://doi.org/10.1017/S0004972723000229 Published online by Cambridge University Press

References

[1] G. E. Andrews and B. C. Berndt, *Ramanujan's Lost Notebook, Part II* (Springer-Verlag, New York, 2005).
[2] A. O. L. Atkin, 'Proof of a conjecture of Ramanujan', *Glasg. Math. J.* **8** (1967), 14–32.
[3] B. C. Berndt, *Ramanujan's Notebook, Part III* (Springer-Verlag, New York, 1991).
[4] B. C. Berndt and K. Ono, 'Ramanujan's unpublished manuscipt on the partition and tau functions with proofs and commentary', *Sém. Lothar. Combin.* **42** (1999), Article no. B42c.
[5] S.-P. Cui and N. S. S. Gu, 'Arithmetic properties of ℓ-regular partitions', *Adv. in Appl. Math.* **51**(4) (2013), 507–523.
[6] F. Diamond and J. Shurman, *A First Course in Modular Forms*, Graduate Texts in Mathematics, 228 (Springer-Verlag, New York, 2005).
[7] L. Euler, *Introductio in Analysin Infinitorum*, Vol. 2 (MM Bousquet Hanson, Lausanne, 1748).
[8] B. Gordon and K. Ono, 'Divisibility of certain partition functions by powers of primes', *Ramanujan J.* **1**(1) (1997), 25–34.
[9] M. Merca, 'Ramanujan-type congruences modulo 4 for partitions into distinct parts', *An. Şt. Univ. Ovidius Constanţa* **30**(3) (2022), 185–199.
[10] K. Ono, *The Web of Modularity: Arithmetic of the Coefficients of Modular Forms and q-Series*, CBMS Regional Conference Series in Mathematics, 102 (American Mathematical Society, Providence, RI, 2004).
[11] P. Paule and S. Radu, 'Partition analysis, modular functions, and computer algebra', in: *Recent Trends in Combinatorics*, IMA Volumes in Mathematics and its Applications, 159 (eds. A. Beveridge, J. R. Griggs, L. Hogben, G. Musiker and P. Tetali) (Springer, Cham, 2016), 511–543.
[12] C.-S. Radu, 'An algorithmic approach to Ramanujan–Kolberg identities', *J. Symbolic Comput.* **68**(1) (2015), 225–253.
[13] N. A. Smoot, 'On the computation of identities relating partition numbers in arithmetic progressions with eta quotients: an implementation of Radu's algorithm', *J. Symbolic Comput.* **104** (2021), 276–311.
[14] J. Sturm, 'On the congruence of modular forms', in: *Number Theory* (eds. D. V. Chudnovsky, G. V. Chudnovsky, H. Cohn and M. B. Nathanson) (Springer, Berlin, 1987) 275–280.
[15] G. N. Watson, 'Ramanujan's vermutung über zerfällungsanzahlen', *J. reine angew. Math.* **179** (1938), 97–128.

JULIA Q. D. DU, School of Mathematical Sciences,
Hebei International Joint Research Center for Mathematics and Interdisciplinary Science, Hebei Normal University, Shijiazhuang 050024, PR China
e-mail: qddu@hebtu.edu.cn

DAZHAO TANG, School of Mathematical Sciences,
Chongqing Normal University, Chongqing 401331, PR China
e-mail: dazhaotang@sina.com