On the general solution of Mathieu's equation.

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§ 1. *Description of paper.*

The differential equation of Mathieu, or "equation of the elliptic cylinder functions,"

$$\frac{d^2y}{dz^2} + (a + 16q\cos 2z)y = 0 \qquad\qquad (1)$$

occurs in many physical and astronomical problems. From the general theory of linear differential equations, we learn that its solution is of the type

$$y = \mathrm{A}e^{\mu z}\phi(z) + \mathrm{B}e^{-\mu z}\psi(z)$$

where A and B denote arbitrary constants, μ is a constant depending on the constants a and q of the differential equation, and $\phi(z)$ and $\psi(z)$ are periodic functions of z. For certain values of a and q the constant μ vanishes, and the solution y is then a purely periodic function of z; but in general μ is different from zero.

While the general character of the solution from the function-theory point of view is thus known, its actual analytical determination presents great difficulties. The chief impediment is that the constant μ cannot readily be found in terms of a and q. It was in order to determine μ that Hill invented the celebrated method known by his name, which introduced infinite determinants into analysis. But it is much to be desired that μ should be found in some more manageable form than as a root of a determinant with an infinite number of rows and columns.

It has occurred to me as possible that the difficulty may be overcome by introducing another parameter in place of a. This suggestion may be illustrated by reference to a well-known feature of the theory of elliptic functions. In elliptic functions the parameter used by the earlier workers was the modulus k; but expansions in terms of k are awkward and unmanageable, and it was shown by Jacobi that the true parameter of the subject is the quantity $exp\left(-\frac{\pi K'}{K}\right)$: the introduction of this made possible the theory of the theta-functions and q-series, which furnish the most satisfactory expansions of elliptic functions.

In the present paper it is shown that the solution of Mathieu's equation (1) can be effected by introducing in place of a a new parameter which will be denoted by σ. The parameter μ whose value is required, and the parameter a itself, will be expressed in terms of σ and the parameter q: so that when a and q are given we can first find σ from them, and then find μ from σ and q, and ultimately obtain the solution y of the equation.

§ 2. *The solution.*

As already mentioned, there are certain values of a for which the solution of Mathieu's equation is purely periodic; in particular, if

$$a = 1 + 8q - 8q^2 - 8q^3 + \dots$$

then the equation (1) has a solution

$$y = \sin z + q\sin 3z + q^2(\sin 3z + \tfrac{1}{3}\sin 5z) + q^3(\tfrac{1}{3}\sin 3z + \tfrac{4}{9}\sin 5z + \tfrac{1}{18}\sin 7z) + \dots;$$

and if

$$a = 1 - 8q - 8q^2 + 8q^3 + \dots$$

then the equation (1) has a solution

$$y = \cos z + q\cos 3z + q^2(-\cos 3z + \tfrac{1}{3}\cos 5z) + q^3(\tfrac{1}{3}\cos 3z - \tfrac{4}{9}\cos 5z + \tfrac{1}{18}\cos 7z) + \dots$$

The form of these series suggests that they may be degenerate cases of a general solution of Mathieu's equation, having the form

$$y = e^{\mu z}u$$

where

$$u = \sin(z-\sigma) + a_3\cos(3z-\sigma) + b_3\sin(3z-\sigma) + a_5\cos(5z-\sigma) + b_5\sin(5z-\sigma) + \dots,$$

where σ is a new parameter. The first of the two special solutions above would then correspond to $\sigma = 0$, and the second of them to $\sigma = \frac{\pi}{2}$.

It will be observed that there is no term in $\cos(z-\sigma)$; this really constitutes the definition of σ, and the possibility of obtaining series which remain convergent for all real values of σ

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depends on our choosing σ in this way. The coefficient of $\sin(z-\sigma)$ is taken to be unity, which amounts to fixing the arbitrary constant by which the solution is multiplied.

Let us then try to satisfy the differential equation (1) by an expression of this form, the parameters μ and a being expressed in terms of σ and q by series proceeding in ascending powers of q, thus:—

$$\mu = q\kappa(\sigma) + q^2\lambda(\sigma) + q^3\mu(\sigma) + q^4\nu(\sigma) + \dots$$
$$a = 1 + q\alpha(\sigma) + q^2\beta(\sigma) + q^3\gamma(\sigma) + q^4\delta(\sigma) + \dots$$

We shall write

$$u = \sin(z-\sigma) + qA(z, \sigma) + q^2B(z, \sigma) + q^3C(z, \sigma) + \dots$$

where, as we have seen, there are to be no terms in $\sin(z-\sigma)$ or $\cos(z-\sigma)$ in $A(z, \sigma)$, $B(z, \sigma)$, $C(z, \sigma)$, ...

Substituting these expansions in the differential equation, we obtain

$$\begin{aligned} &-\sin(z-\sigma) + qA'' + q^2B'' + q^3C'' + \dots \\ &+ 2(q\kappa + q^2\lambda + q^3\mu + \dots)\{\cos(z-\sigma) + qA' + q^2B' + q^3C' + \dots\} \\ &+ \{(q\kappa + q^2\lambda + q^3\mu + \dots)^2 + 1 + q\alpha + q^2\beta + q^3\gamma + \dots + 16q\cos 2z\} \\ &\qquad\qquad \{\sin(z-\sigma) + qA + q^2B + q^3C + \dots\} = 0. \end{aligned}$$

Equating to zero the terms independent of q in this equation, we have a mere identity. Equating to zero the terms involving the first power of q, we have

$$A'' + A + 2\kappa\cos(z-\sigma) + \alpha\sin(z-\sigma) + 16\cos 2z\sin(z-\sigma) = 0$$

or

$$\begin{aligned} A'' + A + (2\kappa - 8\sin 2\sigma)\cos(z-\sigma) + (\alpha - 8\cos 2\sigma)\sin(z-\sigma) \\ + 8\sin(3z-\sigma) = 0. \end{aligned}$$

Since A is not to contain any terms in $\cos(z-\sigma)$ or $\sin(z-\sigma)$, this gives at once the three equations

$$\alpha = 8\cos 2\sigma$$
$$\kappa = 4\sin 2\sigma$$
$$A = \sin(3z-\sigma).$$

Next equating to zero the terms involving the square of q, we have

$$\begin{aligned} B'' + B + 2\kappa A' + 2\lambda\cos(z-\sigma) + \kappa^2\sin(z-\sigma) + \alpha A + \beta\sin(z-\sigma) \\ + 16A\cos 2z = 0 \end{aligned}$$

or

$$\begin{aligned} &B'' + B + 2\lambda\cos(z-\sigma) + (16\sin^2 2\sigma + 8 + \beta)\sin(z-\sigma) \\ &\qquad + 24\sin 2\sigma\cos(3z-\sigma) + 8\cos 2\sigma\sin(3z-\sigma) + 8\sin(5z-\sigma) = 0. \end{aligned}$$

Applying the conditions that B is not to contain any terms in $\sin(z-\sigma)$ or $\cos(z-\sigma)$, this gives

$$\lambda = 0$$

$$\beta = -16 + 8\cos 4\sigma$$

$$B = 3\sin 2\sigma\cos(3z-\sigma) + \cos 2\sigma\sin(3z-\sigma) + \tfrac{1}{3}\sin(5z-\sigma).$$

The coefficients of the higher powers of q in the series can be found in the same way.

We thus obtain the following theorem :

Mathieu's equation (1) *is satisfied by a function which we shall denote by* $\Lambda(z)$, *which can be expressed in the form*

$$\Lambda(z) = e^{\mu z}u(z),$$

where $u(z)$ *is a purely periodic function of* z, *and* μ *is given by the power-series in* q

$$\mu = 4q\sin 2\sigma - 12q^3\sin 2\sigma - 12q^4\sin 4\sigma + \ldots ;$$

in this equation σ *is a parameter connected with the parameters* a *and* q *of the differential equation by the relation*

$$a = 1 + 8q\cos 2\sigma + (-16 + 8\cos 4\sigma)q^2 - 8q^3\cos 2\sigma + (\tfrac{256}{3} - 88\cos 4\sigma)q^4 + \ldots ;$$

and $u(z)$ *is given by the Fourier series*

$$u(z) = \sin(z-\sigma) + a_3\cos(3z-\sigma) + b_3\sin(3z-\sigma) + a_5\cos(5z-\sigma) + b_5\sin(5z-\sigma) + a_7\cos(7z-\sigma) + \ldots$$

where the coefficients are given by the power-series in q

$$b_3 = q + q^2\cos 2\sigma + (-\tfrac{14}{3} + 5\cos 4\sigma)q^3 + (-\tfrac{74}{9}\cos 2\sigma + 7\cos 6\sigma)q^4 + \ldots$$

$$a_3 = 3q^2\sin 2\sigma + 3q^3\sin 4\sigma + (-\tfrac{274}{9}\sin 2\sigma + 9\sin 6\sigma)q^4 + \ldots$$

$$b_5 = \tfrac{1}{3}q^2 + \tfrac{4}{9}q^3\cos 2\sigma + (-\tfrac{155}{54} + \tfrac{82}{27}\cos 4\sigma)q^4 + \ldots$$

$$a_5 = \tfrac{14}{9}q^3\sin 2\sigma + \tfrac{44}{27}q^4\sin 4\sigma + \ldots$$

$$b_7 = \tfrac{1}{18}q^3 + \tfrac{1}{12}q^4\cos 2\sigma + \ldots$$

$$a_7 = \tfrac{35}{108}q^4\sin 2\sigma + \ldots$$

$$b_9 = \tfrac{1}{180}q^4 + \ldots$$

....

Moreover, since a is an even function of σ, it is evident that *a second solution of the equation is obtained by merely changing* σ *to* $-\sigma$ *in the above formulae*: so that *the general solution of Mathieu's equation (1) is*

$$y = A\Lambda(z, \sigma, q) + B\Lambda(z, -\sigma, q)$$

where A *and* B *are arbitrary constants.*

When σ has either of the special values 0 and $\frac{\pi}{2}$, μ vanishes and one solution of Mathieu's equation is purely periodic: the solution $\Lambda(z, -\sigma, q)$ then ceases to be distinct from $\Lambda(z, \sigma, q)$, and we have the "logarithmic case" as with Bessel functions.

The above series are better adapted for numerical calculation than the infinite determinant.

§ 3. *The general terms of the series.*

Although for purposes of practical computing it is not a matter of much importance to have expressions for the general terms of the series, it may be of interest to determine some of them.

We shall first obtain the constants μ and a in a finite form in terms of a_3 and b_3. Substitute

$$y = e^{\mu z}\{\sin(z-\sigma) + a_3\cos(3z-\sigma) + b_3\sin(3z-\sigma) + \ldots\}$$

in the differential equation (1), and equate to zero the coefficient of $\cos(z-\sigma)$ in the equation so obtained: we thus find

$$\mu = 4q\sin2\sigma - 4qa_3.$$

Similarly equating to zero the coefficient of $\sin(z-\sigma)$, we have

$$a = 1 + 8q\cos2\sigma - \mu^2 - 8qb_3.$$

Thus μ and a are expressed in a finite form in terms of a_3 and b_3.

Next we shall show how the coefficients a_{2r+1} and b_{2r+1} may themselves be determined. It is obvious from what has preceded that b_{2r+1} and a_{2r+1} will be of the form

$$b_{2r+1} = \mathrm{A}_r q^r + \mathrm{B}_r q^{r+1}\cos2\sigma + (\mathrm{C}_r + \mathrm{D}_r\cos4\sigma)q^{r+2} + \ldots$$

$$a_{2r+1} = \mathrm{K}_r q^{r+1}\sin2\sigma + \mathrm{L}_r q^{r+2}\sin4\sigma + (\mathrm{M}_r\sin2\sigma + \mathrm{N}_r\sin6\sigma)q^{r+3} + \ldots$$

where A_r, B_r, ..., K_r, L_r, ... are purely numerical coefficients, independent of q and σ.

But if we equate to zero the coefficients of $\sin\{(2r+1)z-\sigma\}$ in the equation last used, we have

$$\{-(2r+1)^2 + \mu^2 + a\}b_{2r+1} - 2\mu(2r+1)a_{2r+1} + 8q(b_{2r-1} + b_{2r+3}) = 0$$

or

$$\{-4r(r+1) + 8q\cos2\sigma - 8qb_3\}b_{2r+1} - 8q(2r+1)(\sin2\sigma - a_3)a_{2r+1} + 8q(b_{2r-1} + b_{2r+3}) = 0$$

and similarly by equating to zero the coefficient of $\cos\{(2r+1)z-\sigma\}$, we have

$$\{-4r(r+1) + 8q\cos2\sigma - 8qb_3\}a_{2r+1} + 8q(2r+1)(\sin2\sigma - a_3)b_{2r+1} + 8q(a_{2r-1} + a_{2r+3}) = 0.$$

These are linear difference equations to determine a_{2r+1} and b_{2r+1}; they may evidently be combined into the single linear difference-equation

$$\{-4r(r+1)+8q\cos 2\sigma-8qb_3-8qi(2r+1)(\sin 2\sigma-a_3)\}x_{2r+1} \\ +8q(x_{2r-1}+x_{2r+3})=0$$

to determine the complex coefficient $x_{2r+1}=b_{2r+1}+i\,a_{2r+1}$.

By aid of these formulae the coefficients which have been denoted by A_r, B_r, ...K_r, ... above can be determined: we find for instance that

$$A_r=\frac{2^r}{(r+1)!\,r!},\quad B_r=\frac{2^{2r+1}}{(r+1)!\,(r+1)!}$$

$$K_r=\frac{2^{r+1}}{r!\,(r+1)!}\left\{\frac{1}{r+1}+\frac{2}{r}+\frac{2}{r-1}+\frac{2}{r-2}+\dots+\frac{2}{3}+\frac{2}{2}+1\right\}.$$

The series in the last equation can, as is well known, be summed by means of the logarithmic derivate of the Gamma-Function.