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POSITIVE SOLUTIONS TO A NON-RADIAL SUPERCRITICAL KLEIN–GORDON-TYPE EQUATION

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Abstract We find positive solutions to a nonlinear equation of Klein–Gordon type. Our analysis is carried out by truncating the related functional and estimating mountain pass solutions by Moser's iterative scheme.

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1. Introduction

In this paper we deal with the equation

$$-\Delta_m u + |u|^{m-2}u - a(x)(\lambda|u|^{p-1}u - |u|^{q-1}u) = 0 \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $N \geqslant 3$, $1 < m < N$, $m-1 < p < m^* - 1 \leqslant q$, $m^* = mN/(N-m)$ and λ is a positive parameter. We assume throughout this note that

$$0 \leqslant a(x) < K \quad \text{for } x \in \mathbb{R}^N \text{ and } K > 0 \text{ a constant}, \tag{1.2}$$

$$a(x) > 0 \quad \text{for } x \in B \subset \mathbb{R}^N, \text{ where } B \text{ is a ball} \tag{1.3}$$

and

$$\lim_{|x|\to\infty} a(x) = 0. \tag{1.4}$$

A similar problem was treated in [**2**] where $m = 2$ and $a \equiv 1$ for the equation

$$-\Delta u - g(u) = 0 \quad \text{in } \mathbb{R}^N.$$

The nonlinearity g is subcritical, for instance, $g(u) = u^p - u^q - u$ with $1 < p, q < (N+2)/(N-2)$. The authors were able to handle more general nonlinearities $g(u)$ under suitable assumptions.

Also, in [**1**] the authors solve (1.1) with $m = 2$, $a \equiv 1$, $1 < p < (N+2)/(N-2)$ and $q > p$. This supercritical problem is treated in radial coordinates by minimizing $\int \frac{1}{2}|\nabla u|^2$ subject to the constraint $\int G(u) = 1$, where G is the primitive of g with $G(0) = 0$. They recover compactness by working in $H^1_{\mathrm{rad}}(\mathbb{R}^N)$ and using the Strauss lemma [**7**]. The Lagrange multiplier is ruled out by rescaling the equation, since $a \equiv 1$. Their result is sharp in the sense that they determine a constant $\lambda_0 > 0$ such that (1.1) has a positive (radial) solution if and only if $\lambda > \lambda_0$. Their approach also applies if one considers the m-Laplacian instead of the Laplacian operator. But if one incorporates a weight function $a(x)$ as in (1.1), even a radial one, then their techniques are not applicable, because there is no way to rescale the problem when one finds a constrained minimizer.

The functional setting to attack the supercritical problem (1.1) is by considering

$$J : W^{1,m}(\mathbb{R}^N) \cap L^{1+q}(\mathbb{R}^N) \to \mathbb{R}$$

defined by

$$J(u) = \frac{1}{m}\int (|\nabla u|^m + |u|^m) - \int a(x)\biggl(\frac{\lambda}{p+1}|u|^{p+1} - \frac{1}{q+1}|u|^{q+1}\biggr).$$

For ease of notation, all integrals throughout the paper are computed in $\mathbb{R}^N$, unless otherwise mentioned. The functional J has no clear geometry, for instance, it does not satisfy the 'mountain pass theorem' assumptions in $W^{1,m}(\mathbb{R}^N)$. Note that zero is not a local minimizer in the case $q > m^* - 1$. And, even if $q = m^* - 1$, there is a lack of compactness. Unlike [**1**, **2**], we cannot work with radial functions, since $a(x)$ may be non-radial. We then truncate the nonlinearity $g(x,u) = a(x)(\lambda |u|^{p-1}u - |u|^{q-1}u)$ in order to make the problem appropriately subcritical. The principal part, $-\Delta_m u$, of the equation makes difficult the obtainment of estimates, since we cannot bootstrap using the classical L^p theory involving linear elliptic operators. The new (truncated) functional $\hat{J}$ satisfies the mountain pass geometry. In this way, there is a Palais–Smale (PS) sequence. Since $a(x)$ decays to zero, this allows us to show that the sequence has non-trivial limit u, by [**5**]. u is then a critical point of $\hat{J}$. To conclude that u is indeed a solution of the original problem (1.1), we follow a Moser iterative scheme to find an L^∞ bound for u (see [**6**]). Actually, one concludes that $\|u\|_{L^\infty(\mathbb{R}^N)}$ is small for sufficiently large λ.

There are works related to ours where, say, non-autonomous subcritical problems are addressed. In [**3**] the authors found exponentially decaying solutions by a weighted-space approach to solving the equation. The equations considered in [**4**, **8**] are more closely related to ours, except for the fact that, in their context, $a(x)$ does not tend to zero at infinity.

2. Statements and proofs

We begin with a non-existence result. Henceforth, we assume that (1.2)–(1.4) hold.

Theorem 2.1. *There is a constant $\tilde{\lambda} > 0$ such that, for $\lambda \leqslant \tilde{\lambda}$, there is no non-trivial solution.*

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Proof of Theorem 2.1. Multiply Equation (1.1) by a possible positive solution u. Then

$$\begin{aligned}\int |\nabla u|^m &= \int a(x)(\lambda|u|^{p+1} - |u|^{q+1}) - |u|^m \\ &= \int_{u\leqslant\lambda^{1/(q-p)}} a(x)(\lambda|u|^{p+1} - |u|^{q+1}) \\ &\qquad\qquad - \int |u|^m + \int_{u>\lambda^{1/(q-p)}} a(x)(\lambda|u|^{p+1} - |u|^{q+1}) \\ &\leqslant \int_{u\leqslant\lambda^{1/(q-p)}} a(x)(\lambda|u|^{p+1} - |u|^{q+1}) - \int |u|^m \\ &\leqslant \int_{u\leqslant\lambda^{1/(q-p)}} K(\lambda|u|^{p+1} - |u|^{q+1}) - |u|^m.\end{aligned}$$

Let $h(\lambda, s) = \lambda K s^{p+1} - K s^{q+1} - s^m$ (see (1.2)). A simple calculation shows that there exists a $\tilde{\lambda}$ such that, for $0 \leqslant \lambda \leqslant \tilde{\lambda}$, $h(\lambda, s) \leqslant 0$ for every $s \geqslant 0$. Consequently,

$$\int |\nabla u|^m \leqslant \int_{u\leqslant\lambda^{1/(q-p)}} h(\lambda, u)\,\mathrm{d}x \leqslant 0.$$

Hence, $u \equiv 0$ for $\lambda \leqslant \tilde{\lambda}$. □

Remark 2.2. Note that the previous theorem is true for $1 < m < \infty$.

The existence result reads as follows.

Theorem 2.3. *There is a constant $\lambda^* > 0$ such that for $\lambda > \lambda^*$ there is a weak solution $u > 0$ belonging to $W^{1,m}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.*

Proof of Theorem 2.3. Define

$$f(\lambda, u) = \begin{cases} 0 & \text{if } u \leqslant 0, \\ \left(1 - \dfrac{u^{q-p}}{\lambda}\right) u^p & \text{if } 0 < u \leqslant \varepsilon, \\ \left(1 - \dfrac{\varepsilon^{q-p}}{\lambda}\right) u^p & \text{if } u > \varepsilon. \end{cases} \tag{2.1}$$

We also assume that $\lambda > 1$, because this allows us to take ε in the interval $(0, 1)$ uniformly in λ. Here ε is chosen in such a way that

$$(1 - \varepsilon^{q-p}) u^p \leqslant f(\lambda, u) \leqslant u^p \tag{2.2}$$

and

$$\theta F(\lambda, u) \leqslant f(\lambda, u) u \quad \text{for } u > 0 \text{ for some constant } \theta > m, \tag{2.3}$$

where F is the primitive of f with $F(\lambda, 0) = 0$. We now study the modified equation

$$-\Delta_m u + |u|^{m-2} u - \lambda a(x) f(\lambda, u) = 0. \tag{2.4}$$

The corresponding functional $\hat{J} : W^{1,m}(\mathbb{R}^N) \to \mathbb{R}$ is defined by

$$\hat{J}(u) = \frac{1}{m}\int |\nabla u|^m + |u|^m - \int \lambda a(x) F(\lambda, u).$$

The functional $\hat{J}$ satisfies the mountain pass geometry. Hence there is a PS sequence $u_n \in W^{1,m}(\mathbb{R}^N)$, that is, $\hat{J}(u_n) \to c$ and $\hat{J}'(u_n) \to 0$. Our aim is to prove that a subsequence of u_n converges to some u, thus finding a critical point of $\hat{J}$. We need also to verify that $u \not\equiv 0$.

By multiplying Equation (2.4) by u_n, using the fact that $\hat{J}(u_n) \to c$ and by (2.3), we can conclude that $u_n \rightharpoonup u$ in $W^{1,m}(\mathbb{R}^N)$. Since $\hat{J}'(u_n) \to 0$, by (2.2) one has

$$\|u_n\|^m_{W^{1,m}(\mathbb{R}^N)} \leqslant C\|u_n\|^{p+1}_{L^{p+1}(\mathbb{R}^N)} + \delta_n \|u_n\|_{W^{1,m}(\mathbb{R}^N)},$$

where $\delta_n \to 0$. By a variant of the result of [**5**], there is a sequence x_n such that

$$\int_{B_1(x_n)} |u_n|^m > \bar{c} > 0$$

for some constant $\bar{c}$. Passing to a subsequence if necessary, there are only two cases to analyse. If there is a constant $K > 0$ such that $|x_n| \leqslant K$, then there is a sufficiently large R such that

$$\int_{B_R(0)} |u_n|^m > \bar{c} > 0,$$

implying that $u \not\equiv 0$. On the other hand, we may have $|x_n| \to \infty$. In this case, let $v_n(x) = u_n(x + x_n)$. Hence, $v_n \rightharpoonup v$ and

$$\int_{B_1(0)} |v_{n_k}|^m > \bar{c} > 0;$$

thus, $v \not\equiv 0$. By (1.4), v satisfies

$$-\Delta_m v + |v|^{m-2}v = 0 \text{ in } \mathbb{R}^N \quad \text{and} \quad v \in W^{1,m}(\mathbb{R}^N), \tag{2.5}$$

implying that $v \equiv 0$ in $\mathbb{R}^N$, which constitutes a contradiction.

Moreover, since $u \not\equiv 0$ by (2.1) and (2.4), we conclude that $u > 0$ in $\mathbb{R}^N$.

By Lemma 2.5 below, the norm $\|u\|_{L^\infty(\mathbb{R}^N)}$ is small for sufficiently large λ. Thus, u is indeed a positive solution of the original problem (1.1). □

Remark 2.4. It is an open question as to whether $\tilde{\lambda} = \lambda^*$. In general $\tilde{\lambda}$ depends on m; for $m = 2$ and $a \equiv 1$ the equality $\tilde{\lambda} = \lambda^*$ is valid (see [**1**]).

Lemma 2.5. *Let u be a mountain pass solution of (1.1). Then*

$$\|u\|_{L^\infty(\mathbb{R}^N)} \leqslant C\lambda^{-1/(p+1-m)}$$

for some constant $C > 0$ independent of λ and u.

Proof of Lemma 2.5. The critical value corresponding to a mountain pass solution u is given by

$$c = \min_{\gamma\in\Gamma}\max_{t\in[0,1]} \hat{J}(\gamma(t)) = \hat{J}(u),$$
$$\Gamma = \{\gamma \in C^0([0,1], W^{1,m}(\mathbb{R}^N)) : \gamma(0) = 0,\ \gamma(1) = e\},$$

where $\hat{J}(0) = 0$ and $\hat{J}(e) \leqslant 0$. In order to choose some value e, let $\varphi \in W^{1,m}(\mathbb{R}^N)$ with $\operatorname{supp}(\varphi) \subset \{a > 0\}$ and $\varphi \geqslant 0$. Hence,

$$\begin{aligned}\hat{J}(t\varphi) &= t^m\|\varphi\|^m_{W^{1,m}(\mathbb{R}^N)} - \lambda\int_\Omega a(x)F(\lambda, t\varphi)\\ &\leqslant t^m\|\varphi\|^m_{W^{1,m}(\mathbb{R}^N)} - \frac{t^{p+1}}{p+1}\lambda(1-\varepsilon^{q-p})\int_\Omega a(x)\varphi^{p+1}.\end{aligned}$$

We choose $t = t_0$, in such a way that $\hat{J}(t_0\varphi) \leqslant 0$, so that

$$t_0^{p+1-m} = \frac{(p+1)\|\varphi\|^m_{W^{1,m}(\mathbb{R}^N)}}{\lambda(1-\varepsilon^{q-p})\int_\Omega a(x)\varphi^{p+1}}.$$

Let $e = t_0\varphi$. Then

$$\|e\|_{W^{1,m}(\mathbb{R}^N)} = t_0\|\varphi\|_{W^{1,m}(\mathbb{R}^N)} \leqslant \frac{k}{\lambda^{1/(p+1-m)}},$$

where $k > 0$ is a constant independent of λ. Hence,

$$\begin{aligned}c &\leqslant \max_{t>0}\hat{J}(t\varphi)\\ &\leqslant \max_{t>0}\left[t^m\|e\|^m_{W^{1,m}(\mathbb{R}^N)} - \frac{t^{p+1}}{p+1}\lambda(1-\varepsilon^{q-p})\int_\Omega a(x)|e|^{p+1}\right]\\ &\leqslant t_0^m\|e\|^m_{W^{1,m}(\mathbb{R}^N)} - \frac{t_0^{p+1}}{p+1}\lambda(1-\varepsilon^{q-p})\int_\Omega a(x)|e|^{p+1}\\ &\leqslant t_0^m\|e\|^m_{W^{1,m}(\mathbb{R}^N)} \leqslant \left(\frac{k_1}{\lambda^{1/(p+1-m)}}\right)^m,\end{aligned}$$

where k_1 is independent of λ.

Since u is a solution,

$$\left(\frac{1}{m} - \frac{1}{\theta}\right)\|u\|^m_{W^{1,m}(\mathbb{R}^N)} \leqslant \hat{J}(u) = c < \left(\frac{k_1}{\lambda^{1/(p+1-m)}}\right)^m$$

and

$$\|u\|_{L^{m^*}(\mathbb{R}^N)} \leqslant k\|u\|_{W^{1,m}(\mathbb{R}^N)} \leqslant \frac{k_1}{\lambda^{1/(p+1-m)}}. \tag{2.6}$$

We are going to perform a Moser iterative scheme (see, for example, [**6**]). Here we need to keep track of the dependence on λ.

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Note that $m < p+1 < m^*$ and, by (1.3) and (2.2),

$$-\Delta_m u + u^{m-1} \leqslant \lambda u^p \quad \text{in } \mathbb{R}^N. \tag{2.7}$$

For $M > 0$ and $k > 0$, define $v_M(y) = \inf\{u(y), M\}$ and $v(y) = (v_M(y))^{km+1}$. Using v as a test function in (2.7), we obtain

$$(km+1)\int v_M^{km}|\nabla v_M|^m + \int v_M^{m+km} \leqslant \lambda \int v_M^{p+km+1},$$

$$\frac{km+1}{(k+1)^m}\int |\nabla (v_M)^{k+1}|^m \leqslant \lambda \int v_M^{(k+1)m+p+1-m},$$

$$\frac{1}{c_1}\frac{km+1}{(k+1)^m}\biggl(\int |(v_M)^{(k+1)m^*}|\biggr)^{m/m^*} \leqslant \lambda \biggl(\int v_M^{(k+1)ml}\biggr)^{1/l}\biggl(\int v_M^{m^*}\biggr)^{(p+1-m)/m^*}.$$

From now on c_i, $i = 1,2,3,4$, denote positive constants independent of λ. Note that

$$l = \frac{m^*}{m^* - (p+1-m)}$$

and observe that $m < p+1 < m^*$, so that $0 < p+1-m < m^* - m$. Then $m < m^* - (p+1-m) < m^*$. From now on the norm spaces correspond to functions defined in $\mathbb{R}^N$. Thus,

$$\|v_M\|_{L^{m^*(k+1)}} \leqslant c_2^{1/(k+1)}\biggl(\frac{k+1}{(km+1)^{1/m}}\biggr)^{1/(k+1)} \lambda^{1/((1+k)m)}\|v_M\|_{L^{(k+1)ml}}\|v_M\|_{L^{m^*}}^{(p+1-m)/(m(k+1))}.$$

Letting $M \to \infty$ yields

$$\|u\|_{L^{m^*(k+1)}} \leqslant (c_2\lambda^{1/m}\|u\|_{L^{m^*}}^{(p+1-m)/m})^{1/(k+1)}\|u\|_{L^{(k+1)ml}}\biggl(\frac{k+1}{(km+1)^{1/m}}\biggr)^{1/(k+1)}.$$

Define k_1 as $(k_1+1)ml = m^*$. Note that $k_1 + 1 = m^*/ml > 1$ and therefore

$$\|u\|_{L^{m^*(k_1+1)}} \leqslant (c_2\lambda^{1/m}\|u\|_{L^{m^*}}^{(p+1-m)/m})^{1/(k_1+1)}\|u\|_{L^{m^*}}\biggl(\frac{k+1}{(km+1)^{1/m}}\biggr)^{1/(k_1+1)}.$$

By induction, we define $(k_n+1)ml = m^*(k_{n-1}+1)$. Then $k_n + 1 = (m^*/ml)^n$ and

$$\|u\|_{L^{m^*(k_n+1)}} \leqslant (c_2\lambda^{1/m}\|u\|_{L^{m^*}}^{(p+1-m)/m})^{\sum_{j=1}^n 1/(k_j+1)}\prod_{j=1}^n\biggl(\biggl(\frac{k_j+1}{(k_jm+1)^{1/m}}\biggr)^{1/\sqrt{k_j+1}}\biggr)^{1/\sqrt{k_j+1}}\|u\|_{L^{m^*}}$$

and hence

$$\|u\|_{L^{m^*(k_n+1)}} \leqslant (c_2\lambda^{1/m}\|u\|_{L^{m^*}}^{(p+1-m)/m})^{\sum_{j=1}^n 1/(k_j+1)} c_3^{\sum_{j=1}^n 1/\sqrt{k_j+1}}\|u\|_{L^{m^*}}.$$

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Since

$$\sum_{j=1}^{\infty} \frac{1}{k_j+1} = \sum_{j=1}^{\infty} \left(\frac{ml}{m^*}\right)^j = \frac{1}{1-(ml/m*)} - 1 = \frac{m^*}{m^*-ml} - 1 = \frac{m}{m^*-(p+1)},$$

we obtain

$$\begin{aligned}\|u\|_{L^\infty} &\leqslant c_4 \lambda^{1/(m^*-(p+1))} \|u\|_{L^{m^*}}^{(p+1-m)/(m^*-(p+1))+1} \\ &= c_4 \lambda^{1/(m^*-(p+1))} \|u\|_{L^{m^*}}^{(m^*-m)/(m^*-(p+1))}.\end{aligned}$$

By (2.6), we obtain the desired estimate. □

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