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ON CERTAIN APPLICATIONS OF THE KHUKHRO–MAKARENKO THEOREM

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Abstract. Some recent results of Khukhro and Makarenko on the existence of characteristic $\mathfrak{X}$-subgroups of finite index in a group G, for certain varieties $\mathfrak{X}$, are used to obtain generalisations of some well-known results in the literature pertaining to groups G, in which all proper subgroups satisfy some condition or other related to the property 'soluble-by-finite'. In addition, a partial generalisation is obtained for the aforementioned results on the existence of characteristic subgroups.

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1. Introduction. Let F be a free group of countable rank with basis $\{x_1, x_2, \dots\}$. Then an outer commutator word of weight 1 is x_1, and an outer commutator word ω of weight $t > 1$ is a word of the form

$$\omega(x_1, \dots, x_t) = [u(x_1, \dots, x_r), v(x_{r+1}, \dots, x_t)],$$

where u, v are outer commutator words of weight r, $t - r$ respectively. Let ω be an outer commutator word of weight t. We denote by $\mathfrak{X}_\omega$ the class of groups G satisfying $\omega(g_1, \dots, g_t) = 1$ for all $g_1, \dots, g_t \in G$, i.e. $\omega(G) = 1$.

Some recent results of Khukhro and Makarenko (see especially Lemma 2.1) establish that, for certain group-theoretic properties $\mathfrak{Y}$, the existence of an $\mathfrak{Y}$-subgroup H of finite index in a group G ensures that there is a *characteristic* $\mathfrak{Y}$-subgroup C of finite index in G. In the present paper we shall use these results to obtain generalisations of some well-known results on groups G, in which all proper subgroups satisfy certain conditions, in several cases the condition in question being either 'almost in the variety $\mathfrak{X}_\omega$' for some outer commutator word ω (see for example Theorem 2.4) or '$\mathfrak{X}_\omega$-by-Chernikov' (see for example Theorem 2.5). We shall also obtain a generalisation of a result on barely transitive p-groups (see Theorem 2.3). Recall that a group of permutations G of an infinite set Ω is called a *barely transitive* group if G acts transitively on Ω and every orbit of every proper subgroup is finite. Equivalently, G is barely transitive if G has a subgroup H such that $|G : H|$ is infinite, $\bigcap_{g\in G} H^g = 1$ and $|K : K \cap H|$ is finite for every proper subgroup K of G, where the subgroup H is called a *point stabiliser*. Finally, in Section 4 of the paper, we obtain some partial generalisations of the Khukhro–Makarenko results.

We shall use the following notation for the given classes of groups.

$\mathfrak{A}$: Abelian groups,
$\mathfrak{N}$: Nilpotent groups,
$\mathfrak{S}$: Soluble groups,
$\mathfrak{S}_d$: Soluble groups of derived length at most d,
$\mathfrak{C}$: Chernikov groups,
$\mathfrak{R}$: Groups of finite (Prüfer) rank,
$\mathfrak{F}$: Finite groups,
$\mathfrak{D}$: Divisible (radicable) groups,
$\mathfrak{T}$: Periodic groups,
$\mathfrak{L}$: $(\mathfrak{T} \cap \mathfrak{D} \cap \mathfrak{A})$-groups.

We also denote the class of all $\mathfrak{X}$-by-$\mathfrak{Y}$-groups by $\mathfrak{X}\mathfrak{Y}$, and $\mathfrak{X}\mathfrak{X}$-groups by $\mathfrak{X}^2$.

2. $\mathfrak{X}_\omega\mathfrak{C}$-groups.

We will use the following very useful result, referred to here as the Khukhro–Makarenko theorem.

LEMMA 2.1 ([**9**, Theorem 1], [**11**, Theorem 1] or [**13**]). *If a group G has a subgroup H of finite index n satisfying the identity $\chi(H) = 1$, where χ is an outer commutator word of weight w, then G has also a characteristic subgroup C of finite (n, w)-bounded index satisfying the same identity $\chi(C) = 1$.*

Before we give an application of Lemma 2.1, we prove the following lemma.

LEMMA 2.2. *Let G be a group and let ω be an outer commutator word of weight $t \geq 2$; then $G^{(t-1)} \leq \omega(G)$. In particular,*

(i) *if $\omega(G) = 1$, then G is in $\mathfrak{S}_{t-1}$, i.e. $\mathfrak{X}_\omega \leq \mathfrak{S}_{t-1}$,*
(ii) *if G is a perfect group, then $\omega(G) = G$.*

Proof. We proceed by induction on t. If $t = 2$, then $G^{(t-1)} = G^{(1)} = G' = \omega(G)$. Now assume that $t \geq 3$; then there exist outer commutator words σ, τ of weight $1 \leq t_1, t_2 < t$, respectively, such that $t = t_1 + t_2$ and $\omega = [\sigma, \tau]$, and then $\omega(G) = [\sigma(G), \tau(G)]$. By induction hypothesis, we have $G^{(t_1-1)} \leq \sigma(G)$ and $G^{(t_2-1)} \leq \tau(G)$. Put $m = \max\{t_1, t_2\}$, then

$$G^{(m)} = [G^{(m-1)}, G^{(m-1)}] \leq [G^{(t_1-1)}, G^{(t_2-1)}] \leq [\sigma(G), \tau(G)] = \omega(G).$$

Clearly $t_1 + t_2 \geq m + 1$ and thus $t - 1 \geq m$. So $G^{(t-1)} \leq G^{(m)} \leq \omega(G)$ and the induction is complete.

(i) If $\omega(G) = 1$, then $G^{(t-1)} = 1$. So G is in $\mathfrak{S}_{t-1}$.

(ii) Assume that G is a perfect group. Since $G^{(t-1)} \leq \omega(G)$, we have $G^{(t-1)} = G$, and hence $G = \omega(G)$, as desired. □

As an application of the Khukhro–Makarenko theorem we present the following result.

THEOREM 2.3. *Let G be a locally finite barely transitive p-group with a point stabiliser H and let ω be an outer commutator word of weight t. If $H \in \mathfrak{X}_\omega$, then $G' \neq G$ and $G' \in \mathfrak{X}_\omega$.*

Proof. Let N be a proper normal subgroup of G; then $N \cap H \in \mathfrak{X}_\omega$. Since $|N : N \cap H|$ is finite, by Lemma 2.1, N has a characteristic subgroup $K \in \mathfrak{X}_\omega$ such that $N/K \in \mathfrak{F}$. It is well known that G has no proper subgroup of finite index, so $N/K \leq Z(G/K)$. It follows that $N' \leq K$ and that $N' \in \mathfrak{X}_\omega$. Since there exists a chain $\{N_i : i \in I\}$ of proper normal subgroups of G such that $G = \bigcup_{i \in I} N_i$, it follows that

$$G' = \bigcup_{i \in I} N_i'.$$

Consequently, we have $G' \in \mathfrak{X}_\omega$ and $G \neq G'$ by Lemma 2.2(ii). □

Theorem 2.3 generalises [**1**] and [**2**, Theorem 2], and by using Lemma 2.2(i) we can obtain the same results as those in [**1**] and [**2**, Theorem 2]. The structure of imperfect locally finite barely transitive groups is described in [**7**].

Let $v(x_1, \ldots, x_s)$ and $u(x_1, \ldots, x_t)$ be two words. Then the composite of v and u, $v \circ u$ is defined as follows:

$$v \circ u = v(u(x_1, \ldots, x_t), \ldots, u(x_{(s-1)t+1}, \ldots, x_{st})).$$

If v is an outer commutator word and u is a word, then it is well known that $v \circ u(G) = v(u(G))$ for any group G (see for example [**16**, Lemma 2.5]).

We will use this definition to describe the structure of certain groups.

Let $\mathfrak{Y}$ be a class of groups. Recall that a group G is called a minimal non-$\mathfrak{Y}$-group if every proper subgroup of G is a $\mathfrak{Y}$-group, but G itself is not. The minimal non-$\mathfrak{Y}$-groups are denoted by $MN\mathfrak{Y}$.

Now define the word θ as $\theta(x, y) = [x, y]$, which will be used in the sequel.

THEOREM 2.4. *Let G be an $MN\mathfrak{X}_\omega\mathfrak{F}$-group, where ω is an outer commutator word of weight $t > 1$. If G has no infinite simple images, then the following properties hold.*

(i) *G has no proper subgroup of finite index and no simple images.*
(ii) *$N' \in \mathfrak{X}_\omega$ for every proper normal subgroup N of G.*
(iii) *G is not perfect, $G \in \mathfrak{X}_\omega(\mathfrak{L} \cap \mathfrak{C})$ and $G' \in \mathfrak{X}_\omega$. In particular, $G \in \mathfrak{S}_t$.*
(iv) *$(\omega \circ \theta)(G) = 1$, i.e. $G \in \mathfrak{X}_{\omega\circ\theta}$.*

Proof. We first assume that G has a proper subgroup K of finite index. Since $K \in \mathfrak{X}_\omega\mathfrak{F}$, K has a normal subgroup $L \in \mathfrak{X}_\omega$ such that $K/L \in \mathfrak{F}$. Hence, $\mathrm{core}_G L \in \mathfrak{X}_\omega$ and has finite index in G, i.e. $G \in \mathfrak{X}_\omega\mathfrak{F}$. But this is a contradiction. So G has no proper subgroup of finite index and it has no simple images. Thus (i) holds.

Now let N be a proper normal subgroup of G. Since $N \in \mathfrak{X}_\omega\mathfrak{F}$, N has a characteristic subgroup $S \in \mathfrak{X}_\omega$ of finite index in N by Lemma 2.1. Put $\overline{G} := G/S$, then $\overline{G} = C_{\overline{G}}(\overline{N})$, since G/S has no proper subgroup of finite index and so we have $[G, N] \leq S$. Since $\mathfrak{X}_\omega$ is subgroup-closed, $N' \in \mathfrak{X}_\omega$, and thus (ii) holds.

Now assume that G is perfect. Since G has no simple images, it is a union of a chain of proper normal subgroups. If N is a proper normal subgroup of G, then $N' \in \mathfrak{X}_\omega$ by (ii) and so $G = G'$ is a union of $\mathfrak{X}_\omega$-groups. So $G \in \mathfrak{X}_\omega$, a contradiction.

Thus, G is not perfect and G/G' has a proper subgroup R/G' such that $G/R \in \mathfrak{L} \cap \mathfrak{C}$. Now by Lemma 2.1, R has a characteristic subgroup $W \in \mathfrak{X}_\omega$ such that $G/W \in \mathfrak{C}$. Since G/W has no proper subgroup of finite index, we have $G/W \in \mathfrak{L} \cap \mathfrak{C}$.

Consequently, $G \in \mathfrak{X}_\omega(\mathfrak{L} \cap \mathfrak{C})$. In particular, $G' \leq W$ and hence $G' \in \mathfrak{X}_\omega$. In particular, $G \in \mathfrak{S}_t$ by Lemma 2.2(i). So (iii) holds.

Finally, since $G' \in \mathfrak{X}_\omega$, we have $(\omega \circ \theta)(G) = \omega(G') = 1$, and (iv) holds. □

The following is the $\mathfrak{X}_\omega\mathfrak{C}$ version of Theorem 2.4.

THEOREM 2.5. *Let G be an $MN\mathfrak{X}_\omega\mathfrak{C}$-group. If G has no infinite simple images, then the following are satisfied.*

(i) *G has no proper subgroup of finite index and no simple images.*
(ii) *$N' \in \mathfrak{X}_{\omega\circ\theta}$ for every proper normal subgroup N of G, i.e. $N \in \mathfrak{X}_{\omega\circ\theta^2}$.*
(iii) *G is not perfect and $G \in \mathfrak{X}_{\omega\circ\theta}(\mathfrak{L} \cap \mathfrak{C})$. In particular, $G' \in \mathfrak{X}_{\omega\circ\theta}$ and $G \in \mathfrak{S}_{t+1}$.*

Proof. By a similar argument to that used in the proof of Theorem 2.4, G has no proper subgroup of finite index. So (i) holds.

Now let N be a proper normal subgroup of G, then it has a normal subgroup $S \in \mathfrak{X}_\omega$ such that $N/S \in \mathfrak{C}$. So N/S has a normal subgroup $R/S \in \mathfrak{L} \cap \mathfrak{C}$ such that $N/R \in \mathfrak{F}$. Since R/S is in $\mathfrak{A}$, $R \in \mathfrak{X}_{\omega\circ\theta}$. By Lemma 2.1 N has a characteristic subgroup $M \in \mathfrak{X}_{\omega\circ\theta}$ such that $N/M \in \mathfrak{F}$ and hence $N' \leq M$, i.e. $N' \in \mathfrak{X}_{\omega\circ\theta}$. So (ii) holds.

Suppose next that G has a non-trivial $\mathfrak{C}$-image G/N. Then N has a normal subgroup $S \in \mathfrak{X}_\omega$ such that $N/S \in \mathfrak{C}$ and N/S has a normal subgroup $M/S \in \mathfrak{L} \cap \mathfrak{C}$ such that $N/M \in \mathfrak{F}$. So $N \in \mathfrak{X}_{\omega\circ\theta}\mathfrak{F}$. By Lemma 2.1 N has a characteristic subgroup $T \in \mathfrak{X}_{\omega\circ\theta}$ with $N/T \in \mathfrak{F}$. This implies that $G/T \in \mathfrak{C}$, and hence $G/T \in \mathfrak{L} \cap \mathfrak{C}$ by (i) and $G \in \mathfrak{X}_{\omega\circ\theta}(\mathfrak{L} \cap \mathfrak{C})$ in this case.

Now if G is perfect, then as in the proof of Theorem 2.4, G is a union of proper normal subgroups and so we have $G \in \mathfrak{X}_{\omega\circ\theta^2}$, and hence $\omega(G) = 1$, a contradiction. So G is not perfect and G/G' has a proper normal subgroup R/G' such that $G/R \in \mathfrak{L} \cap \mathfrak{C}$. By the previous argument $G \in \mathfrak{X}_{\omega\circ\theta}(\mathfrak{L} \cap \mathfrak{C})$ and so $G' \in \mathfrak{X}_{\omega\circ\theta}$ and $G \in \mathfrak{S}_{t+1}$ by Lemma 2.2(i). Thus, (iii) holds. □

3. Applications to $MN\mathfrak{S}_n\mathfrak{C}$ and $MN\mathfrak{S}_n\mathfrak{F}$-groups. Since a group is in $\mathfrak{S}\mathfrak{C}$ if and only if it is in $\mathfrak{S}\mathfrak{F}$, we see that a group is in $MN\mathfrak{S}\mathfrak{C}$ if and only if it is in $MN\mathfrak{S}\mathfrak{F}$.

We know that the celebrated example of Heineken and Mohamed (see [**15**, Theorem 6.2.1]) is an $MN\mathfrak{A}\mathfrak{F}$-group which is in $\mathfrak{A}\mathfrak{C}$. So an $MN\mathfrak{S}_n\mathfrak{F}$-group (for a positive integer n) is not in general an $MN\mathfrak{S}_n\mathfrak{C}$-group.

The locally graded groups with all proper subgroups in $\mathfrak{S}\mathfrak{F}$ are classified by [**6**, Theorem C], as follows

THEOREM 3.1. *Let G be a locally graded group with all proper subgroups in $\mathfrak{S}\mathfrak{F}$. Then either*

(i) *G is locally soluble, or*
(ii) *$G \in \mathfrak{S}\mathfrak{F}$, or*
(iii) *G is $\mathfrak{S}$-by-$PSL(2, \mathfrak{F})$, or*
(iv) *G is $\mathfrak{S}$-by-$Sz(\mathfrak{F})$,*

where F is an infinite locally finite field with no infinite proper subfields.

By the remark above, we see that Theorem 3.1 also gives a classification of the locally graded groups with all proper subgroups in $\mathfrak{S}\mathfrak{C}$.

If G is a countable locally graded simple group with all subgroups in $\mathfrak{S}\mathfrak{F}$ (or in $\mathfrak{S}\mathfrak{C}$), then a super-inert subgroup R (see [**6**] for the definition) of G either has non-trivial Hirsch–Plokin radical or is in $\mathfrak{F}$, hence G is locally finite [**6**, Theorem 2]. So by [**12**]

G is isomorphic either to $PSL(2, \mathbb{F})$ or to $Sz(\mathbb{F})$ for some infinite locally finite field $\mathbb{F}$ containing no infinite proper subfield.

THEOREM 3.2. *There are infinite locally finite simple $MN\mathfrak{S}_2\mathfrak{F}$ and $MN\mathfrak{S}_3\mathfrak{F}$-groups.*

Proof. Let $G := PSL(2, \mathbb{F})$ or $G := Sz(\mathbb{F})$ for some infinite locally finite field $\mathbb{F}$ containing no infinite proper subfield. In the first case every proper subgroup is either in $\mathfrak{A}^2$ or in $\mathfrak{F}$ and so in $\mathfrak{S}_2\mathfrak{F}$ by [**4**, Example 3]. Clearly $G \notin \mathfrak{S}_2\mathfrak{F}$. So $G \in MN\mathfrak{S}_2\mathfrak{F}$.

In the second case every proper subgroup of G is in $\mathfrak{F}$ or is $\mathfrak{N}_2$-by-locally cyclic (i.e. in $\mathfrak{S}_3\mathfrak{F}$) by the proof of [**5**, Lemma 2]. Consequently, G is in $MN\mathfrak{S}_3\mathfrak{F}$. □

Let G be a group, H a subgroup of G; then the *isolator* $I_G(H)$ of H in G is defined as

$$I_G(H) = \{x \in G \mid \text{ there is a non-zero integer } n \text{ such that } x^n \in H\}.$$

We prove the following general lemma.

LEMMA 3.3. *Let ω be an outer commutator word of weight t, and let H be a subgroup of the locally nilpotent torsion-free group G. Then*

$$\omega(I_G(H)) \leq I_G(\omega(H)).$$

Proof. First let U and V be subgroups of G. Then with the notation of [**14**, Section 2.3] we have $I_G(U) \sim U$ and $I_G(V) \sim V$. It follows that $[I_G(U), I_G(V)] \sim [U, V]$ by [**14**, 2.3.5]. So we see that

$$[I_G(U), I_G(V)] \leq I_G([U, V]).$$

Now we proceed by induction on t. If $t = 1$, then the result is immediate. If $t > 1$, then $\omega = [\varphi, \delta]$ for some outer commutator words φ and δ of weights $1 \leq t_1 < t$, $1 \leq t_2 < t$ such that $t_1 + t_2 = t$. By induction hypothesis and the above remark, we have

$$\begin{aligned}\omega(I_G(H)) = [\varphi(I_G(H)), \delta(I_G(H))] &\leq [I_G(\varphi(H)), I_G(\delta(H))] \\ &\leq I_G([\varphi(H), \delta(H)]) = I_G(\omega(H)),\end{aligned}$$

and the proof is complete. □

THEOREM 3.4. *Let G be a locally nilpotent torsion-free group.*

(i) *If all proper subgroups of G are in $\mathfrak{X}_\omega\mathfrak{T}$, then $G \in \mathfrak{X}_\omega$.*

(ii) *If all proper subgroups of G are in $\mathfrak{X}_\omega\mathfrak{R}$, then $G \in \mathfrak{X}_\omega(\mathfrak{R} \cap \mathfrak{N})$. In particular, G is in $\mathfrak{S}$.*

Proof. (i) Let K be a proper subgroup of G. Then K has a normal subgroup $N \in \mathfrak{X}_\omega$ such that $K/N \in \mathfrak{T}$, and so $I_K(N) = K$. By Lemma 3.3

$$\omega(K) = \omega(I_K(N)) \leq I_K(\omega(N)) = I_K(1) = 1.$$

This means that every proper subgroup K of G is in $\mathfrak{X}_\omega$.

If G is not finitely generated, then every finitely generated subgroup of G is in $\mathfrak{X}_\omega$, and thus $G \in \mathfrak{X}_\omega$. Otherwise, G is finitely generated, and by [**18**, 5.2.21] it has a normal subgroup $H \in \mathfrak{X}_\omega$ of finite index, since it is nilpotent. Hence, $I_G(H) = G$ and as above $\omega(G) = 1$. So $G \in \mathfrak{X}_\omega$.

(ii) Assume for a contradiction that G is not in $\mathfrak{X}_\omega\mathfrak{R}$, and first suppose that G has a proper normal subgroup of N such that G/N is in $\mathfrak{R}$. Then N has a normal subgroup M such that $M \in \mathfrak{X}_\omega$ and $N/M \in \mathfrak{R}$. By [**10**, Theorem 3], we may assume that M is characteristic in N so that M is normal in G. Clearly G/M is in $\mathfrak{R}$, so $G \in \mathfrak{X}_\omega\mathfrak{R}$, a contradiction. Therefore, G has no proper images which are in $\mathfrak{R}$ and hence it is perfect.

Let H be any proper normal subgroup of G and let K be a characteristic subgroup of H such that $K \in \mathfrak{X}_\omega$ and $H/K \in \mathfrak{R}$. If T/K denotes the torsion subgroup of H/K, then $T \in \mathfrak{X}_\omega\mathfrak{T}$ and hence $T \in \mathfrak{X}_\omega$ by (i). Since torsion-free locally nilpotent $\mathfrak{R}$-groups are in $\mathfrak{N}$ [**18**, Theorem 6.36], we have $H/T \in \mathfrak{N}$. So H/T has a finite characteristic series whose factors are torsion-free $\mathfrak{A} \cap \mathfrak{R}$-groups. If U is such a factor, then $G/C_G(U)$ is nilpotent by [**17**, Part 2, Lemma 6.37] and hence $G = C_G(U)$ since G is perfect. We deduce that H/T is contained in the hypercentre of G/T, which equals the centre, as G is perfect. Thus, $H/T \leq Z(G/T)$ and so $H' \leq T$. We deduce that $H' \in \mathfrak{X}_\omega$. As before, since there exists a chain $\{N_i : i \in I\}$ of proper normal subgroups of G such that $G = \bigcup_{i \in I} N_i$, it follows that $G = G' = \bigcup_{i \in I} N_i'$. Consequently, we have $G \in \mathfrak{X}_\omega$, a contradiction. Therefore, $G \in \mathfrak{X}_\omega\mathfrak{R}$.

Let N be a normal subgroup of G such that $N \in \mathfrak{X}_\omega$ and $G/N \in \mathfrak{R}$. If T/N denotes the torsion subgroup of G/N, then again by (i) $T \in \mathfrak{X}_\omega$, and since G/T is a locally nilpotent torsion-free $\mathfrak{R}$-group, it is in $\mathfrak{N}$. Therefore, $G \in \mathfrak{X}_\omega(\mathfrak{R} \cap \mathfrak{N})$. By Lemma 2.2(i), we deduce that G is in $\mathfrak{S}$, as claimed. □

Let us define the outer commutator word ϕ_j for every $j \geq 0$ as follows:
$\phi_0(x) = x$ and for $i \geq 1$

$$\phi_i(x_1, \ldots, x_{2^i}) = [\phi_{i-1}(x_1, \ldots, x_{2^{i-1}}), \phi_{i-1}(x_{2^{i-1}+1}, \ldots, x_{2^i})].$$

Then G is in $\mathfrak{S}$ if and only if there is a positive integer n such that $\phi_n(G) = 1$.

THEOREM 3.5. *Let G be a group without infinite simple images. Then the following are satisfied.*

(i) *If every proper subgroup of G is in $\mathfrak{S}_n\mathfrak{F}$ for some fixed positive integer n, then either $G \in \mathfrak{S}_n\mathfrak{F}$ or $G \in \mathfrak{S}_n(\mathfrak{L} \cap \mathfrak{C})$. So if G is an $MN\mathfrak{S}_n\mathfrak{F}$-group, then $G \in \mathfrak{S}_n\mathfrak{C} \cap \mathfrak{S}_{n+1}$.*

(ii) *If every proper subgroup of G is in $\mathfrak{S}_n\mathfrak{C}$ for some fixed positive integer n, then $G \in \mathfrak{S}_n\mathfrak{C}$ or $G \in \mathfrak{S}_{n+1}(\mathfrak{L} \cap \mathfrak{C})$. So if G is an $MN\mathfrak{S}_n\mathfrak{C}$-group, then $G \in \mathfrak{S}_{n+1}\mathfrak{C} \cap \mathfrak{S}_{n+2}$.*

Proof. (i) Take $\omega = \phi_n$. If $G \notin \mathfrak{S}_n\mathfrak{F}$, then G is an $MN\mathfrak{S}_n\mathfrak{F}$-group. By Theorem 2.4 (iii) $G \in \mathfrak{X}_{\phi_n}(\mathfrak{L} \cap \mathfrak{C})$.

(ii) Again take $\omega = \phi_n$ so that $\omega \circ \theta = \phi_{n+1}$. If $G \notin \mathfrak{S}_n\mathfrak{C}$, then by Theorem 2.5 (iii) $G \in \mathfrak{X}_{\phi_{n+1}}(\mathfrak{L} \cap \mathfrak{C})$, and the proof is complete. □

The following lemma will be generalised in Section 4 (see Lemma 4.1), but since the ‘soluble’ version of the lemma is useful here, we shall prove it.

LEMMA 3.6 (c.f. [**3**, Proposition 1]). *Let G be in $\mathfrak{T} \cap \mathfrak{N}$, $A \in \mathfrak{S}_n$ $(n \geq 1)$ a normal subgroup of G such that $G/A \in \mathfrak{L}$. Then also $G \in \mathfrak{S}_n$.*

Proof. We proceed by induction on n. If $n = 1$, then A is in $\mathfrak{A}$ and hence $A \leq C_G(A)$. So $T := G/C_G(A) \in \mathfrak{L}$ is isomorphic to a subgroup of Aut A. By [**3**, Lemma 1], $A \leq Z(G)$, and by [**15**, Section 5.3.5] G is in $\mathfrak{A}$. Now let $n > 1$ and consider $G/A^{(n-1)}$.

Then

$$\frac{G/A^{(n-1)}}{A/A^{(n-1)}} \in \mathfrak{L} \text{ and } A/A^{(n-1)} \in \mathfrak{S}_{n-1}.$$

By induction hypothesis $G/A^{(n-1)} \in \mathfrak{S}_{n-1}$ and thus $G^{(n-1)} = A^{(n-1)}$. This implies that $G^{(n)} = 1$ and $G \in \mathfrak{S}_n$, as desired. □

THEOREM 3.7. *Let G be a locally graded $\mathfrak{T}$-group and suppose that every proper subgroup of G is in $\mathfrak{S}_n\mathfrak{C}$ for some fixed positive integer n. If G contains a normal $\mathfrak{N}$-subgroup N such that $G/N \in \mathfrak{C}$, then $G \in \mathfrak{S}_n\mathfrak{C}$.*

Proof. Assume for a contradiction that G is an $MN\mathfrak{S}_n\mathfrak{C}$-group. Since G has no proper subgroup of finite index, we have $G/N \in \mathfrak{L}$ or $G = N$. Hence, $G' \neq G$ and thus $1 \neq G/G' \in \mathfrak{L}$. If $N = G$, then we have the contradiction that G is in $\mathfrak{A}$ by [**18**, Section 5.2.5], since G is in $\mathfrak{T} \cap \mathfrak{N}$. So $N \neq G$ and hence N has a normal subgroup $S \in \mathfrak{S}_n$ such that $N/S \in \mathfrak{C}$. So N/S contains a maximal $\mathfrak{L}$-subgroup R/S such that $N/R \in \mathfrak{F}$. By Lemma 3.6, $R \in \mathfrak{S}_n$. Therefore, we can assume by Lemma 2.1 that R is characteristic in N and hence R is normal in G. So G/R is in $\mathfrak{C}$. Consequently, $G \in \mathfrak{S}_n\mathfrak{C}$, a contradiction, and the proof is complete. □

4. $\mathfrak{N}\mathfrak{C}$-groups with certain characteristic subgroups.

LEMMA 4.1. *Let G be in $\mathfrak{T} \cap \mathfrak{N}$, N a normal subgroup of G, ω an outer commutator word of weight $t \geq 2$ such that $\omega(N) = 1$. If G/N is in $\mathfrak{A} \cap \mathfrak{D}$, then $\omega(G) = 1$.*

Proof. We proceed by induction on t. If $t = 2$, then $\omega(x, y) = [x, y]$ and

$$\omega(N) = [N, N] = 1,$$

i.e. N is in $\mathfrak{A}$. So $G/C_G(N)$ is in $\mathfrak{A} \cap \mathfrak{D}$ and isomorphic to a subgroup of Aut N. By [**3**, Lemma 1] $G = C_G(N)$ and thus $N \leq Z(G)$. Applying [**18**, 5.3.5] we have that G is in $\mathfrak{A}$. Let $t > 2$; then $\omega = [\psi, \phi]$ for some outer commutator words ψ, ϕ of weight $1 \leq t_1, t_2 < t$ such that $t = t_1 + t_2$. Now G/N is in $\mathfrak{L}$ and $\psi(N/\psi(N)) = 1$. If $t_1 > 1$, then by induction hypothesis $\psi(G/\psi(N)) = 1$, i.e. $\psi(G) \leq \psi(N)$. Clearly $\psi(N) \leq \psi(G)$ and it follows that $\psi(G) = \psi(N)$. If also $t_2 > 1$, then similarly $\phi(G) = \phi(N)$, and we have

$$\omega(G) = [\psi(G), \phi(G)] = [\psi(N), \phi(N)] = 1,$$

as required. So we may assume that $t_2 = 1$ and hence $t_1 > 1$, since $t > 2$. (If $t_1 = 1$, then a similar argument works.) Then $\omega(N) = [\psi(N), N] = 1$ and hence $N \leq C_G(\psi(N))$. We also have that $\psi(N)$ is in $\mathfrak{A}$. Then $G/C_G(\psi(N))$ is in $\mathfrak{A} \cap \mathfrak{D}$ and isomorphic to a subgroup of Aut $\psi(N)$. So by [**3**, Lemma 1] $\psi(N) \leq Z(G)$; in other words $[\psi(N), G] = 1$. It follows that

$$\omega(G) = [\psi(G), G] = [\psi(N), G] = 1,$$

and the proof is complete. □

THEOREM 4.2. *Let G be a $\mathfrak{T}$-group and let $N \in \mathfrak{N}_c \cap \mathfrak{X}_\omega$ be a normal subgroup of G such that $G/N \in \mathfrak{C}$ for some outer commutator word ω. Then G contains a characteristic* *(even invariant under all surjective endomorphisms) subgroup* $S \in \mathfrak{N}_c \cap \mathfrak{X}_\omega$ *such that* $G/S \in \mathfrak{C}$.

Proof. Let $W := \langle N^\alpha \mid \alpha \in \mathrm{Aut}\, G\rangle$, then W is characteristic in G and $W/N \in \mathfrak{C}$. By [**8**, Lemma 4.7] W is in $\mathfrak{N}$. We also have that W/N has a normal $\mathfrak{A} \cap \mathfrak{D}$-subgroup $R/N \in \mathfrak{R}$ such that W/R is in $\mathfrak{F}$. Now by Lemma 4.1 we have $R \in \mathfrak{N}_c \cap \mathfrak{X}_\omega$. By Lemma 2.1 W has characteristic (even invariant under all surjective endomorphisms) subgroups $S_1 \in \mathfrak{N}_c$ and $S_2 \in \mathfrak{X}_\omega$ such that W/S_i is in $\mathfrak{F}$ for $i = 1,\ 2$. Put $S = S_1 \cap S_2$, then $|W : S| < \infty$ and S is contained in $\mathfrak{N}_c \cap \mathfrak{X}_\omega$. Since W is characteristic in G, we see that S is characteristic in G, and since $G/W \in \mathfrak{C}$ and W/S is finite, we have $G/S \in \mathfrak{C}$. The proof is complete. □

If we take $\omega = \gamma_{c+1}$, then

$$\mathfrak{N}_c \cap \mathfrak{X}_\omega = \mathfrak{N}_c \cap \mathfrak{N}_c = \mathfrak{N}_c.$$

Hence, we obtain the following result.

Corollary 4.3. *Let* G *be a* $\mathfrak{T}$*-group and let* $N \in \mathfrak{N}_c$ *be a normal subgroup of* G *such that* $G/N \in \mathfrak{C}$. *Then* G *contains a characteristic (even invariant under all surjective endomorphisms) subgroup* $S \in \mathfrak{N}_c$ *such that* $G/S \in \mathfrak{C}$.

Corollary 4.3 sharpens [**8**, Lemma 4.7] and generalises [**3**, Lemma 3] and [**9**, Corollary 1(i)] in the periodic case.

In [**8**, p. 321] Hartley gives an example that shows that the ‘periodicity’ condition cannot be removed from the hypothesis of Corollary 4.3 and defined Chernikov-subnormality ($\mathfrak{C}$-subnormality, in short) as follows:

A subgroup N of a group G is called $\mathfrak{C}$-*subnormal* in G if there is a finite series

$$N = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_r = G$$

such that $N_{i+1}/N_i \in \mathfrak{C}$ for $0 \le i \le r-1$.

Corollary 4.4. *Let* G *be a* $\mathfrak{T}$*-group containing a* $\mathfrak{C}$*-subnormal subgroup* $N \in \mathfrak{N}_c$. *Then* G *contains a characteristic (even invariant under all surjective endomorphisms) subgroup* $S \in \mathfrak{N}_c$ *such that* $G/S \in \mathfrak{C}$.

Proof. The result follows by Corollary 4.3 and a simple induction. □

We can give an immediate application of Corollary 4.3 by considering the following result due to Hartley.

Theorem 4.5 [**8**, Theorem B]. *If* G *is a locally finite group admitting an involutory automorphism* ϕ *such that* $C_G(\phi)$ *is in* $\mathfrak{C}$, *then both* $[G, \phi]'$ *and* $G/[G, \phi]$ *are in* $\mathfrak{C}$.

As Shumyatsky mentions in [**19**, p. 160], if we take $N = C_{[G,\phi]}([G, \phi]')$, then $N \in \mathfrak{N}_2$, $G/N \in \mathfrak{C}$ and N is ϕ-invariant. So by Corollary 4.3, G has a characteristic subgroup $S \in \mathfrak{N}_2$ such that $G/S \in \mathfrak{C}$.

We record here the following theorem, which is an immediate consequence of Lemma 2.1.

Theorem 4.6. *Let* G *be a group and let* $N \in \mathfrak{X}_\omega$ *be a normal subgroup of* G *for some outer commutator word* ω *such that* $G/N \in \mathfrak{C}$. *Then* G *contains a characteristic (even invariant under all surjective endomorphisms) subgroup* $S \in \mathfrak{X}_{\omega\circ\theta}$ *such that* G/S *is finite.*

Proof. Since $G/N \in \mathfrak{C}$, there exists a normal $\mathfrak{A} \cap \mathfrak{D}$-subgroup R/N of G/N such that G/R is in $\mathfrak{F}$. Since $N \in \mathfrak{X}_\omega$ and R/N is in $\mathfrak{A}$, we have $R \in \mathfrak{X}_{\omega\circ\theta}$. By Lemma 2.1 G has a characteristic subgroup (even invariant under all surjective endomorphisms) $S \in \mathfrak{X}_{\omega\circ\theta}$ such that G/S is in $\mathfrak{F}$, and the result is established. □

Of course, if we replace the condition $G/N \in \mathfrak{C}$ with $G/N \in \mathfrak{A}\mathfrak{F}$ in Theorem 4.6, then the result remains true.

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