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COMPLETE CONVERGENCE FOR WEIGHTED SUMS OF SEQUENCES OF AANA RANDOM VARIABLES

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Abstract. Let $\{X_n, n \geq 1\}$ be an asymptotically almost negatively associated (AANA) sequence of random variables. Some complete convergence and Marcinkiewicz–Zygmund type strong laws of large numbers for an AANA sequence of random variables are obtained. The results obtained generalize the results of Kim, Ko and Lee (Kim, T. S., Ko, M. H. and Lee, I. H. 2004. On the strong laws for asymptotically almost negatively associated random variables. Rocky Mountain J. of Math. 34, 979–989.).

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1. Introduction. Let $\{X, X_n, n \geq 1\}$ be a sequence of independent identically distributed (i.i.d.) random variables. The Marcinkiewicz–Zygmund strong laws of large numbers (SLLNs) provides

$$\frac{1}{n^{1/\alpha}} \sum_{i=1}^{n} (X_i - EX_i) \to 0 \quad \text{a.s.} \quad \text{for} \quad 1 \leq \alpha < 2,$$

and

$$\frac{1}{n^{1/\alpha}} \sum_{i=1}^{n} X_i \to 0 \quad \text{a.s.} \quad \text{for} \quad 0 < \alpha < 1,$$

if and only if $E|X|^\alpha < \infty$. The case $\alpha = 1$ is due to Kolmogorov.

As for asymptotically almost negatively associated (AANA) random variables, Chandra and Ghosal [**2**, **3**] gave the following definition.

Definition [**2**, **3**]. A finite family of random variables $\{X_i, 1 \leq i \leq n\}$ is said to be asymptotically almost negatively associated (AANA) if there is a non-negative sequence $q(m) \to 0$ such that

$$\begin{aligned} &\mathrm{Cov}(f(X_m), g(X_{m+1}, \ldots, X_{m+k})) \\ &\leq q(m)(\mathrm{Var}(f(X_m))\mathrm{Var}(g(X_{m+1}, \ldots, X_{m+k})))^{1/2} \end{aligned} \tag{1.1}$$

for all $m, k \geq 1$ and for all coordinatewise increasing continuous functions f and g whenever the right-hand side of (1.1) is finite.

In 2004, Kim, Ko and Lee established the Marcinkiewicz–Zygmund type SLLNs for an AANA sequence of random variables. We can see the following theorems.

THEOREM A. *Let* $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ *be a sequence of real numbers with* $\sup_{n\geq 1} \sum_{i=1}^{n} |a_{ni}| < \infty$, *and let* $\{X, X_n, n \geq 1\}$ *be a sequence of identically distributed AANA random variables with* $EX = 0$ *and* $E|X|^t < \infty$, *for* $0 < t < 2$. *Let* $\sum_{k=1}^{\infty} q^2(k) < \infty$. *Then,*

$$\frac{1}{n^{1/t}} \sum_{i=1}^{n} a_{ni} X_i \to 0 \quad \text{a.s.} \quad \text{for} \quad 0 < t < 2.$$

THEOREM B. *Let* $\{X, X_n, n \geq 1\}$ *be a sequence of identically distributed AANA random variables. Let* $\sum_{k=1}^{\infty} q^2(k) < \infty$. *If* $E|X|^t < \infty$, *for* $0 < t < 2$, *then*

$$\frac{1}{n^{1/t}} \sum_{i=1}^{n} (X_i - EX_i) \to 0 \quad \text{a.s.} \quad \text{for} \quad 1 \leq t < 2$$

and

$$\frac{1}{n^{1/t}} \sum_{i=1}^{n} X_i \to 0 \quad \text{a.s.} \quad \text{for} \quad 0 < t < 1.$$

The main purpose of this paper is to establish some complete convergence and Marcinkiewicz–Zygmund type SLLNs for an AANA sequence of random variables. The results obtained generalize the results of **[8]**.

2. Main Results. Throughout this paper, C will represent a positive constant though its value may change from one appearance to the next, and $a_n = O(b_n)$ will mean $a_n \leq C b_n$. Also, $a_n \ll b_n$ will mean $a_n = O(b_n)$.

In order to prove our results, we need the concept of complete convergence, the concept of the Hsu–Robbins–Erdös law of large numbers (see **[6]** and **[7]**) and the following lemmas.

DEFINITION 2.1. **[7]** Let $\{X, X_n, n \geq 1\}$ be a random variables sequence, for any $\varepsilon > 0$, if

$$\sum_{n=1}^{\infty} P(|X_n - X| > \varepsilon) < \infty,$$

holds, then we call $\{X_n, n \geq 1\}$ complete convergence to X.

Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables and denote $S_n = \sum_{i=1}^{n} X_i$. The Hsu–Robbins–Erdös law of large numbers (see **[6]** and **[7]**) states that

$$\forall \varepsilon > 0, \qquad \sum_{n=1}^{\infty} P(|S_n| > \varepsilon n) < \infty,$$

is equivalent to $EX = 0$ and $EX^2 < \infty$.

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This is a fundamental theorem in probability theory and has been intensively investigated by many authors in the past decades. One of the most important results is Baum and Katz's [**1**] law of large numbers, which states that, for $p < 2$ and $r \geq p$,

$$\forall \varepsilon > 0, \qquad \sum_{n=1}^{\infty} n^{\frac{r}{p}-2} P\big(|S_n| > \varepsilon n^{\frac{1}{p}}\big) < \infty,$$

if and only if $E|X|^r < \infty$, $r \geq 1$ and $EX = 0$.

There have been many extensions in various directions. Two of them are [**4**, **5**] and [**9**].

The following Lemma can be obtained easily from the definition of AANA random variables.

LEMMA 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of AANA random variables. Then $\{f_n(X_n), n \geq 1\}$ is still a sequence of AANA random variables, where $f_n(\cdot)$, $n = 1, 2, \ldots,$ are non-decreasing functions.*

LEMMA 2.2. [**4**, **5**] *Let $\{X_n, n \geq 1\}$ be a sequence of AANA random variables with $EX_k = 0$ and $EX_k^2 < \infty$, $k \geq 1$. Let $A^2 = \sum_{m=1}^{n-1} q^2(m)$ and $\sigma_k^2 = EX_k^2$, $k \geq 1$. Then, for $\varepsilon > 0$,*

$$P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} X_i\right| > \varepsilon\right) \leq 2\varepsilon^{-2}\big(A + (1 + A^2)^{1/2}\big)^2 \sum_{k=1}^{n} \sigma_k^2.$$

THEOREM 2.1. *Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be a sequence of real numbers with $\sum_{i=1}^{n} a_{ni}^2 = O(n)$. Let $\{X_i, i \geq 1\}$ be an AANA sequence of random variables with $EX_i = 0$. Let $\sum_{k=1}^{\infty} q^2(k) < \infty$. If $E|X|^t < \infty$, for $0 < t < 2$ and $P(|X_i| > x) \leq P(|X| > x)$, $x > 0$. Then*

$$\forall \varepsilon > 0, \quad \sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\leq k\leq n} \frac{1}{n^{\frac{1}{t}}}\left|\sum_{i=1}^{k} a_{ni}X_i\right| > \varepsilon\right) < \infty. \tag{2.1}$$

Proof of Theorem 2.1. Without loss of generality, we can assume that $a_{ni} \geq 0$ for $1 \leq i \leq n$, $n \geq 1$.

For any $i \geq 1$, define $X_i^{(n)} = X_i I(|X_i| \leq n^{\frac{1}{t}}) + n^{\frac{1}{t}} I(X_i > n^{\frac{1}{t}}) - n^{\frac{1}{t}} I(X_i < -n^{\frac{1}{t}})$, $T_j^{(n)} = \frac{1}{n^{\frac{1}{t}}} \sum_{i=1}^{j} (a_{ni}X_i^{(n)} - Ea_{ni}X_i^{(n)})$, then $\forall \varepsilon > 0$,

$$\begin{aligned} &P\left(\max_{1\leq k\leq n} \frac{1}{n^{\frac{1}{t}}}\left|\sum_{i=1}^{k} a_{ni}X_i\right| > \varepsilon\right) \\ &\leq P\left(\max_{1\leq j\leq n} |X_j| > n^{\frac{1}{t}}\right) + P\left(\max_{1\leq j\leq n}\left|T_j^{(n)}\right| > \varepsilon - \max_{1\leq j\leq n}\frac{1}{n^{\frac{1}{t}}}\left|\sum_{i=1}^{j} Ea_{ni}X_i^{(n)}\right|\right). \end{aligned} \tag{2.2}$$

By $\sum_{i=1}^{n} a_{ni}^2 = O(n)$ and Cauchy's inequality

$$\frac{1}{n}\sum_{i=1}^{n} |a_{ni}| \leq \left(\frac{1}{n}\sum_{i=1}^{n} a_{ni}^2\right)^{1/2},$$

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we have

$$\sum_{i=1}^{n} |a_{ni}| = O(n).$$

First, we show that

$$\max_{1\le j\le n} \frac{1}{n^{\frac{1}{t}}} \left| \sum_{i=1}^{j} E a_{ni} X_i^{(n)} \right| \to 0 \quad as \quad n \to \infty. \tag{2.3}$$

In fact, (i) when $t \geq 1$, by $EX_i = 0$, $E|X|^t < \infty$, for $0 < t < 2$, $P(|X_i| > x) \leq P(|X| > x)$, $x > 0$ and $\sum_{i=1}^{n} |a_{ni}| = O(n)$, then

$$\begin{aligned}
&\max_{1\le j\le n} \frac{1}{n^{\frac{1}{t}}} \left| \sum_{i=1}^{j} E a_{ni} X_i^{(n)} \right| \\
&= \max_{1\le j\le n} \frac{1}{n^{\frac{1}{t}}} \left| \sum_{i=1}^{j} a_{ni} E\big[X_i I\big(|X_i| \le n^{\frac{1}{t}}\big) + n^{\frac{1}{t}} I\big(|X_i| > n^{\frac{1}{t}}\big) - n^{\frac{1}{t}} I\big(|X_i| < -n^{\frac{1}{t}}\big)\big] \right| \\
&\le \max_{1\le j\le n} \frac{1}{n^{\frac{1}{t}}} \left| \sum_{i=1}^{j} E a_{ni} X_i I\big(|X_i| \le n^{\frac{1}{t}}\big) \right| + \sum_{j=1}^{n} a_{nj} P\big(|X_j| > n^{\frac{1}{t}}\big) \\
&\le \max_{1\le j\le n} \frac{1}{n^{\frac{1}{t}}} \left| \sum_{i=1}^{j} E a_{ni} X_i I\big(|X_i| > n^{\frac{1}{t}}\big) \right| + \sum_{j=1}^{n} a_{nj} P\big(|X_j| > n^{\frac{1}{t}}\big) \\
&\ll \frac{n}{n^{\frac{1}{t}}} E|X| I\big(|X| > n^{\frac{1}{t}}\big) + \sum_{j=1}^{n} a_{nj} P\big(|X_j| > n^{\frac{1}{t}}\big) \\
&\le \frac{n}{n^{\frac{1}{t}}} \big(n^{\frac{1}{t}}\big)^{1-t} E|X|^t I\big(|X| > n^{\frac{1}{t}}\big) + nP\big(|X| > n^{\frac{1}{t}}\big) \\
&\le E|X|^t I\big(|X| > n^{\frac{1}{t}}\big) + nP\big(|X| > n^{\frac{1}{t}}\big) \\
&\to 0 \quad as \quad n \to \infty.
\end{aligned}$$

(ii) When $0 < t < 1$, by $EX_i = 0$, $E|X|^t < \infty$, for $0 < t < 2$, $P(|X_i| > x) \le P(|X| > x)$, $x > 0$ and $\sum_{i=1}^{n} |a_{ni}| = O(n)$, then

$$\begin{aligned}
&\max_{1\le j\le n} \frac{1}{n^{\frac{1}{t}}} \left| \sum_{i=1}^{j} E a_{ni} X_i^{(n)} \right| \\
&= \max_{1\le j\le n} \frac{1}{n^{\frac{1}{t}}} \left| \sum_{i=1}^{j} a_{ni} E\big[X_i I\big(|X_i| \le n^{\frac{1}{t}}\big) + n^{\frac{1}{t}} I\big(|X_i| > n^{\frac{1}{t}}\big) - n^{\frac{1}{t}} I\big(|X_i| < -n^{\frac{1}{t}}\big)\big] \right| \\
&\le \max_{1\le j\le n} \frac{1}{n^{\frac{1}{t}}} \left| \sum_{i=1}^{j} a_{ni} E X_i I\big(|X_i| \le n^{\frac{1}{t}}\big) \right| + \sum_{j=1}^{n} a_{nj} P\big(|X_j| > n^{\frac{1}{t}}\big) \\
&\ll \frac{n}{n^{\frac{1}{t}}} E|X| I\big(|X| \le n^{\frac{1}{t}}\big) + \sum_{j=1}^{n} a_{nj} P\big(|X_j| > n^{\frac{1}{t}}\big) \\
&= \frac{n}{n^{\frac{1}{t}}} \sum_{k=1}^{n} E|X| I\big(k-1 < |X|^t \le k\big) + nP\big(|X| > n^{\frac{1}{t}}\big).
\end{aligned}$$

Using $E|X|^t < \infty$, we have

$$\begin{aligned}
&\sum_{k=1}^{\infty} \frac{k}{k^{\frac{1}{t}}} E|X|I(k-1 < |X|^t \le k) \\
&\le \sum_{k=1}^{\infty} \frac{k}{k^{\frac{1}{t}}} k^{\frac{1}{t}} P(k-1 < |X|^t \le k) \\
&\le \sum_{k=1}^{\infty} kP(k-1 < |X|^t \le k) \\
&\le E|X|^t + 1 < \infty.
\end{aligned}$$

By Kronecker's lemma, we get

$$\frac{n}{n^{\frac{1}{t}}} \sum_{k=1}^{n} E|X|I(k-1 < |X|^t \le k) \to 0, \quad n \to \infty,$$

so

$$\max_{1 \le j \le n} \frac{1}{n^{\frac{1}{t}}} \left| \sum_{i=1}^{j} EX_i^{(n)} \right| \to 0, \quad n \to \infty. \tag{2.4}$$

Hence, from (i) and (ii), (2.3) is true.
From (2.2) and (2.3), it follows that for large enough n

$$P\left(\max_{1 \le k \le n} \frac{1}{n^{\frac{1}{t}}} \left| \sum_{i=1}^{k} a_{ni} X_i \right| > \varepsilon \right) \le \sum_{j=1}^{n} P\big(|X_j| > n^{\frac{1}{t}}\big) + P\left(\max_{1 \le j \le n} |T_j^{(n)}| > \frac{\varepsilon}{2} \right).$$

Hence, we only need to prove that

$$\begin{aligned}
I &=: \sum_{n=1}^{\infty} n^{-1} \sum_{j=1}^{n} P\big(|X_j| > n^{\frac{1}{t}}\big) < \infty, \\
II &=: \sum_{n=1}^{\infty} n^{-1} P\left(\max_{1 \le j \le n} \big|T_j^{(n)}\big| > \frac{\varepsilon}{2} \right) < \infty.
\end{aligned} \tag{2.5}$$

From the fact that $E|X|^t < \infty$ and $P(|X_i| > x) \le P(|X| > x),\ x > 0$, it easily follows that

$$\begin{aligned}
I &= \sum_{n=1}^{\infty} n^{-1} \sum_{j=1}^{n} P\big(|X_j| > n^{\frac{1}{t}}\big) \\
&\ll \sum_{n=1}^{\infty} P\big(|X| > n^{\frac{1}{t}}\big) \\
&\le E|X|^t + 1 < \infty.
\end{aligned} \tag{2.6}$$

By Lemma 2.1, $\{a_{ni} X_i^{(n)}, i \ge 1\}$ is still a sequence of AANA random variables.
By Lemma 2.2 and $E|X|^t < \infty, 0 < t < 2$ and $P(|X_i| > x) \le P(|X| > x), x > 0,$ we

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have

$$
\begin{aligned}
II &= \sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\le j\le n} |T_j^{(n)}| > \frac{\varepsilon}{2}\right) \\
&\ll \sum_{n=1}^{\infty} n^{-1} \frac{1}{n^{\frac{2}{t}}} \frac{4}{\varepsilon^2} \sum_{j=1}^{n} E|a_{nj} X_j^{(n)}|^2 \\
&\le C \sum_{n=1}^{\infty} n^{-1} \frac{1}{n^{\frac{2}{t}}} \sum_{i=1}^{n} a_{ni}^2 E\big|X_i I(|X_i| \le n^{1/t}) + n^{1/t} I(|X_i| > n^{1/t}) - n^{1/t} I(|X_i| < -n^{1/t})\big|^2 \\
&\le C \sum_{n=1}^{\infty} n^{-1} \frac{1}{n^{\frac{2}{t}}} \sum_{i=1}^{n} a_{ni}^2 E X_i^2 I(|X_i| \le n^{\frac{1}{t}}) + C \sum_{n=1}^{\infty} n^{-1} \frac{1}{n^{\frac{2}{t}}} n^{\frac{2}{t}} \sum_{j=1}^{n} a_{nj}^2 P(|X_j| > n^{\frac{1}{t}}) \\
&\ll C \sum_{n=1}^{\infty} \frac{1}{n^{\frac{2}{t}}} E X^2 I(|X| \le n^{\frac{1}{t}}) + C \sum_{n=1}^{\infty} n^{-1} n P(|X| > n^{\frac{1}{t}}) \\
&= C \sum_{n=1}^{\infty} \frac{1}{n^{\frac{2}{t}}} \sum_{k=1}^{n} E X^2 I(k-1 < |X|^t \le k) + C \sum_{n=1}^{\infty} P(|X| > n^{\frac{1}{t}}) \\
&\le C \sum_{n=1}^{\infty} \frac{1}{n^{\frac{2}{t}}} \sum_{k=1}^{n} k^{2/t} P(k-1 < |X|^t \le k) + C \sum_{n=1}^{\infty} P(|X| > n^{\frac{1}{t}}) \\
&\le C \sum_{k=1}^{\infty} k^{2/t} P(k-1 < |X|^t \le k) \sum_{n=k}^{\infty} \frac{1}{n^{\frac{2}{t}}} + C \sum_{n=1}^{\infty} P(|X| > n^{\frac{1}{t}}) \\
&\le C \sum_{k=1}^{\infty} k P(k-1 < |X|^t \le k) + C \sum_{n=1}^{\infty} P(|X| > n^{\frac{1}{t}}) \\
&\le C(E|X|^t + 1) < \infty.
\end{aligned} \tag{2.7}
$$

Proof of Theorem 2.1 is now complete.

COROLLARY 2.1. *Under the conditions of Theorem* 2.1 *and let* $T_n = \sum_{i=1}^{n} a_{ni} X_i$,

$$
\lim_{n\to\infty} \frac{|T_n|}{n^{\frac{1}{t}}} = 0 \qquad \text{a.s.}
$$

Proof. By (2.1), we have

$$
\begin{aligned}
\infty &> \sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\le j\le n} |T_j| > \varepsilon n^{\frac{1}{t}}\right) \\
&= \sum_{i=0}^{\infty} \sum_{n=2^i}^{2^{i+1}-1} n^{-1} P\left(\max_{1\le j\le n} |T_j| > \varepsilon n^{\frac{1}{t}}\right) \\
&\ge \frac{1}{2} \sum_{i=1}^{\infty} P\left(\max_{1\le j\le 2^i} |T_j| > \varepsilon 2^{\frac{i+1}{t}}\right).
\end{aligned}
$$

By Borel–Cantelli lemma, we have

$$P\left(\max_{1\le j\le 2^i} |T_j| > \varepsilon 2^{\frac{i+1}{t}} \qquad i.o.\right) = 0.$$

Hence

$$\lim_{i\to\infty} \frac{\max_{1\le j\le 2^i} |T_j|}{2^{\frac{i+1}{t}}} = 0 \qquad \text{a.s.}$$

and using

$$\max_{2^{i-1}\le n<2^i} \frac{|T_n|}{n^{\frac{1}{t}}} \le 2^{\frac{2}{t}} \frac{\max_{1\le j\le 2^i} |T_j|}{2^{\frac{i+1}{t}}},$$

we have

$$\lim_{n\to\infty} \frac{|T_n|}{n^{\frac{1}{t}}} = 0 \qquad \text{a.s.}$$

Hence, we complete the proof of Corollary 2.1. □

REMARK 2.1. Theorem 2.1 and Corollary 2.1 generalise the results of [**8**].

REMARK 2.2. Let $a_{ni} = 1$, by Corollary 2.1, we can prove Theorem B.

REMARK 2.3. Corollary 2.1 is more generalised than Theorem A.

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REFERENCES

1. L. E. Baum and M. Katz, Convergence rates in the law of large numbers, *Trans. Amer. Math. Soc*. **120** (1965), 108–123.

2. T. K. Chandra and S. Ghosal, Extensions of the strong laws of large numbers of Marcinkiewicz and Zygmund for dependent variables, *Acta Math. Hungar.* **32** (1996a), 327–336.

3. T. K. Chandra and S. Ghosal, The strong laws of large numbers for weighted averages under dependence assumptions, *J. Theor. Probab.* **9** (1996b), 797–809.

4. Y. S. Chow and T. L. Lai, Some one-sided theorems on the tail distribution of sample sums with applications to the last time and largest excess of boundary crossings, *Trans. Amer. Math. Soc.* **208** (1975), 51–72.

5. Y. S. Chow and T. L. Lai, Paley-type inequalities and convergence rates related to the law of large numbers and extended renewal theory, Z. Wahrsch. Verw. Geb. **45** (1978) 1–19.

6. P. Erdös, On a theorem of Hsu–Robbins, *Ann. Math. Statist*. **20** (1949), 286–291.

7. P. L. Hsu and H. Robbins, Complete convergence and the law of larege numbers, *Proc. Nat. Acad. Sci. (USA)* **33**(2) (1947), 25–31.

8. T. S. Kim, M. H. Ko and I. H. Lee, On the strong laws for asymptotically almost negatively associated random variables. *Rocky Mountain J. Math*. **34** (224), 979–989.

9. V. V. Petrov, Limit theorems of probability theory sequences of independent random variables (Oxford, Oxford Science Publications, 1995).