ORIENTATION OF A SATELLITE LOCATED AT THE LIBRATION POINT IN THE RESTRICTED THREE-BODY PROBLEM

Grzegorz Duliński, Andrej J. Maciejewski
Institute of Astronomy
Nicolaus Copernicus University
Chopina 12/18, 87-100 Toruń, Poland

1. INTRODUCTION

In this paper the initial results of an investigation of the motion of a rigid body located at the libration point in the planar, restricted three-body problem are given. This problem was analyzed in part by Kane and Marsh (1971), Markeev (1967a,b). However the present investigation is formulated in terms of hamiltonian mechanics. The final results will by used to study nonlinear effects connected with the gravitational influence of the "second" central body.

2. EQUATIONS OF MOTION

Let m_1, m_2, m_3 designate rigid bodies of mass m_1, m_2 and m_3 respectively. We assume that m_3 is so small in comparison with m_1 and m_2 that it has no effects on the motions of m_1 and m_2. Furthermore let m_1 and m_2 have spherically symmetric mass distributions so that the gravitational effects of m_1 and m_2 on other bodies are equivalent to those of point masses. The motion of the mass centers of these two massive bodies are the same as in the elliptic 2-body problem. Then we introduce the following righthanded, orthonormal coordinate systems: inertial, orbital and principal axes (see fig. 1).

For describing an orientation of a third coordinate system relative to the second one we use Euler´s angles: φ, θ, ψ in 1-2-1 sequence (Markley 1978). Next we assume that the third rigid body-satellite- is axially symmetric and is located at the triangular L_4 libration point.

The Lagrange function has the form

$$L = \frac{1}{2}\,\omega^T I\omega - V \tag{1}$$

where

$$\omega^T = (\omega_1, \omega_2, \omega_3), \quad \omega_i = \omega_i(\varphi, \theta, \psi, \dot{\varphi}, \dot{\theta}, \dot{\psi}), \quad \dot{} = \frac{d}{dt}$$

V. V. Markellos and Y. Kozai (eds.), Dynamical Trapping and Evolution in the Solar System, 27–35.
© 1983 by D. Reidel Publishing Company.

Downloaded from https://www.cambridge.org/core. IP address: 34.204.173.45, on 23 Oct 2019 at 11:41:32, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0252921100096883

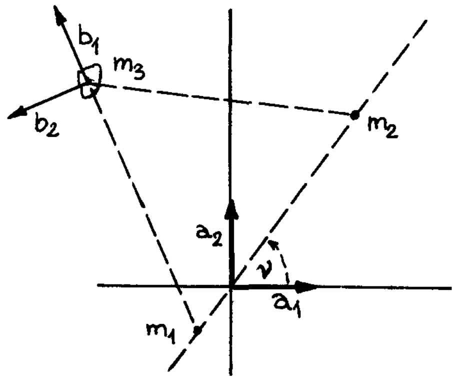

Fig. 1. Geometry of the 3-body problem.
(a_1, a_2, a_3) - inertial coordinate system
(b_1, b_2, b_3) - orbital coordinate system

are the components of the absolute angular velocity in the third coordinate system.

$$I = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_2 \end{bmatrix},$$

is an inertia tensor $(I_1, I_2$ - are the moments of inertia),

$$\omega = N\dot{q} + b,$$

$$N = \begin{bmatrix} c\theta & 0 & 1 \\ s\psi\, s\theta & c\psi & 0 \\ c\psi\, s\theta & -s\psi & 0 \end{bmatrix}, \quad b = \begin{bmatrix} -s\theta\, c\varphi \\ c\psi\, s\varphi + s\psi\, c\theta\, c\varphi \\ -s\psi\, s\varphi + c\psi\, c\theta\, c\varphi \end{bmatrix},$$

$s\theta = \sin\theta$, $c\theta = \cos\theta$, etc.

$$\dot{\nu} = \frac{C}{r^2}, \quad C = G(m_1 + m_2)p, \quad r = \frac{p}{1 + e\cos\nu},$$

G - gravitational constant, p - orbit parameter,
e - eccentricity, r - radius vector, ν - true anomaly
T - transposition of the matrix,
$\dot{q} = (\dot{\varphi}, \dot{\theta}, \dot{\psi})$,

Downloaded from https://www.cambridge.org/core. IP address: 34.204.173.45, on 23 Oct 2019 at 11:41:32, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0252921100096883

$$V = \frac{3}{2}\,\omega_0^2\,(1-e^2)^{-3}\,(1 + e\cos\nu)^3\,(I_1 - I_2) \times \left[(1-\mu)c^2\theta + \frac{1}{4}\mu(c\theta - \sqrt{3}\,s\theta\,s\varphi)^2\right], \qquad (2)$$

$$\omega_0 = \frac{2\pi}{P}, \quad P \text{ - orbital period of the massive bodies},$$

$$\mu = \frac{m_2}{m_1 + m_2}, \qquad 0 \leqq \mu \leqq \frac{1}{2}.$$

Introducing now:

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \quad i = 1,2,3, \quad \dot{q}_i = \dot{\varphi}, \dot{\theta}, \dot{\psi},$$

where L is given by expression (1), we get the hamiltonian

$$H = \frac{p_\varphi^2}{2\,I_2\,s^2\theta} + \frac{p_\theta^2}{2I_2} + \frac{1}{2}\left(\frac{ctg^2\theta}{I_2} + \frac{1}{I_1}\right)p_\psi^2 - \frac{ctg\,\theta}{I_2 s\theta}\,p_\psi p_\varphi$$

$$-\dot{\nu}\,ctg\theta\,c\varphi\,p_\varphi - \dot{\nu}\,s\varphi p_\theta + \dot{\nu}\,\frac{c\varphi}{s\theta}\,p_\psi + V. \qquad (3)$$

It is seen that ψ is a cyclic coordinate, therefore:

$$p_\psi = I_1\omega_1 = \text{constant}.$$

After substitution into equations (3) of the true anomaly $-\nu$ as independent variable instead of time $-t$, and introduction of dimensionless momenta defined by:

$$p_\varphi' = \frac{p_\varphi}{\sigma}, \quad p_\theta' = \frac{p_\theta}{\sigma}, \quad \text{where:} \quad \sigma = I_2(1-e^2)^{-3/2}\,\omega_0,$$

we derive the following form for Hamilton's function:

$$H = \frac{p_\varphi'^2}{2(1 + e\cos\nu)^2 s^2\theta} + \frac{p_\theta'^2}{2(1+e\cos\nu)^2} + \frac{\gamma\,ctg\theta\,p_\varphi'}{(1+e\cos\nu)^2 s\theta}$$

$$- ctg\theta\,c\varphi\,p_\varphi' - s\varphi p_\theta' + \gamma\,\frac{c\varphi}{s\theta} + \frac{\gamma^2\,ctg^2\theta}{(1+e\cos\nu)^2} \qquad (4)$$

$$+ \frac{3}{2}(\alpha-1)\,(1 + e\cos\nu)\left[(1-\mu)c^2\theta + \frac{1}{4}\mu\,(c\theta - \sqrt{3}\,s\theta\,s\varphi)^2\right]$$

Downloaded from https://www.cambridge.org/core. IP address: 34.204.173.45, on 23 Oct 2019 at 11:41:32, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0252921100096883

where:

$$\alpha = \frac{I_1}{I_2}, \quad \beta = \frac{\omega_1}{\omega_0}, \quad \gamma = \alpha\beta(1-e^2)^{3/2}.$$

For $\mu = 0$ we have obtained the same expression as that derived by Markeev (1967b) for the one central body case.

3. PARTICULAR SOLUTION AND THE STABILITY ANALYSIS

Hamilton's equations of motion are as follows:

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta}, \quad \dot{\varphi} = \frac{\partial H}{\partial p_\varphi}, \quad \dot{p}_\theta = -\frac{\partial H}{\partial \theta}, \quad \dot{p}_\varphi = -\frac{\partial H}{\partial \varphi}, \tag{5}$$

where:

$\dot{}$ – denotes differentation with respect to ν,
H is given by equation (4).

The above equations have only one particular, stationary solution such that:

$$\theta = \frac{\pi}{2}, \quad \varphi = \pi, \quad p_\theta = 0, \quad p_\varphi = 0. \tag{6}$$

This is the only known particular solution of equations (5) even though we get $e = 0$. To discuss the properties of the solutions near this particular solution, it is necessary to investigate its stability. This can be done in the $e = 0$ case by analytic methods.

For small e, but larger than zero, we can do it and calculate instability regions studying the parametric resonance in this system (Arnold 1978, Markeev 1978). We introduce new variables q_1, q_2, p_1, p_2 defined as follows:

$$q_1 = \theta - \frac{\pi}{2}, \quad q_2 = \varphi - \pi, \quad p_1 = p_\theta, \quad p_2 = p_\varphi. \tag{7}$$

Then the hamiltonian takes the form

$$H(q, p) = H_2(q, p) + H_3(q, p) + H_4(q, p) + \ldots \tag{8}$$

For $e = 0$ (circular problem) we may write

$$H = \frac{q_1^2}{2}\left[\alpha^2\beta^2 - \alpha\beta + \frac{3}{4}(\alpha-1)(4-3\mu)\right] + \frac{q_2^2}{2}\left[\frac{9}{4}(\alpha-1)\mu + \alpha\beta\right]$$
$$+ \frac{p_1^2}{2} + \frac{p_2^2}{2} + q_1 q_2\left[\frac{-3\sqrt{3}}{4}(\alpha-1)\mu\right] + p_1 q_2 + q_1 p_2(\alpha\beta - 1). \tag{9}$$

Downloaded from https://www.cambridge.org/core. IP address: 34.204.173.45, on 23 Oct 2019 at 11:41:32, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0252921100096883

For this hamiltonian the equations of motion are linear and may be written:

$$\frac{dX}{dt} = I\ H\ X \tag{10}$$

where:

$$X^T = (q_1, q_2, p_1, p_2),$$

$$I H = \begin{bmatrix} 0 & 1 & 1 & 0 \\ a & 0 & 0 & 1 \\ b & c & 0 & -a \\ c & d & -1 & 0 \end{bmatrix},$$

$$a = \alpha\beta - 1, \quad b = \alpha\beta(1-\alpha\beta) - \frac{3}{4}(\alpha-1)(4-3\mu),$$

$$c = \frac{3\sqrt{3}}{4}(\alpha-1)\mu, \quad d = -\frac{9}{4}(\alpha-1)\mu - \alpha\beta,$$

$$I = \begin{bmatrix} 0 & E_n \\ -E_n & 0 \end{bmatrix}, \quad \text{where: } E_n - \text{nxn unit matrix} \quad (n = 2).$$

The characteristic equation of matrix I H has the form

$$\lambda^4 + a' \lambda^2 + b' = 0, \tag{11}$$

where

$$a' = \alpha^2\beta^2 - 2\alpha\beta + 3\alpha - 1,$$

$$b' = (\alpha\beta - 1)(3\alpha + \alpha\beta - 4) + \frac{27}{4}\mu(\mu-1)(\alpha-1)^2 .$$

Stability conditions are as follows:

$$a' > 0, \quad b' > 0, \text{ and } \quad a'^2 - 4b' > 0 . \tag{12}$$

On the parametric plane (γ,α), $(\gamma = \alpha\beta)$, the above conditions designate the stability regions of the solution (6) in linear approximation of the equations of motion. After comparison of these results with those obtained for the one central body case we see that the discussed regions are slightly different. To study the stability of the solution (6) of the complete equations of motion it is necessary to transform the hamiltonian H into normal form. We can do this in the following way.

Let λ_i be a root of the characteristic equation (11). Then we have :

Downloaded from https://www.cambridge.org/core. IP address: 34.204.173.45, on 23 Oct 2019 at 11:41:32, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0252921100096883

$$\lambda_1 = i\omega_1\varepsilon_1, \quad \lambda_2 = i\omega_2\varepsilon_2, \quad \lambda_3 = -i\omega_1\varepsilon_1, \quad \lambda_4 = -i\omega_2\varepsilon_2,$$

where:

$$\omega_1 > \omega_2 > 0, \quad \varepsilon_i = \pm 1, \text{ a sign that will be fixed later.}$$

However, λ_i are the eigenvalues corresponding to the eigenvectors b_i, of the matrix I H, given by:

$$b_i = \frac{1}{\alpha_i} \begin{bmatrix} \lambda_i(1-a) + c \\ \lambda_i^2 - a^2 - b \\ -a\lambda_i^2 + c\lambda_i + a^2 + b \\ \lambda_i^3 - \lambda_i(a+b) - ac \end{bmatrix},$$

where a, b, c are the same as in equation (10), and α_i is a normalized factor defined below. Now we may write b_i as the sum of two vectors:

$$b_j = r_j + i\, s_j, \quad j = 1,2,3,4, \quad i^2 = -1, \tag{13}$$

Linear transformation of the form

$$X = A\,Y$$

where:

$$X^T = (q_1, q_2, p_1, p_2), \qquad Y^T = (y_1, y_2, y_3, y_4)$$

$$A = 2\left[-s_1, -s_2, r_1, r_2\right],$$

transforms the hamiltonian H into

$$H_2 = \sum_{i=1}^{2} \varepsilon_i\omega_i\,(y_i^2 + y_{i+2}^2) \tag{15}$$

Since the matrix A is sympletic, so finally we may write the following relations:

$$(r_k, I\, s_k) = \frac{1}{4}\alpha_k, \quad k = 1,2, \tag{16}$$

where (,) denotes the scalar product.

Then we have a relation for the normalized factor α_i as follows:

$$\frac{1}{4}\alpha_i^2 = \varepsilon_i\omega_i\, f_i(\alpha, \beta, \mu), \tag{17}$$

Downloaded from https://www.cambridge.org/core. IP address: 34.204.173.45, on 23 Oct 2019 at 11:41:32, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0252921100096883

where:

$$f_i(\alpha,\beta,\mu) = \omega_i^2 \left[(\alpha\beta-2)^2 - \frac{3}{2}(\alpha-1)(2-3\mu)\right] + (\alpha\beta-1)(\alpha\beta-2)^2 + 3(\alpha-1)\left[\alpha^2\beta^2 - 3\alpha\beta + 3\alpha\right] + \frac{9}{4}(\alpha-1)\mu\left[(\alpha\beta-1)^2 + 9\alpha - 8\right] + \frac{27}{2}(\alpha-1)\mu^2$$

One can find that $f_i(\alpha,\beta,\mu) > 0$ in the stability region I (see fig. 2). Therefore on this region $\varepsilon_i = +1$, H_2 is a positive definite function and sufficient stability conditions are satisfied on this region. On region II (see fig. 2) we have $f_2(\alpha,\beta,\mu) < 0$ and $f_1(\alpha,\beta,\mu) > 0$, and we choose $\varepsilon_1 = +1$, $\varepsilon_2 = -1$. Therefore on this region the hamiltonian H_2 has not a fixed sign.

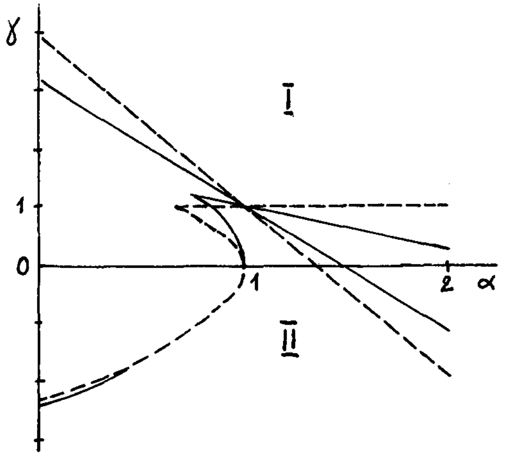

Fig. 2. Stability regions for $\mu = 0$(---) and for $\mu = \frac{1}{2}$ (——).

To make an exact study of the stability problem in this case we may take into account the H_4 term (H_3=0) in equation (8). This term may be written as:

$$H_4(q,p) = \sum_{n_1+n_2+n_3+n_4=4} h_{n_1n_2n_3n_4}\, q_1^{n_1} q_2^{n_2} p_1^{n_3} p_2^{n_4}, \quad (18)$$

Downloaded from https://www.cambridge.org/core. IP address: 34.204.173.45, on 23 Oct 2019 at 11:41:32, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0252921100096883

where:

$$n_i \geqq 0, \quad n_i \in N, \quad (i = 1,2,3,4),$$

$$h_{4000} = 8\alpha^2\beta^2 - 5\alpha\beta - 3(\alpha-1),$$

$$h_{0400} = -\alpha\beta - 9(\alpha-1)\mu,$$

$$h_{2020} = 1,$$

$$h_{3001} = 5\alpha\beta - 2,$$

$$h_{0310} = -1,$$

$$h_{2200} = \alpha\beta - 9(\alpha-1)\mu - \frac{3\sqrt{3}}{4}(\alpha-1)\mu,$$

$$h_{1201} = 1,$$

and the other terms $h_{n_1n_2n_3n_4} = 0$.

Assuming $i_1\omega_1 + i_2\omega_2 \neq 0$ for $0 < |i_1| + |i_2| \leqq 4$,

which in our case corresponds to

$$\omega_1 \neq \omega_2, \quad \omega_1 \neq 2\omega_2, \quad \omega_1 \neq 3\omega_2,$$

and making the transformation (14) as well as the appropriate Birkhoff transformation we get the following for the hamiltonian H:

$$H = \omega_1(q_1^2 + p_1^2) - \omega_2(q_2^2 + p_2^2) + a_{11}(p_1^2 + q_1^2)^2 + a_{12}(p_1^2 + q_1^2)(q_2^2 + p_2^2) + a_{22}(p_2^2 + q_2^2)^2 + \ldots \tag{19}$$

where:

$$a_{ij} = a_{ij}(\alpha,\beta,\mu),$$

$+ \ldots$ – stands for the higher terms in expression (19).

Using Arnold-Moser theorem (Markeev 1978) we notice that on region II (see fig. 2) the solution (6) will be stable everywhere except those points for which the following relations are satisfied:

$$\omega_1 = \omega_2, \quad \omega_1 = 2\omega_2 \text{ and } \omega_1 = 3\omega_2, \text{ and} \tag{20a}$$

Downloaded from https://www.cambridge.org/core. IP address: 34.204.173.45, on 23 Oct 2019 at 11:41:32, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0252921100096883

$$f(\alpha,\beta,\mu) = a_{11}\omega_1^2 + a_{12}\omega_1\omega_2 + a_{22}\omega_2^2 = 0 \ . \qquad (20b)$$

The further investigation including nonlinear effects of higher orders will be presented in another paper(Duliński and Maciejewski,1983).

ACKNOWLEDGEMENT

This research was partly supported by a grant of the Polish Academy of Sciences.

REFERENCES

Arnold, V.I.: 1978, *Mathematical Methods in Classical Mechanics*, Springer Verlag, New York

Diliński, G. and Maciejewski, A.: 1983, in preparation

Kane, T.R. and Marsh, L.E.: 1971, *Celes. Mech.* 4, pp. 78-91

Markeev, A.P.: 1967a, *Kosm. Issled.* 5, pp. 365-376

Markeev, A.P.: 1967b, *Kosm. Issled.* 5, pp. 530-540

Markeev, A.P.: 1978, *Libration Points in Celestial Mechanics and Astrodynamics*, Nauka, Moscow (in Russian)

Markley, F.L.: 1978, in J.R. Wertz (ed.), D. Reidel Publ. Company, ASSL Vol. 73

Downloaded from https://www.cambridge.org/core. IP address: 34.204.173.45, on 23 Oct 2019 at 11:41:32, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0252921100096883