PERMANENTS OF (0, 1)-CIRCULANTS

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1. Introduction. The permanent of an n-square matrix $A = (a_{ij})$ is defined by

$$p(A) = \sum_{\sigma \varepsilon S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$$

where the summation extends over all permutations σ of the symmetric group S_n. A matrix is said to be a (0, 1)-matrix if each of its entries is either 0 or 1. A (0, 1)-matrix of the form $\sum_{j=0}^{n-1} \theta_j P_n^j$, where $\theta_j = 0$ or 1, $j = 1, \ldots, n$, and P_n is the n-square permutation matrix with ones in the (1, 2), (2, 3), ..., (n-1, n), (n, 1) positions, is called a (0, 1)-circulant. Denote the (0, 1)-circulant $\sum_{j=0}^{k-1} P_n^j$ by Q(n, k). It has been conjectured that

$$(1) \qquad p(Q(n, r)) \geq n!\,(r/n)^n .$$

This is a special case of the famous unresolved van der Waerden conjecture [5] which states that if S is a doubly stochastic n-square matrix (i.e. a non-negative matrix whose row sums and column sums are all 1) then $p(S) \geq n!/n^n$.

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In [3] Mendelsohn gave the following asymptotic formula for the permanent of $Q(n, r)$:

for a fixed r

$$p(Q(n, r)) \sim K(r)\, \alpha^{n} \tag{2}$$

where $K(r)$ is a constant depending on r only and α is the root of $x^{r} - 2x^{r-1} + 1 = 0$ in the interval $1 < \alpha < 2$. This formula is of considerable interest since it implies the falsity of (1) and thus of the van der Waerden conjecture. For, according to (2),

$$p(Q(n, r)) \sim K(r)\, \alpha^{n} < K(r)\, 2^{n}$$

and for a fixed $r > 5$ and for a sufficiently large n

$$K(r) \cdot 2^{n} < (r/e)^{n} < r^{n}\, n!/n^{n} .$$

Thus

$$p(Q(n, r)) < n!\,(r/n)^{n}$$

which contradicts (1).

In the present note I show that (2) is false for $r > 4$ and also that both the exact and the asymptotic formulas for $p(Q(n, 3))$ and $p(Q(n, 4))$ given in [3] are incorrect. I give correct formulas for the permanents of $Q(n, 3)$ and $Q(n, 4)$ and I show that for every r and n, $r < n$, there exists a doubly stochastic n-square matrix whose permanent is less than that of $Q(n, r)/r$.

2. Results. Let $f(n, r)$ denote the n'th Fibonacci number of order r, i.e.

$$f(n, r) = \begin{cases} 0, & \text{if } n < 0, \\ 1, & \text{if } n = 0, \\ f(n-1, r) + \dots + f(n-r, r), & \text{if } n > 0 . \end{cases}$$

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THEOREM 1. (i) $f(n, r) = \sum_{i=1}^{r} \alpha_i^{n-1}/(2-\alpha_i)\, g_r'(\alpha_i)$ where $g_r(x) = x^r - x^{r-1} - \dots - x - 1$ and $\alpha_1, \dots, \alpha_r$ $(|\alpha_1| \geq \dots \geq |\alpha_r|)$ are the roots of $g_r(x) = 0$.

(ii) $f(n, r) \sim \alpha_1^{n-1}/(2-\alpha_1)\, g_r'(\alpha_1)$.

Proof. We have, for $n > 0$,

$$f(n, r) = f(n-1, r) + \dots + f(n-r, r).$$

Solving this functional equation we obtain

$$f(n, r) = c_1\alpha_1^n + c_2\alpha_2^n + \dots + c_r\alpha_r^n$$

where the α_j are the roots of $g_r(x) = 0$ and the c_j are constants to be determined. Now, $f(n, r) = 1, 2, 4, \dots, 2^{r-1}$ for $n = 1, 2, 3, \dots, r$. Therefore, in the matrix notation,

$$Ac = v$$

where A is the r-square matrix with α_j^i in its (i, j) position and c and v are column vectors whose i'th entries are c_i and 2^{i-1} respectively. Note that $A = \alpha_1 \dots \alpha_r\ B = (-1)^r B$ where $B = (b_{ij})$ is a Vandermonde matrix with $b_{ij} = \alpha_j^{i-1}$. Therefore

$$c_i = \prod_{\substack{t=1 \\ t \neq i}}^{r} (2-\alpha_t) / \prod_{\substack{t=1 \\ t \neq i}}^{r} (\alpha_i - \alpha_t)$$

$$= g_r(2)/(2-\alpha_i)\, g_r'(\alpha_i)$$

$$= 1/(2-\alpha_i)\, g_r'(\alpha_i).$$

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To complete the proof we show that the equation $g_r(x) = 0$ has a real root in the interval (1, 2) and that the moduli of all other roots are less than 1. We apply a classical result [1] on the localization of characteristic roots of a matrix to the companion matrix of $(x-1)g_r(x) = x^{r+1} - 2x^r + 1$

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 2 \end{pmatrix}$$

and we find that one zero of $(x-1)g_r(x)$ lies inside or on the boundary of the circle $|z-2| = 1$ and the other r zeros lie inside or on the boundary of $|z| = 1$. Since the only point common to both circles is not a zero of $g_r(x)$ we can conclude from a result due to Taussky [4] that one zero of $g_r(x)$ lies inside $|z-2| = 1$ and the other $r-1$ zeros inside $|z| = 1$.

We now construct an n-square (0, 1)-matrix whose permanent and principal subpermanents are Fibonacci numbers of prescribed order. Let $F(n, r)$, $r \leq n+1$, denote the n-square (0, 1)-matrix with 1 in the (i, j) position for $i - 1 \leq j \leq i + r - 2$ and 0 otherwise,

$$F(n,r) = \begin{pmatrix} \overbrace{1 & 1 & 1 & \cdots & 1}^{r-1} & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 1 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & & 1 & 0 \\ 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & & 1 & 1 \\ 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & & 1 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \cdots & & 0 & 1 & 1 & & 1 \\ 0 & \cdots & 0 & \cdots & & 0 & 0 & 1 & & 1 \end{pmatrix}$$

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THEOREM 2.

$$p(F(n,r)) = f(n,r-1) .$$

Proof. It is easy to see that expanding $p(F(n,r))$ by the elements of the first row we obtain in case $n \geq 2(r-1)$

$$p(F(n,r)) = p(F(n-1,r)) + \ldots + p(F(n-r+1,r))$$

and in the case $r - 1 \leq n < 2(r-1)$

$$p(F(n,r)) = p(F(n-1,r)) + \ldots + p(F(r-1,r)) + p(F(r-2,r-1)) + \ldots + p(F(r-(2r-n-1), r-(2r-n-2))).$$

It remains to show that $p(F(s-1,s)) = f(s-1,s-1) = 2^{s-2}$. We use induction on s:

$$\begin{aligned} p(F(s-1,s)) &= p(F(s-2,s-1)) + \ldots + p(F(2,3)) + p(F(1,2)) + 1 \\ &= 2^{s-3} + \ldots + 2 + 1 + 1, \text{ by the induction hypothesis,} \\ &= 2^{s-2} . \end{aligned}$$

COROLLARY. $p(Q(n,r)) > f(n,r-1)$.

THEOREM 3.

(i) $p(Q(n,3)) = f(n-1,2) + 2f(n-2,2) + 2$,

(ii) $p(Q(n,3)) = p(Q(n-1,3)) + p(Q(n-2,3)) - 2$,

(iii) $p(Q(n,3)) = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n + 2$.

Proof. We use Laplace expansion of the permanent on the first two columns.

$$\begin{aligned} p(Q(n,3)) &= 1 + p(F(n-3,3)) + 2p(F(n-2,3)) + p(F(n-4,3)) \\ &\quad + p(F(n-3,3)) + 1 \\ &= p(F(n-1,3)) + 2p(F(n-2,3)) + 2 \\ &= f(n-1,2) + 2f(n-2,2) + 2, \text{ by Theorem 2.} \end{aligned}$$

Thus

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$$\begin{aligned} p(Q(n,3)) &= (f(n-2,2) + 2f(n-3,2) + 2) \\ &\quad + (f(n-3,2) + 2f(n-4,2) + 2) - 2 \\ &= p(Q(n-1,3)) + p(Q(n-2,3)) - 2 . \end{aligned}$$

The last part of the theorem is obtained by solving this linear difference equation using the fact that $p(Q(3,3)) = 6$ and $p(Q(4,3)) = 9$.

THEOREM 4.

(i) $p(Q(n,4)) = 2(f(n-1,3) + 2f(n-2,3) + 3f(n-3,3) + 1)$,

(ii) $p(Q(n,4)) = p(Q(n-1,4)) + p(Q(n-2,4)) + p(Q(n-3,4)) - 4$,

(iii) $p(Q(n,4)) = 2(\alpha_1^n + \alpha_2^n + \alpha_3^n + 1)$

where $\alpha_1 = \left(\sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}} + 1 \right) / 3 = 1.839286\ldots$

and α_2, α_3 are the other two roots of $x^3 - x^2 - x - 1 = 0$.

Proof. We expand $p(Q(n,4))$ using the first three columns, and after some simplification we obtain

$$\begin{aligned} p(Q(n,4)) &= 8p(F(n-3,4)) + 12p(F(n-4,4)) + 14p(F(n-5,4)) \\ &\quad + 8p(F(n-6,4)) + 2p(F(n-7,4)) + 2 \\ &= 8f(n-3,3) + 12f(n-4,3) + 14f(n-5,3) + 8f(n-6,3) \\ &\quad + 2f(n-7,3) + 2 \\ &= 2f(n-1,3) + 4f(n-2,3) + 6f(n-3,3) + 2 . \end{aligned}$$

It follows that

$$\begin{aligned} p(Q(n,4)) &= (2f(n-2,3) + 4f(n-3,3) + 6f(n-4,3) + 2) \\ &\quad + (2f(n-3,3) + 4f(n-4,3) + 6f(n-5,3) + 2) \\ &\quad + (2f(n-4,3) + 4f(n-5,3) + 6f(n-6,3) + 2) - 4 \\ &= p(Q(n-1,4)) + p(Q(n-2,4)) + p(Q(n-3,4)) - 4 . \end{aligned}$$

Part (iii) is obtained by solving the above difference equation in the usual way.

Note that formulas (3), (10), (4) and (11) in [3] are inconsistent with Theorems 3 and 4 and are incorrect.

Theorems 3(ii) and 4(ii) suggest that the formula

$$p(Q(n,r)) = p(Q(n-1,r)) + \dots + p(Q(n-r+1,r)) + \text{constant}$$
$$\sim K(r)\,\alpha^n ,$$

where $K(r)$ and α are defined as in (2), may hold for all r. We show that for $r > 4$ this is not the case.

Let $V_n = (v_{ij})$ be the n-square (0, 1)-matrix with $v_{ij} = 1$ if $|i-j| \leq 2$ and $v_{ij} = 0$ otherwise. We compute directly: $p(V_n) = 1, 2, 6, 14, 31$ for $n = 1, 2, 3, 4, 5$ respectively.

THEOREM 5. *If* $n > 5$ *then*

$$p(V_n) = 2p(V_{n-1}) + 2p(V_{n-3}) - p(V_{n-5}) .$$

Proof. Let Y_{n-1} be the (n-1)-square submatrix obtained from V_n by deleting the first row and the second column. Then expanding $p(V_n)$ by the first row we obtain

$$p(V_n) = p(V_{n-1}) + p(Y_{n-1}) + (p(Y_{n-2}) + p(V_{n-3}) + p(V_{n-4})) .$$

Similarly

$$p(V_{n-1}) = p(V_{n-2}) + p(Y_{n-2}) + (p(Y_{n-3}) + p(V_{n-4}) + p(V_{n-5}))$$

and therefore

$$p(V_n) - p(V_{n-1}) = p(V_{n-1}) + (p(Y_{n-1}) - p(Y_{n-2})) - p(V_{n-2})$$
$$+ (p(Y_{n-2}) - p(Y_{n-3})) + p(V_{n-3}) - p(V_{n-5}) .$$

Now, expanding $p(Y_{n-1})$ by the first column we have

$$p(Y_{n-1}) = p(V_{n-2}) + p(Y^T_{n-2}) ,$$

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where T denotes the transpose, and therefore

$$p(Y_{n-1}) - p(Y_{n-2}) = p(V_{n-2})$$

and

$$p(Y_{n-2}) - p(Y_{n-3}) = p(V_{n-3}) .$$

Hence

$$p(V_n) - p(V_{n-1}) = p(V_{n-1}) + 2p(V_{n-3}) - p(V_{n-5}) .$$

THEOREM 6. *If* $K(r)$ *and* α *are defined as in* (2) *then for a fixed* $r > 4$

$$\lim_{n\to\infty} p(Q(n, r))/K(r)\alpha^n = \infty .$$

Proof. For $n > 5$, by Theorem 5,

$$p(V_n) = 2p(V_{n-1}) + 2p(V_{n-3}) - p(V_{n-5})$$

and therefore

$$p(V_n) = \sum_{j=1}^{r} c_j\beta_j^n$$

where c_j are constants and β_j the roots of $x^5 - 2x^4 - 2x^2 + 1 = 0$. A straightforward computation shows that one real root of this equation is greater than 2.3 while the moduli of the other four roots do not exceed 1.2. It follows that for sufficiently large n

$$p(V_n) > 2.2^n .$$

Now observe that for $r \geq 5$

$$P_n^{-2}Q(n, r) \geq V_n ,$$

i.e. all the entries in the (0, 1)-circulant $P_n^{-2}Q(n, r)$ are greater

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than or equal to the corresponding entries in the (0, 1)-matrix V_n. Hence

$$p(P_n^{-2}Q(n,r)) = p(Q(n,r)) \geq p(V_n)$$

and for a sufficiently large n and $r \geq 5$

$$p(Q(n,r))/K(r)\alpha^n > p(Q(n,r))/2^n \geq p(V_n)/2^n > 1.1^n \to \infty .$$

We now show that for $r < n$ the permanent of the doubly stochastic matrix $Q(n,r)/r$ cannot be minimal in the polyhedron of doubly stochastic n-square matrices.

THEOREM 7. *For any* n *and* r, $r < n$, *there exists a doubly stochastic* n-*square matrix* S *such that*

$$p(S) < p(Q(n,r)/r) .$$

Proof. For $r = 1$, $n > 1$, and $r = 2$, $n > 2$, we have

$$p(Q(n,1)) = 1 > p(Q(n,n)/n) = n!/n^n ,$$

$$p(Q(n,2)/2) = 1/2^{n-1} > p(Q(n,n)/n) = n!/n^n .$$

We now prove the theorem for $r > 2$. If $p(Q(n,r)/r) \leq p(A)$, $r < n$, for all doubly stochastic n-square matrices A then, by a result due to M. Marcus and M. Newman [2], no permanental minor of a non-zero element of $Q(n,r)/r$ can exceed the permanental minor of a zero element. We show that $Q(n,r)$, and thus $Q(n,r)/r$, does not possess this property. Let Q_{ij} denote the submatrix of $Q(n,r)$ obtained by deleting the i'th row and the j'th column of $Q(n,r)$. Then $p(Q_{11})$ is the permanental minor of a non-zero element while $p(Q_{21})$ is the permanental minor of a zero element of $Q(n,r)$. We show that, for $r > 2$, $p(Q_{11}) > p(Q_{21})$. Now, $Q_{11} = Q_{21} + E_{1r}$ where E_{ij} denotes the $(n-1)$-square (0, 1)-matrix with 1 in the (i,j) position and 0 elsewhere. The $(n-1)$-square (0, 1)-matrix Q_{11} has 1's in its $(2,2)$, $(3,3)$, ..., $(r-1,r-1)$, $(r,r+1)$, $(r+1,r+2)$, ..., $(n-2,n-1)$, $(n-1,1)$ positions and thus the permanental minor of the $(1,r)$ element of Q_{11} is positive. Hence

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$$p(Q_{11}) = p(Q_{21} + E_{1r}) > p(Q_{21}) .$$

A table of $p(Q(n,r))$ is appended. Note that the entries differ from the corresponding entries in a table given in [3].

n/r	3	4	5	6	7
3	6				
4	9	24			
5	13	44	120		
6	20	80	265	720	
7	31	144	579	1,854	5,040
8	49	264	1,265	4,738	14,833
9	78	484	2,783	12,072	43,387
10	125	888	6,208	30,818	126,565
11	201	1,632	13,909	79,118	369,321
12	324	3,000	31,337	204,448	1,081,313
13	523	5,516	70,985	528,950	3,182,225
14	845	10,144	161,545	1,370,674	9,411,840
15	1,366	18,656	369,024	3,557,408	27,888,139

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