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# COMPLEX NUMBERS WITH THREE RADIX EXPANSIONS

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**1. Introduction.** This paper deals with the lack of uniqueness of the representations of the complex numbers in positional notation using Gaussian integers as bases.

Kátai and Szabó [3] proved that all the complex numbers can be written in radix form using the base $-n + i$ with the natural numbers $0, 1, 2, \ldots, n^2$ as digits. They remarked that they did not assert the uniqueness of these representations but gave no further indications of any multiple expansions. The geometry of these complex bases [2] indicates that some numbers have two expansions in a given base, while a few numbers even have three different expansions.

We give a complete description of all the multiple representations in the base $-n + i$ in terms of the complex base expansions. Numbers with two or three expansions correspond to certain infinite directed paths in a state graph which will be constructed for each complex base. The direction of this path leaving a given state is determined by the digits in the $k$th place of the different expansions.

The base $-1 + i$ provides a binary expansion of all the complex numbers and an example with three representations in this base is the number

$$(1 - 2i)/5 = (0.\overline{001})_{-1+i} = (1.\overline{100})_{-1+i} = (111.\overline{010})_{-1+i},$$

where the bar over a sequence of digits indicates that the sequence is to be repeated indefinitely. In each base, except $-2 + i$, the numbers with three different representations have expansions that are ultimately periodic with period three. The corresponding infinite paths in the state graph become ultimately trapped in a 3-cycle. In the exceptional base $-2 + i$, some numbers with three expansions have periods two and one, such as

$$(9 + 2i)/25 = (0.03\overline{40})_{-2+i} = (1.22\overline{04})_{-2+i} = (14.40\overline{2})_{-2+i}.$$

In the examples given above, the three expansions each have different integer parts. Geometrically, the points in the complex plane with two or more expansions with different integer parts lie on the boundary of a snowflake region of unit area surrounding each of the Gaussian integers [2]. These regions tile the plane and normally there are six points on

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the boundary of each region that also lie in two other regions; these points correspond to the numbers with three expansions with different integer parts. We show algebraically that for each base $-n + i$ $(n \neq 2)$ there are precisely six such numbers having a given integer part. We also show that for the base $-2 + i$ there are a countable number of such points. Geometrically this happens because the snowflake region corresponding to the base $-2 + i$ has a much more jagged reentrant form than that for the other bases and each region has a countable number of places where its width is zero, see [**1**], Fig. 7.

**2. Expansions in complex bases.** If $n$ is a fixed integer then the set of natural numbers $\{0, 1, 2, \ldots, n^2\}$ forms a complete residue system for the Gaussian integers modulo $-n + i$. If $n$ is positive, the number $b = -n + i$ can be used to uniquely represent the Gaussian integers with this complete residue system as digit set; that is, each Gaussian integer $z$ can be expressed uniquely as

$$z = \sum_{j=0}^{l} a_j b^j \quad \text{where } a_j \in \{0, 1, 2, \ldots, n^2\}.$$

These numbers, together with their conjugates $-n - i$, are the only numbers suitable as bases for all the Gaussian integers using natural numbers as digits [**3**].

Each of these bases can be used to represent all the complex numbers by means of infinite expansions involving negative powers of the base. We say that the complex number $z$ can be written in the base $b = -n + i$ if it can be expressed in the form

$$z = \sum_{j=-\infty}^{l} a_j b^j \quad \text{with } a_j \in \{0, 1, 2, \ldots, n^2\};$$

we denote this expansion by

$$z = (a_l a_{l-1} \ldots a_1 a_0 \cdot a_{-1} a_{-2} \ldots)_b.$$

We will often omit the subscript $b$ if the base is clear from the context. The digits to the left of the radix point define a Gaussian integer $(a_l a_{l-1} \ldots a_1 a_0)_b$ called the *integer part* of the expansion.

We now investigate the relationship between the integer parts of two expansions that represent the same number. Suppose $q$ and $r$ are two expansions of the same complex number in the base $b = -n + i$. Let

$$q = q_l q_{l-1} \ldots q_1 q_0 \cdot q_{-1} q_{-2} \ldots$$

and

$$r = r_l r_{l-1} \ldots r_1 r_0 \cdot r_{-1} r_{-2} \ldots$$

where some of the leading digits may be zero. Then

$$b^{-k}q = q_l \ldots q_k \cdot q_{k-1} \ldots$$

and

$$b^{-k}r = r_l \ldots r_k \cdot r_{k-1} \ldots$$

also represent the same complex number. For any positive or negative integer $k$, let $q(k) = (q_l q_{l-1} \ldots q_{k+1} q_k)_b$ be the integer part of $b^{-k}q$.

PROPOSITION 1. *Two expansions $q$ and $r$ represent the same complex number in the base $b = -n + i$ if and only if, for all integers $k$, either*

$$q(k) - r(k) = 0, \pm 1, \pm(n - 1 + i), \pm(n + i), \text{ if } n \neq 2$$

*or*

$$q(k) - r(k) = 0, \pm 1, \pm(1 + i), \pm(2 + i), \pm i, \pm(2 + 2i), \text{ if } n = 2.$$

*Proof.* Let $s = q(k) - r(k)$. Then, since $b^{-k}q = b^{-k}r$,

$$s = (0 \cdot r_{k-1} r_{k-2} \ldots)_b - (0 \cdot q_{k-1} q_{k-2} \ldots)_b = \sum_{t=1}^{\infty} (r_{k-t} - q_{k-t}) b^{-t}.$$

Now $|r_{k-t} - q_{k-t}| \leqq n^2$ so

$$|s| \leqq n^2 \sum_{t=1}^{\infty} |b|^{-t} = \frac{n^2}{|b| - 1} = \sqrt{n^2 + 1} + 1 < n + 2.$$

For $b = -n + i$, $z$ a Gaussian integer and $j$ a natural number, let

$$R_b^{\,j}(z) = \{zb^j + a_{j-1}b^{j-1} + a_{j-2}b^{j-2} + \ldots + a_1 b + a_0 \mid 0 \leqq a_t \leqq n^2\}.$$

Then $R_b^{\,j}(0)$ is the set of Gaussian integers representable in the base $b$ using at most $j$ digits. For $b = -1 + i$ and $-2 + i$, these regions $R_b^{\,j}(0)$ are depicted in [**2**], Figs. 2 and 4. Each set $R_b^{\,j}(z)$ is a complete residue system for the Gaussian integers modulo $b^j$. For fixed $j$ and $b$, the sets $R_b^{\,j}(z)$ partition the Gaussian integers into congruent jigsaw pieces.

Now, for any $j \geqq 0$,

$$b^j s + (q_{k-1} \ldots q_{k-j})_b - (r_{k-1} \ldots r_{k-j})_b = \sum_{t=1}^{\infty} (r_{k-j-t} - q_{k-j-t}) b^{-t}$$

which has absolute value less than $n + 2$. However $b^j s + (q_{k-1} \ldots q_{k-j})_b \in R_b^{\,j}(s)$ and $(r_{k-1} \ldots r_{k-j})_b \in R_b^{\,j}(0)$. Therefore the regions $R_b^{\,j}(s)$ and $R_b^{\,j}(0)$ are within a distance $n + 2$ of each other for all $j \geqq 0$. We will use the shape and configuration of these regions to determine the possible values for $s$.

For the base $b = -1 + i$, [**2**], Fig. 7 shows the regions $R^8_{-1+i}(z)$ divided by the factor $(-1 + i)^8 = 16$; the regions are labelled by $z$ in this figure. The two regions $R^8_{-1+i}(s)$ and $R^8_{-1+i}(0)$ are within distance 3 of each other if the region labelled $s$ is within 3/16 of the region containing the origin in [**2**], Fig. 7. Hence the only possible values of $s$ are $0, \pm 1, \pm i$ or $\pm(1 + i)$.

The form of the regions $R_b{}^1(z)$ is shown in Figure 1 and the construction of $R_b{}^j(z)$ for higher values of $j$ is described in [**1**]. For $n \geqq 3$ it is sufficient to look at the regions $R_b{}^3(z)$ to determine that $s = 0, \pm 1, \pm(n - 1 + i)$ or $\pm(n + i)$. The general pattern of these regions is seen in $R^3_{-3+i}(z)$ shown in Figure 2.

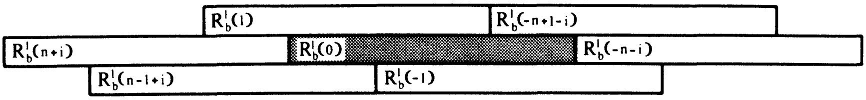


FIGURE 1.

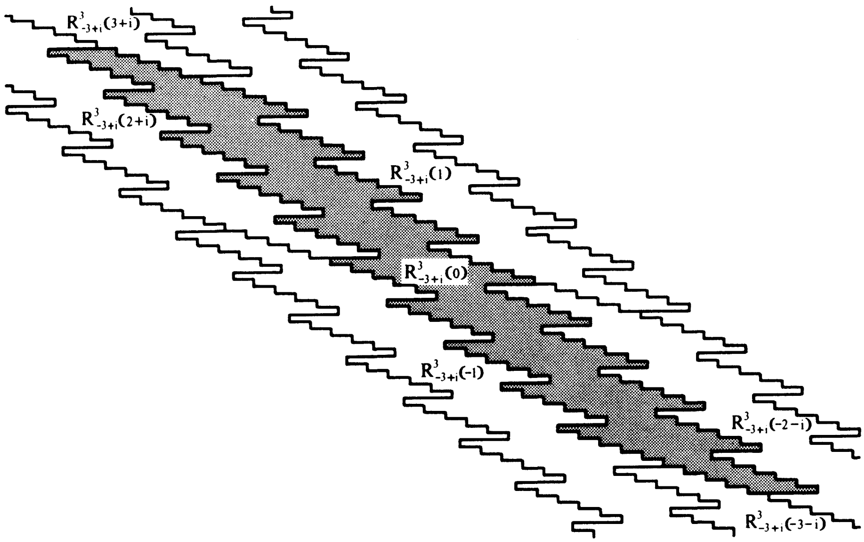


FIGURE 2.

The base $b = -2 + i$ is an exception due to the strongly reentrant form of the regions $R^j_{-2+i}(z)$ (c.f. [**1**], Fig. 7 which shows $R^j_{-2+i}(z)$ divided by $(-2 + i)^j$ for large $j$) and $s$ cannot be restricted to seven values. However by comparing the regions $R^2_{-2+j}(z)$ within distance 4 of each other in Figure 3 it follows that $s = 0, \pm 1, \pm(1 + i), \pm(2 + i), \pm i$ or $\pm(2 + 2i)$.

This completes the proof of the necessity of Proposition 1. The sufficiency follows because the boundedness of $q(k) - r(k)$ implies $q$ and $r$ must converge to the same value.

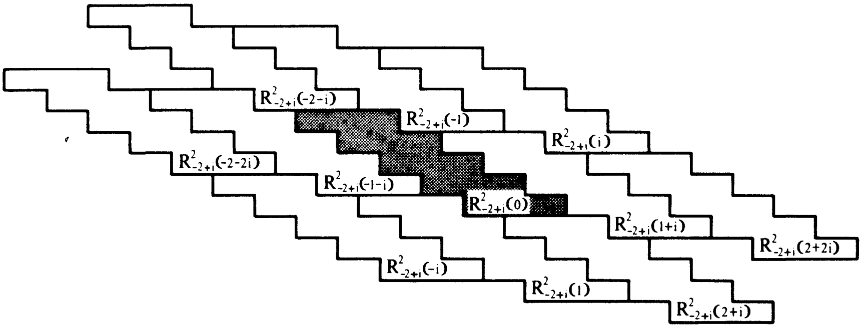


FIGURE 3.

**3. Description of the states.** Let

$$p = p_l p_{l-1} \ldots p_1 p_0 \cdot p_{-1} p_{-2} \ldots$$
$$q = q_l q_{l-1} \ldots q_1 q_0 \cdot q_{-1} q_{-2} \ldots$$
$$r = r_l r_{l-1} \ldots r_1 r_0 \cdot r_{-1} r_{-2} \ldots$$

be three expansions representing the same complex number in the base $b = -n + i$. For any integer $k$, let

$$S(k) = (p(k) - q(k), q(k) - r(k), r(k) - p(k)).$$

These three differences must satisfy Proposition 1 for each $k$. Now $p(k)$ is obtained from $p(k + 1)$ by shifting $p(k + 1)$ left one place and adding $p_k$; that is $p(k) = bp(k + 1) + p_k$. Given $S(k + 1)$ there are only certain values of $(p_k, q_k, r_k)$ for which the three differences in $S(k)$ satisfy Proposition 1. For each base we will produce a state graph whose states are the allowable values of $S(k)$, subject to Proposition 1, and whose transition function from the state $S(k + 1)$ to the state $S(k)$ is determined by these certain values of $(p_k, q_k, r_k)$. Each directed edge in the state graph will be labelled by the permissible values of $(p_k, q_k, r_k)$, written in column form. Each triple $p, q, r$ of expansions representing the same complex number will correspond to an infinite path in this state graph through the states $S(l + 1), S(l), S(l - 1), \ldots$.

The possible types of states are described by the symbols in Figure 4. These symbols show the relative position of $p(k)$, $q(k)$ and $r(k)$ in the state $S(k)$. One square to the east of another denotes a difference of 1; one square north denotes a difference of $n - 1 + i$ and one square northeast a difference of $n + i$. For the case $n = 2$ only, one square northwest denotes a difference of $n - 2 + i = i$.

A state is in fact a triple of numbers, occurring in Proposition 1, whose sum is zero. The symbols in Figure 4 describe the following states:

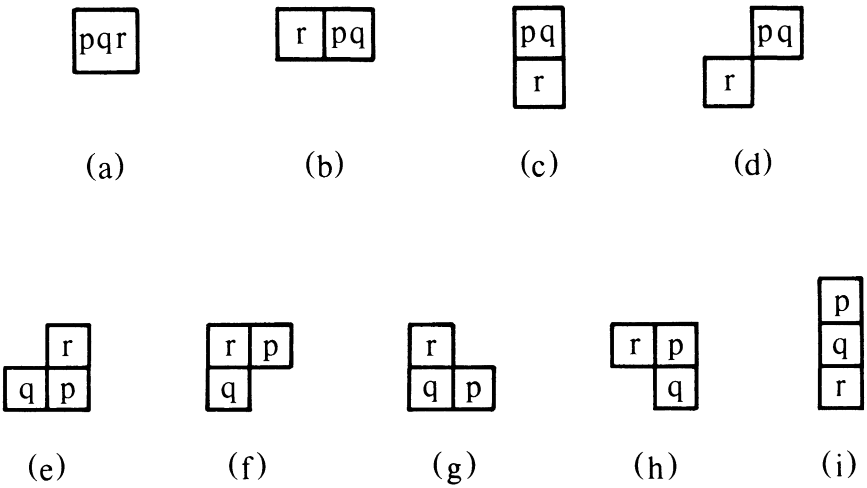


FIGURE 4. The types of states.

(a) $S(k) = (0, 0, 0)$ (the initial state)
(b) $S(k) = (0, 1, -1)$
(c) $S(k) = (0, n - 1 + i, -n + 1 - i)$
(d) $S(k) = (0, n + i, - n - i)$
(e) $S(k) = (1, -n - i, n - 1 + i)$
(f) $S(k) = (n + i, -n + 1 - i, -1)$
(g) $S(k) = (1, -1 - i, i)$ (for $n = 2$ only)
(h) $S(k) = (1 + i, -i, -1)$ (for $n = 2$ only)
(i) $S(k) = (1 + i, 1 + i, -2 - 2i)$ (for $n = 2$ only).

There are theoretically other possible types of states when $n = 2$, such as $S(k) = (0, i, -i)$. However we shall see in Section 6 that these are essentially contained in the types given in Figure 4.

If $n \neq 2$, it is not possible to have four different numbers $p(k)$, $q(k)$, $r(k)$, $t(k)$, all of whose differences satisfy Proposition 1. If $n = 2$, it is possible to have four such numbers, but we show in Section 6 that they cannot lead to four different expansions of a given number.

**4. Multiple expansions in the base $-1 + i$.** We now indicate how the state graph for the base $-1 + i$, shown in Figure 5, is constructed.

Let $p, q$ and $r$ be three expansions representing the same complex number in the base $-1 + i$. The initial state $S(l + 1) = (0, 0, 0)$ is indicated by an unlabelled arrow at the top of the state graph. Suppose that the expansions $p$, $q$ and $r$ agree down to the $(k + 1)$st position; that is $p_j = q_j = r_j$ for $j \geqq k + 1$. This corresponds to the state $S(k + 1) = (0, 0, 0)$. The base $-1 + i$ provides a binary representation so $p_k, q_k, r_k \in$

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$\{0, 1\}$ and either

(i) $p_k = q_k = r_k$

or

(ii) two of $p_k, q_k, r_k$ are equal and one is different.

In case (i), $S(k) = S(k+1) = (0, 0, 0)$ and so the $k$th state is the same. This is indicated by the loop at the top of Figure 5. The digits in the $k$th position are

$$\begin{pmatrix} p_k \\ q_k \\ r_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

which is abbreviated to $\begin{matrix}0\\0+\\0\end{matrix}$ in Figure 5. In case (ii) relabel $p, q, r$, if necessary, so that $p_k = q_k$. Then the next state is either

(iia) $S(k) = (0, 1, -1)$ with $(p_k, q_k, r_k) = (1, 1, 0)$

or

(iib) $S(k) = (0, -1, 1)$ with $(p_k, q_k, r_k) = (0, 0, 1)$.

These two states are shown in the second level of the state graph in Figure 5. Case (iia) is the state shown in Figure 4(b), while case (iib) corresponds to a similar state in which $pq$ and $r$ are reversed.

Now consider the successor to the state $S(k+1) = (0, 1, -1)$. This means $p(k+1) = q(k+1) = r(k+1) + 1$ and multiplying by the base $-1 + i$ we obtain

$$p(k) - p_k = q(k) - q_k = r(k) - r_k - 1 + i.$$

The only possible binary values for the digits in the $k$th position for which Proposition 1 is satisfied are $(p_k, q_k, r_k) = (1, 1, 0)$. Therefore there is only one directed edge leaving the state $(0, 1, -1)$ in Figure 5 and the next state is $S(k) = (0, i, -i)$, which is the one shown in Figure 4(c). If $(p_k, q_k, r_k)$ takes any values other than $(1, 1, 0)$ then the expansions $p$, $q$ and $r$ cannot all represent the same complex number in the base $-1 + i$.

If $S(k+1) = (0, i, -i)$, there are five possible values of $(p_k, q_k, r_k)$ for which Proposition 1 is satisfied; namely

(ia) $(p_k, q_k, r_k) = (0, 0, 0)$ with $S(k) = (0, 1 + i, -1 - i)$,
(ib) $(p_k, q_k, r_k) = (1, 1, 1)$ with $S(k) = (0, 1 + i, -1 - i)$,
(ii) $(p_k, q_k, r_k) = (1, 1, 0)$ with $S(k) = (0, -i, i)$,
(iiia) $(p_k, q_k, r_k) = (1, 0, 0)$ with $S(k) = (1, -1 - i, i)$ and
(iiib) $(p_k, q_k, r_k) = (0, 1, 0)$ with $S(k) = (-1, -i, 1 + i)$.

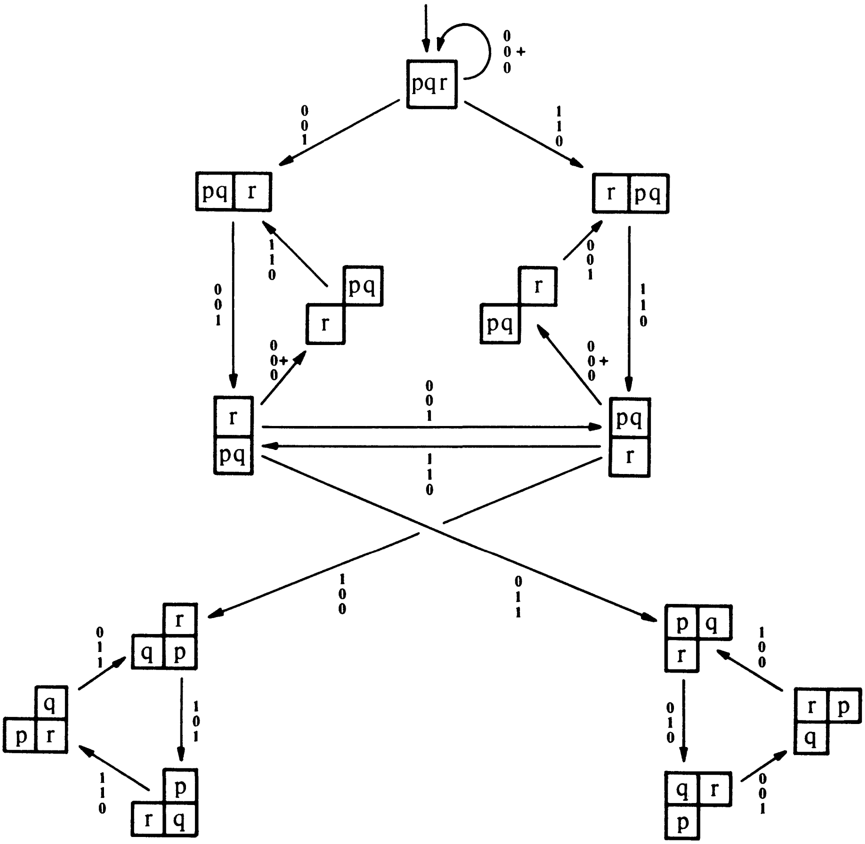

FIGURE 5. The state graph for the base $-1 + i$.

Cases (ia) and (ib) lead to the same state. In the last case (iiib), the labels $p$ and $q$ can be interchanged and this then is equivalent to the case (iiia). Hence there are three possible successors to the state $(0, i, -i)$ shown in Figure 5.

By continuing in this manner and considering the successors to the states until all the possibilities have been exhausted we obtain the state graph shown in Figure 5. Proposition 1 now yields the following result.

THEOREM 2. *The three expansions, $p$, $q$ and $r$ will represent the same complex number in the base $-1 + i$ if and only if they can be obtained from an infinite directed path in the state graph of Figure 5 starting with the state* $(0, 0, 0)$ *and, if necessary, relabelling $p$, $q$ and $r$.*

If the infinite path remains in the initial state then $p$, $q$ and $r$ have identical expansions; that is $p_k = q_k = r_k$ for all $k$. If the path remains in the next six states then $p$ and $q$ have identical expansions but $r$ has a different expansion which represents the same complex number as $p$ and $q$. Finally if the path enters one of the bottom six states it is trapped and must cycle with period three. This implies the following corollary.

COROLLARY 3. *The numbers with three representations in the base* $-1+i$ *have expansions that are ultimately periodic with period three whose periodic digits are* $\overline{001}$ *or* $\overline{110}$.

COROLLARY 4. *For a given Gaussian integer* $z$, *there are precisely six numbers that have three expansions in the base* $-1+i$, *all with different integer parts, one part being* $z$. *The fractional parts of these six numbers are* $(.\overline{001})_{-1+i} = (1-2i)/5$, $(.\overline{010})_{-1+i} = (1+3i)/5$, $(.\overline{100})_{-1+i} = (-4-2i)/5$, $(.\overline{011})_{-1+i} = (2+i)/5$, $(.\overline{110})_{-1+i} = (-3+i)/5$ *and* $(.\overline{101})_{-1+i} = (-3-4i)/5$.

*Proof.* The only sets of three different expansions obtainable from Figure 5 which have different integer parts, one of which is zero, are as follows.

$$\begin{array}{lll} p = 000.\overline{001}; & p = 0000.\overline{010}; & p = 00000.\overline{100}; \\ q = 001.\overline{100} & q = 0011.\overline{001} & q = 00110.\overline{010} \\ r = 111.\overline{010} & r = 1110.\overline{100} & r = 11101.\overline{001} \end{array}$$

$$\begin{array}{lll} p = 0001.\overline{110}; & p = 00011.\overline{101}; & p = 111.\overline{110} \\ q = 0000.\overline{011} & q = 00000.\overline{110} & q = 110.\overline{011} \\ r = 1110.\overline{101} & r = 11101.\overline{011} & r = 000.\overline{101} \end{array}$$

The values of each of these expansion can be calculated in the usual way by multiplying an expansion by the cube of the base and subtracting the expansion from it. This yields the stated values and proves the corollary.

Note that there are only countably many complex numbers with three different expansions and they all have rational real and imaginary parts. However there are an uncountable number of choices of paths that leave the initial state but remain in the next six states and hence there are uncountably many complex numbers with two different expansions. This agrees with the geometric view. The numbers with two expansions containing different integer parts, one being zero, form the boundary curve of a snowflake region such as shown in [**1**], Fig. 6. Examples of such numbers with irrational real or imaginary parts may be constructed as follows. Let $0\cdot s_{-1}s_{-2}s_{-3}\ldots$ be an irrational binary number in base 2. Then

$$q = 0\cdot 0s_{-1}10s_{-2}10s_{-3}\ldots$$
$$r = 1\cdot 1s_{-1}01s_{-2}01s_{-3}\ldots$$

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are two aperiodic expansions representing the same complex number in base $-1 + i$.

**5. Multiple expansions in the base $-n + i$.** In this section the multiple expansions in the base $-n + i$ are determined for $n \geqq 3$. The state graph shown in Figure 6 is similar to that for the base $-1 + i$ except that there are a few more directed edges and more choices for the digits at each stage. The digits $p_k, q_k, r_k \in \{0, 1, 2, \ldots, n^2\}$ and in Figure 6 the notation

a means that $p_k = a + h$ for any integer $h$
b+ $q_k = b + h$
c $r_k = c + h$

such that $a + h$, $b + h$ and $c + h$ are allowable digits for the base $-n + i$; that is, $0 \leqq a + h \leqq n^2$, $0 \leqq b + h \leqq n^2$ and $0 \leqq c + h \leqq n^2$.

The construction of the state graph is similar to that for the base $-1 + i$ and, as before, Proposition 1 leads to the following results.

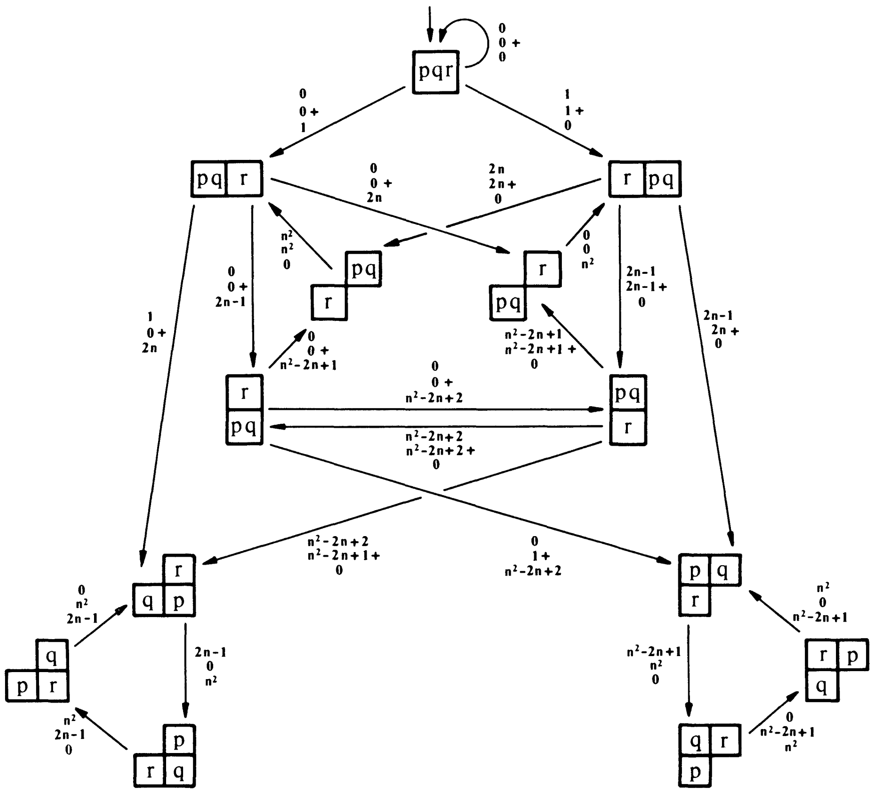

FIGURE 6. The state graph for the base $-n + i$ where $n \geqq 3$.

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THEOREM 5. *Let $n \geqq 3$. Then three expansions $p$, $q$ and $r$ will represent the same complex number in the base $-n + i$ if and only if they can be obtained from an infinite path in the state graph of Figure 6 starting with the state $(0, 0, 0)$ and, if necessary, relabelling $p$, $q$ and $r$.*

COROLLARY 6. *The numbers with three expansions in the base $-n + i$ (for $n \geqq 3$) have expansions that are ultimately periodic with period three whose periodic digits are $\overline{n^2 - 2n + 1\ 0\ n^2}$ or $\overline{2n - 1\ n^2\ 0}$.*

COROLLARY 7. *For a given Gaussian integer $z$, there are precisely six numbers that have three expansions in the base $-n + i$ ($n \geqq 3$), all with different integer parts, one part being $z$.*

The six sets of expansions in this corollary, with one integer part zero, are as follows.

$$\begin{aligned}
p &= 0\quad 0\quad 0\ \cdot\overline{n^2 - 2n + 1\quad 0\quad n^2}\\
q &= 0\quad 0\quad 1\ \cdot\overline{n^2\quad n^2 - 2n + 1\quad 0}\\
r &= 1\quad 2n - 1\quad n^2 - 2n + 2\cdot\overline{0\quad n^2\quad n^2 - 2n + 1}
\end{aligned}$$

$$\begin{aligned}
p &= 1\quad 2n - 1\cdot\overline{n^2 - 2n + 1\quad 0\quad n^2}\\
q &= 0\quad 2n\ \cdot\overline{n^2\quad n^2 - 2n + 1\quad 0}\\
r &= 0\quad 0\ \cdot\overline{0\quad n^2\quad n^2 - 2n + 1}
\end{aligned}$$

$$\begin{aligned}
p &= 1\quad 2n - 1\quad n^2 - 2n + 1\cdot\overline{0\quad n^2\quad n^2 - 2n + 1}\\
q &= 1\quad 2n\quad n^2\ \cdot\overline{n^2 - 2n + 1\quad 0\quad n^2}\\
r &= 0\quad 0\quad 0\ \cdot\overline{n^2\quad n^2 - 2n + 1\quad 0}
\end{aligned}$$

$$\begin{aligned}
p &= 0\quad 1\ \cdot\overline{2n - 1\quad n^2\quad 0}\\
q &= 0\quad 0\ \cdot\overline{0\quad 2n - 1\quad n^2}\\
r &= 1\quad 2n\cdot\overline{n^2\quad 0\quad 2n - 1}
\end{aligned}$$

$$\begin{aligned}
p &= 0\quad 1\quad 2n - 1\cdot\overline{n^2\quad 0\quad 2n - 1}\\
q &= 0\quad 0\quad 0\ \cdot\overline{2n - 1\quad n^2\quad 0}\\
r &= 1\quad 2n\quad n^2\ \cdot\overline{0\quad 2n - 1\quad n^2}
\end{aligned}$$

$$\begin{aligned}
p &= 1\quad 2n - 1\quad n^2 - 2n + 2\cdot\overline{2n - 1\quad n^2\quad 0}\\
q &= 1\quad 2n - 1\quad n^2 - 2n + 1\cdot\overline{0\quad 2n - 1\quad n^2}\\
r &= 0\quad 0\quad 0\ \cdot\overline{n^2\quad 0\quad 2n - 1}
\end{aligned}$$

In particular, the digit set for the base $-3 + i$ is $\{0, 1, 2, \ldots, 9\}$ and so this base yields a decimal representation of the complex numbers. Therefore two numbers with three different decimal expansions are

$$(0.\overline{409})_{-3+i} = (1.\overline{940})_{-3+i} = (155.\overline{094})_{-3+i} = (-23 - 10i)/17$$

and

$$(1.\bar{5}\bar{9}\bar{0})_{-3+i} = (0.\bar{0}\bar{5}\bar{9})_{-3+i} = (16.\bar{9}\bar{0}\bar{5})_{-3+i} = (4+i)/17.$$

**6. Multiple expansions in the base $-2 + i$.** The base $-2 + i$ yields a more complicated state graph because of the extra possibilities in Proposition 1 and hence the extra states shown in Figure 1(g), (h) and (i). The state graph shown in Figure 7 is constructed in a similar way to the previous ones.

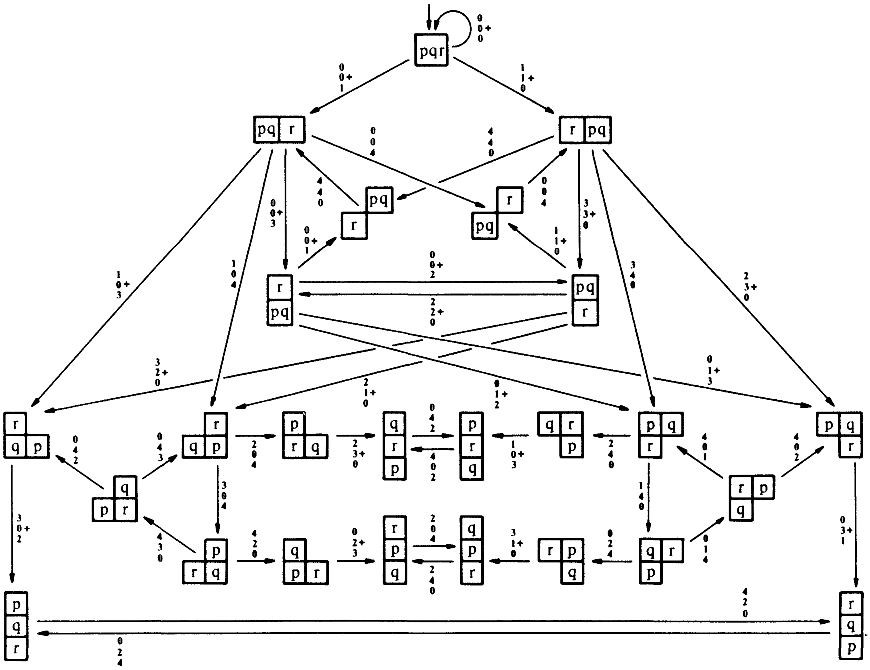

FIGURE 7. The state graph for the base $-2 + i$.

As we mentioned earlier, there are some states which satisfy Proposition 1 that have not been included. For example, suppose $S(k+1) = (1, -1-i, i)$, the state shown in Figure 1(g). Then $p_k, q_k, r_k \in \{0, 1, 2, 3, 4\}$ and there are six sets of possible values of these digits in the $k$th position that satisfy Proposition 1. They are as follows.

(ia) $(p_k, q_k, r_k) = (3, 0, 2)$ with $S(k) = (1+i, 1+i, -2-2i)$

(ib) $(p_k, q_k, r_k) = (4, 1, 3)$ with $S(k) = (1+i, 1+i, -2-2i)$

(iia) $(p_k, q_k, r_k) = (2, 0, 1)$ with $S(k) = (i, 2+i, -2-2i)$

(iib) $(p_k, q_k, r_k) = (3, 1, 2)$ with $S(k) = (i, 2+i, -2-2i)$

(iic) $(p_k, q_k, r_k) = (4, 2, 3)$ with $S(k) = (i, 2+i, -2-2i)$

(iii) $(p_k, q_k, r_k) = (4, 0, 3)$ with $S(k) = (2+i, i, -2-2i)$.

The cases (ia) and (ib) are shown in Figure 7 as the farthest left vertical directed edge. The other cases are not shown in Figure 7 because there are no successor states to $S(k) = (i, 2 + i, -2 - 2i)$ or $(2 + i, i, -2 - 2i)$ satisfying Proposition 1.

There are three possible types of states with four different expansions $p, q, r, t$, namely

(i) $p(k) = q(k) + 1 = r(k) - 1 - i = t(k) - i$

(ii) $p(k) = q(k) + i = r(k) + 1 + i = t(k) + 2 + 2i$

(iii) $p(k) = q(k) + 1 + i = r(k) + 2 + i = t(k) + 2 + 2i.$

All the six differences between $p(k)$, $q(k)$, $r(k)$ and $t(k)$ satisfy Proposition 1. The possible successors to the state shown in case (i) are the states in cases (ii) and (iii). However, these latter states have no successors satisfying Proposition 1 because they contain the states mentioned in the previous paragraph, namely

$$(p(k) - q(k), q(k) - t(k), t(k) - p(k)) = (i, 2 + i, -2 - 2i)$$

and

$$(p(k) - r(k), r(k) - t(k), t(k) - p(k)) = (2 + i, i, -2 - 2i).$$

Hence four different expansions in the base $-2 + i$ cannot occur.

One type of state that can occur, but is not shown, is $S(k) = (0, i, -i)$. However, any path entering this state ends up cycling with period two between the states $(0, -2 - 2i, 2 + 2i)$ and $(0, 2 + 2i, -2 - 2i)$. Hence $p$ and $q$ have the same expansions but the number these expansions represent in fact has three different expansions. Therefore $q$ can be replaced by a third expansion, different from $p$ and $r$, such that the path corresponding to this new triple $p, q, r$ does appear in Figure 7. For example, the following expansions represent the same complex number in the base $-2 + i$.

$$p = 0.120\overline{04}$$
$$q = 0.120\overline{04}$$
$$r = 0.001\overline{40}.$$

However $q$ can be replaced by another expansion representing the same complex number; namely

$$p = 0.120\overline{04}$$
$$q = 0.133\overline{2}$$
$$r = 0.001\overline{40}.$$

Eliminating such cases we obtain the following result.

THEOREM 8. *The three expansions $p$, $q$ and $r$ will represent the same complex number in the base $-2 + i$ if and only if they can be obtained from*

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*an infinite path in the state diagram of Figure 7 starting with the state* (0, 0, 0), *if necessary relabelling* $p$, $q$ *and* $r$ *and in some cases, if* $p = q$, *by replacing* $q$ *by another expansion.*

COROLLARY 9. *The numbers with three representations in the base* $-2 + i$ *have expansions that are ultimately periodic with period three, two or one, whose periodic digits are* $\overline{104}$, $\overline{340}$, $\overline{04}$ *or* $\overline{2}$.

COROLLARY 10. *For a given Gaussian integer* $z$, *there are a countable number of complex numbers that have three expansions in the base* $-2 + i$ *all with different integer parts, one part being* $z$.

As before there are six such numbers with period three. However, for this base, there are a countable number with period one and two. For example, a countable collection of such expansions, with one integer part being zero, is

$$p = 01.(340)^s 22\overline{04}$$
$$q = 00.(034)^s 03\overline{40}$$
$$r = 14.(403)^s 40\overline{2}$$

for any nonnegative integer $s$.

Corollary 10 has the following geometric interpretation. For each Gaussian integer $z$, there is a snowflake region of unit area consisting of all the complex numbers with a base $-2 + i$ expansion having integer part $z$ [**1**], Figure 7. The boundary of this region is a fractal curve consisting of numbers with at least two expansions in base $-2 + i$ with different integer parts. This boundary intersects itself a countable number of times, giving rise to a countable number of points lying on the boundary of three such snowflake regions.

## REFERENCES

**1.** W. J. Gilbert, *The fractal dimension of sets derived from complex bases*, submitted.

**2.** ——— *Fractal geometry derived from complex bases*, The Mathematical Intelligencer *4* (1982), 78–86.

**3.** I. Kátai and J. Szabó, *Canonical number systems for complex integers*, Acta Sci. Math. (Szeged) *37* (1975), 255–260.

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