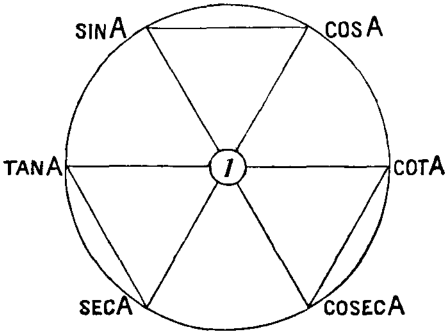
Trigonometrical Mnemonic.

By John Alison, M.A.



If the functions be written as in the above figure, then

1. Of any three functions written consecutively on a circular arc or on a diameter, the mean is the product of the extremes. Thus $\sin A = \tan A \cos A$, $1 = \sin A \operatorname{cosec} A$, and so on.
2. In each triangle the square of the function written at the vertex turned downwards is equal to the sum of the squares of the functions written at the other two vertices. Thus $1^2 = \sin^2 A + \cos^2 A$, and so on.

Notes on Integration by parts and by successive reduction.

By George A. Gibson, M.A.

My object in the following notes is to call attention to some points in integration by successive reduction which may be of use in directing the choice of the particular form for the reduced integral in any given case. The remarks apply chiefly to the binomial differential $x^m(a+bx^n)^p dx$ which has been discussed from a different point of view by Dr Muir in vol. iii., p. 100 of the Proceedings.

Let $f(x)$ be a function of x which may be written as the product of two functions u, v of x. Let $u' = \frac{du}{dx}$, $u_1 = \int u dx$, $v_1 = \int v dx$ &c. Then the integral of $f(x) \cdot dx$ may be expressed in either of the forms

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$$\text{(i)} \quad \int f(x)dx = \int u.vdx = u_1.v - \int u_1.v'dx$$

$$\text{(ii)} \quad \int f(x)dx = \int u.vdx = u.v_1 - \int u'.v_1dx.$$

Now an integral to be found by successive reduction can often be treated by both of these formulae. Take $f(x) = x^m(a+bx^n)^p$ and put $u = x^m$, $v = (a+bx^n)^p$; we have therefore by (i)

$$\int x^m(a+bx^n)^p dx = \frac{x^{m+1}(a+bx^n)^p}{m+1} - \frac{bnp}{m+1}\int x^{m+n}(a+bx^n)^{p-1}dx \quad \text{(A)}$$

Or we may put $u = x^{m-n+1}$, $v = x^{n-1}(a+bx^n)^p$ and we get by (ii)

$$\int x^m(a+bx^n)^p dx = \frac{x^{m-n+1}(a+bx^n)^{p+1}}{bn(p+1)} - \frac{m-n+1}{bn(p+1)}.\int x^{m-n}(a+bx^n)^{p+1}dx \quad \text{(B)}$$

For present purposes, the indices m and p are the most important quantities of which the given integral is a function, and we may therefore denote $\int x^m(a+bx^n)^p dx$ by $\phi(m,\ p)$. Equation (A) then expresses $\phi(m,\ n)$ in terms of $\phi(m+n,\ p-1)$ and (B) in terms of $\phi(m-n,\ p+1)$; and it is clear that the operation of integrating by parts will generally alter both m and p, increasing one and decreasing the other. It should be noted however that m must become $m \pm n$ and p, $p \mp 1$. But out of (A) and (B) we can readily get an integral in which only one index is altered, but altered by being *reduced.* For since $x^{m+n} = x^m.\dfrac{(a+bx^n)-a}{b}$ we get

$$\begin{aligned}\phi(m+n,\ p-1) &= \frac{1}{b}\int x^m(a+bx^n)^p dx - \frac{a}{b}\int x^m(a+bx^n)^{p-1}dx \\ &= \frac{1}{b}\cdot\phi(m,\ p) - \frac{a}{b}\cdot\phi(m,\ p-1).\end{aligned}$$

Substituting this value of $\phi(m+n,\ p-1)$ in (A) we get

$$\phi(m,\ p) = \frac{x^{m+1}(a+bx^n)^p}{m+np+1} + \frac{anp}{m+np+1}\cdot\phi(m,\ p-1) \quad \text{(C)}$$

In the same way since $(a+bx^n)^{p+1} = (a+bx^n)^p(a+bx)$, we have $\phi(m-n,\ p+1) = a\phi(m-n,\ p) + b\phi(m,\ p)$.

Then we get from (B)

$$\phi(m,\ p) = \frac{x^{m-n+1}(a+bx^n)^{p+1}}{b(m+np+1)} - \frac{a(m-n+1)}{b(m+np+1)}\cdot\phi(m-n,\ p) \quad \text{(D)}$$

Should it be desired to express $\phi(m, p)$ in terms of an integral in which one of the quantities m, p is unaltered while the other is increased, the previous work shows that the integrals we are in search of must be either $\phi(m+n, p)$ or $\phi(m, p+1)$ and it is usually easier to begin with one of these latter forms and *work backwards*, that is, first break up the factor with increased index and then integrate by parts. Thus we have

$$\phi(m+n, p)=\int x^m \cdot \frac{(a+bx^n)-a}{b} \cdot (a+bx^n)^p dx$$
$$=\frac{1}{b}\int x^m(a+bx^n)^{p+1}dx-\frac{a}{b}\cdot\phi(m, p).$$

Integrate $\int x^m(a+bx^n)^{p+1}dx$ by parts, and we get

$$\phi(m+n, p)=\frac{x^{m+1}(a+bx^n)^{p+1}}{b(m+1)}-\frac{n(p+1)}{m+1}\cdot\phi(m+n, p)-\frac{a}{b}\phi(m, p)$$

therefore $$\phi(m, p)=\frac{x^{m+1}(a+bx^n)^{p+1}}{a(m+1)}-\frac{b(m+np+n+1)}{a(m+1)}\cdot\phi(m+n, p) \qquad \text{(E)}$$

In the same way by beginning with the form $\phi(m, p+1)$ we get

$$\phi(m, p)=-\frac{x^{m+1}(a+bx^n)^{p+1}}{an(p+1)}+\frac{m+np+n+1}{an(p+1)}\cdot\phi(m, p+1) \qquad \text{(F)}$$

The process applied to $\int x^m(a+bx^n)^p dx$ is equally well suited for the class of integrals of the type $\int \sin^m x\cdot\cos^n x dx$. Thus take $u=\sin^m x\cdot\cos x$ and $v=\cos^{n-1}x$: then we have

$$\int\sin^m x\cdot\cos^n x dx=\frac{\sin^{m+1}x\cdot\cos^{n-1}x}{m+1}+\frac{n-1}{m+1}\int\sin^{m+2}x\cdot\cos^{n-2}x\cdot dx.$$

Denoting $\int\sin^m x\cos^n x dx$ by $\psi(m, n)$ we see that $\psi(m, n)$ may be expressed in terms of $\psi(m+2, n-2)$, $\psi(m-2, n+2)$, $\psi(m, n-2)$, $\psi(m-2, n)$, $\psi(m+2, n)$, $\psi(m, n+2)$. Indeed it would be better for teaching purposes to go through the various reductions for this integral, and then to consider the more general case of the binomial differential.

The results reached may perhaps be expressed as follows, it being understood that the "factors" referred to are of the kind considered in the note :—

In integrating a product of two factors by parts, we get in the first instance a *second integral* in which one of the original factors appears with a lower index and the other with a higher, the decrease and the increase of the indices however following a definite law. This *second integral* may be altered by breaking up the factor whose index has been increased; this breaking up will give the original integral and another in which the index of one factor is the same as in the original integral, while the index of the second factor is lower than at first. Lastly, if a form be desired in which one of the original indices is increased, the other index remaining unaltered, we first observe by how much it is possible to increase the index; we take this integral with increased index, break it up into two integrals, and then integrate by parts.

Question proposée au Concours Général pour la classe de Mathématiques Spéciales, Juin, 1886.

Solution analytique par M. PAUL AUBERT.

Étant donnés une surface du second ordre S et deux points A et B, on mène par le point B une sécante qui rencontre la surface aux points C, C′, et le plan polaire du point A au point D. Soient M et M′ les points où la droite AD rencontre les plans qui touchent la surface aux points C et C′. La sécante BD tournant autour du point B, on demande

1°. *Le lieu décrit par les points M et M′.*

2°. *Ce lieu se compose de deux surfaces du second ordre, dont l' une est indépendante de la position du point B, et l' autre Σ dépend de la position de ce point. Chercher ce que devient la surface Σ quand, dans la construction qui donne les points de cette surface, on fait jouer au point A le rôle du point B, et inversement.*

3°. *Le point A restant fixe, déterminer les positions occupées par le point B quand la surface Σ n' a pas un centre unique à distance finie.*

1°. Rapportons la surface à un tétraèdre de référence ayant pour arêtes opposées la droite AB et la droite polaire conjuguée A′B′.