Determinants, IV, p. 148) in referring to a paper in which $(-)^{\frac{1}{2}n(n-1)}|J|$ arises as the Jacobian of the functions $a_1, a_2, \ldots, a_n$.

The identity (1) may be extended to

$$\mathcal{A}(\ldots.;\ x)\,\mathcal{H}(\ldots.,\ a,\ \beta,\ \ldots.;\ x) = \mathcal{H}(a,\ \beta,\ \ldots.;\ x).$$

and similarly we have

$$\mathcal{A}(\ldots.,\ a,\ \beta,\ \ldots.;\ x)\,\mathcal{H}(\ldots.;\ x) = \mathcal{A}(a,\ \beta,\ \ldots.:\ x).$$

[In particular, (4) is an immediate consequence of

$$\mathcal{A}(a,\ \beta,\ \ldots.,\ \kappa;\ x)\,\mathcal{H}(a;\ x) = \mathcal{A}(\beta,\ldots.,\kappa;\ x).]$$

Further, $\mathcal{A}(\ldots.,\ a,\ \beta,\ldots.;\ x)\,\mathcal{H}(\ldots.,\lambda,\ \mu,\ldots.;\ x)$
$$= \mathcal{A}(a,\ \beta,\ \ldots.;\ x)\,\mathcal{H}(\lambda,\ \mu,\ \ldots.;\ x).$$

Generalizations of (2) may hence be obtained.

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A note on the "problème des rencontres."

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1. This celebrated problem is treated in nearly all the textbooks on probability; for example in Bertrand's *Calcul des Probabilités*, 1889, pp. 15-17, in Poincaré's of the same title, 1896, pp. 36-38, and in most of the recent textbooks. The problem may be stated in abstract terms as follows: Among the $n!$ permutations $(a_1 a_2 a_3 \ldots a_n)$ of the natural order $(123\ldots n)$, how many have no a_j equal to j? The problem has been clothed in many picturesque (and highly unlikely) "representations"; for example, by imagining n letters placed at random in n addressed envelopes, and inquiring what is the chance that no letter is in its correct envelope; or by imagining n gentlemen returning at random to their n houses; and so on, *ad risum*. Various derivations have also been given of the probability in question, namely

$$p(0;\ n) = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-)^n \frac{1}{n!} \qquad (1)$$

https://doi.org/10.1017/S0950184300002846 Published online by Cambridge University Press

the first $n+1$ terms of the expansion of e^{-1}, to which function the probability converges with rapidity as n increases.

A more general question is: what is the probability that of the indices α_j exactly x are such that $\alpha_j = j$?—for example, that x letters and no more have found their way into the proper envelopes. This probability, easily deducible from the preceding, is

$$p(x;\ n) = \frac{1}{x!}\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-)^{n-x}\frac{1}{(n-x)!}\right\}. \qquad (2)$$

In this note we derive these probabilities in a simple and direct manner, we show that the distribution of x tends rapidly to the Poissonian type

$$\psi(x;\ \mu) = e^{-\mu}\mu^x/x! \qquad (3)$$

with mean $\mu = 1$, and we also show that the factorial moment generating function (f.m.g.f.) of the distribution of x is

$$1 + \frac{\alpha}{1!} + \frac{\alpha^2}{2!} + \ldots + \frac{\alpha^n}{n!}, \qquad (4)$$

so that the factorial moments up to order n are all equal to 1, all of higher order being 0. It is well known that the f.m.g.f. of the Poissonian distribution (3) is $e^{\mu\alpha}$.

2. *A special permanent.* Consider the determinant $|\,a_{ij}\,|$ of order n, or better still, to avoid all question of sign, the permanent $\overset{+}{|}\,a_{ij}\,\overset{+}{|}$. The $n!$ terms in the expansion of this permanent are of the form $a_{1\alpha}a_{2\beta}a_{2\gamma}\ldots a_{n\nu}$, and our problem is: what proportion of these contain no diagonal element a_{jj}? To enumerate such terms we put each $a_{ij} = 1$, and so we have $p(0;n)$ in the form

$$\frac{1}{n!}\overset{+}{\left|\begin{matrix} 0 & 1 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 0 & 1 & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & 1 & 1 & 1 & \ldots & 0 \end{matrix}\right|}^{+} = \frac{1}{n!}\overset{+}{\left|\begin{matrix} 1+\lambda & 1 & 1 & 1 & \ldots & 1 \\ 1 & 1+\lambda & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1+\lambda & 1 & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & 1 & 1 & 1 & \ldots & 1+\lambda \end{matrix}\right|}^{+}, \qquad (1)$$

where $\lambda = -1$. Expanding the permanent on the right according to powers of λ and cofactors, and noting that any minor permanent of order r and made of unit elements is equal to $r!$, we have at once

$$p(0;\ n)=\frac{1}{n!}\left\{n!+n\lambda(n-1)!+\binom{n}{2}\lambda^2(n-2)!+\ldots+\lambda^n\right\}$$

$$=1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\ldots+(-)^n\frac{1}{n!},\ \text{since}\ \lambda=-1. \qquad (2)$$

It may be noted in passing that the expansion of the permanent in λ gives the first $n+1$ terms of e^{λ}.

Similarly, to find $p(x;\ n)$, we may choose the x coincidences $a_j=j$ in $\binom{n}{x}$ ways, and multiply by the probability of the $n-x$ non-coincidences $a_j\neq j$. This gives

$$p(x;\ n)=\frac{1}{n!}\binom{n}{x}\frac{1}{(n-x)!}\left\{1-\frac{1}{1!}+\frac{1}{2!}-\ldots+(-)^{n-x}\frac{1}{(n-x)!}\right\}$$

$$=\frac{1}{x!}\left\{1-\frac{1}{1!}+\frac{1}{2!}-\ldots+(-)^{n-x}\frac{1}{(n-x)!}\right\}. \qquad (3)$$

The same result could have been obtained by expanding $\binom{n}{x}$ permanents made of unit elements, each having $n-x$ diagonal elements increased by λ.

3. *The probability distribution.* How closely the probability distribution $p(x;\ n)$ approximates to the Poissonian $\psi(x;\ 1)$ can be seen even for so small a value as $n=5$.

x	0	1	2	3	4	5	6	7	Σ
$5!\ p(x;\ 5)$	44	45	20	10	0	1	0	0	120
$5!\ \psi(x;\ 1)$	44·15	44·15	22·07	7·36	1·84	0·37	0·06	0·01	120 .

The sequence $p(0;\ 1)$, $p(0;\ 2)$, $p(0;\ 3)$, ... is well known in combinatory analysis (see for example MacMahon, *Combinatory Analysis*, Vol. I, p. 102) and gives 1, 0, 2, 9, 44, 265, 1854, 14833, 133496, ... obeying the recurrence relation $u_{n+1}=nu_n+(-1)^n$.

It will next be proved that the f.m.g.f., that is, the power series in a which has, as coefficient of $a^r/r!$, the r^{th} factorial moment $\sum_x p(x;\ n)\,.\,x(x-1)(x-2)\ldots(x-r+1)$, is equal to

$$\sum_x p(x;\ n)\,(1+a)^x=1+\frac{a}{1!}+\frac{a^2}{2!}+\ldots+\frac{a^n}{n!}. \qquad (1)$$

An intuitive proof may be given by setting out in the form of a triangle, a multiplication table from the respective terms of the expansions of e and e^{-1}, as shown below for the case $n=5$.

https://doi.org/10.1017/S0950184300002846 Published online by Cambridge University Press

	1	-1	$\frac{1}{2}$	$-\frac{1}{6}$	$\frac{1}{24}$	$-\frac{1}{120}$
1	1	-1	$\frac{1}{2}$	$-\frac{1}{6}$	$\frac{1}{24}$	$-\frac{1}{120}$
1	1	-1	$\frac{1}{2}$	$-\frac{1}{6}$	$\frac{1}{24}$	
$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{12}$		
$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{12}$			
$\frac{1}{24}$	$\frac{1}{24}$	$-\frac{1}{24}$				
$\frac{1}{120}$	$\frac{1}{120}$					

(2)

By what has preceded, the probabilities $p(x; n)$, for $x = 0, 1, 2 \ldots, n$, are the sums of elements in the respective rows of such a triangle. By mere inspection, too, the entries in the successive north-eastward sloping diagonals of the table are terms in the expansions of $(1-1)^x/x!$, $x = 0, 1, 2, \ldots, n$; and the sum of all entries in the triangle for any n is equal to 1.

To construct the f.m.g.f. we multiply the rows by 1, $1 + a$, $(1 + a)^2, \ldots, (1 + a)^n$ respectively and sum. The diagonals mentioned sum to $(1 + a - 1)^x/x!$, that is, to $a^x/x!$; and so the f.m.g.f. is

$$1 + \frac{a}{1!} + \frac{a^2}{2!} + \ldots + \frac{a^n}{n!}, \tag{3}$$

as stated.

In conclusion it calls for remark that MacMahon (*ibid.* p. 100) used a "guiding determinant" as an enumerant in this *problème des rencontres*; while in a similar connection, in his "Researches in the Theory of Determinants," *Trans. Camb. Phil. Soc.* **23** (1924), pp. 89-135, esp. 106-107, he uses a permanent but does not expand it.

Other combinatory problems, such as the *problème des ménages* of Lucas, which is the same, regarded abstractly, as a problem of knots treated by Cayley, Tait, Muir, Netto and MacMahon (*Combinatory Analysis*, vol. 1, p. 256), can also be solved by permanents, but the method of the recurrence relation is just as effective.

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https://doi.org/10.1017/S0950184300002846 Published online by Cambridge University Press