ON HERMITE-FEJÉR TYPE INTERPOLATION ON THE CHEBYSHEV NODES

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Given \( f \in C[-1, 1] \), let \( H_{n,3}(f, x) \) denote the \((0,1,2)\) Hermite-Fejér interpolation polynomial of \( f \) based on the Chebyshev nodes. In this paper we develop a precise estimate for the magnitude of the approximation error \(|H_{n,3}(f, x) - f(x)|\). Further, we demonstrate a method of combining the divergent Lagrange and \((0,1,2)\) interpolation methods on the Chebyshev nodes to obtain a convergent rational interpolatory process.

1. INTRODUCTION

Suppose \( f \) is a continuous real-valued function defined on the interval \([-1, 1]\), and let

\[
X = \{x_{k,n} : k = 1, 2, \ldots, n; n = 1, 2, 3, \ldots\}
\]

be a triangular matrix such that, for all \( n \),

\[
1 \geq x_{1,n} > x_{2,n} > \ldots > x_{n,n} \geq -1.
\]

Then for each integer \( m \geq 1 \) there is a unique polynomial \( H_{n,m}(X, f, x) \) of degree at most \( mn - 1 \) which satisfies

\[
\begin{align*}
H_{n,m}(X, f, x_{k,n}) &= f(x_{k,n}), \\
H_{n,m}^{(j)}(X, f, x_{k,n}) &= 0, \quad j = 1, 2, \ldots, m - 1,
\end{align*}
\]

for all \( k \). \( H_{n,m}(X, f, x) \) is referred to as the \((0,1,\ldots,m-1)\) Hermite-Fejér interpolation polynomial of \( f(x) \).

\( H_{n,1}(X, f, x) \) is the well-known Lagrange interpolation polynomial of \( f(x) \). A classic result of Faber [1] states that for any matrix \( X \), there exists an \( f \in C[-1,1] \) so that \( H_{n,1}(X, f, x) \) does not tend uniformly to \( f(x) \). On the other hand, if \( T \) denotes the matrix of Chebyshev nodes

\[
T = \{x_{k,n} = \cos \left( \frac{2k - 1}{2n} \pi \right) : k = 1, 2, \ldots, n; n = 1, 2, 3, \ldots\},
\]

and if the modulus of continuity \( \omega(\delta; f) = \omega(\delta) \) of \( f \) is defined by

\[
\omega(\delta; f) = \omega(\delta) = \max |f(x_1) - f(x_2)|,
\]
where the max is taken over all pairs $x_1, x_2$ in $[-1, 1]$ for which $|x_1 - x_2| \leq \delta$, then there exists a number $c_1$ (independent of $f$, $n$ and $x$) so that

$$|H_{n,1}(T, f, x) - f(x)| \leq c_1 \log n \omega(1/n)$$

for all $n \geq 2$ (see Szabados and Vértesi [9, Chapter 1]). Thus the Chebyshev nodes are a good choice if uniform approximation by Lagrange interpolation polynomials is required, and so they have been much studied. We point out that results sharper than (1.1) are known. For example, Kis [4] showed that there exists a number $c_2$ such that

$$|H_{n,1}(T, f, x) - f(x)| \leq c_2 \left[ \log n \omega\left(\frac{\sqrt{1 - x^2}}{n}\right) + \sum_{i=1}^{n} \frac{1}{i} \omega\left(\frac{i}{n^2}\right) \right].$$

A famous result of Fejér [2] states that if $f \in C[-1, 1]$, the $(0,1)$ Hermite-Fejér interpolation polynomials $H_{n,2}(T, f, x)$ converge uniformly to $f(x)$ as $n$ tends to infinity. A brief history of estimates of the rate of convergence is given in Goodenough and Mills [3], who also proved that there exist absolute constants $c_3$ and $c_4$ so that

$$|H_{n,2}(T, f, x) - f(x)| \leq \frac{c_3}{n} T_n(x)^2 \sum_{k=1}^{n} \left[ \omega\left(\frac{\sqrt{1 - x^2}}{k}\right) + \omega\left(\frac{1}{k^2}\right) \right] + c_4 \omega\left(\frac{|T_n(x)|}{n}\right)$$

for all $f \in C[-1, 1]$, $n \geq 2$, and $-1 \leq x \leq 1$. Here $T_n(x)$ denotes the Chebyshev polynomial $T_n(x) = \cos(n \arccos x)$, $-1 \leq x \leq 1$, whose zeros are $\cos((2k-1)\pi/2n)$, $k = 1, 2, \ldots, n$. Thus the estimate (1.3) reflects the fact that $H_{n,2}(T, f, x)$ interpolates $f(x)$ at the zeros of $T_n(x)$. Note also that because $|T_n(x)| \leq 1$, $-1 \leq x \leq 1$, it follows from (1.3) that the approximation error $\|H_{n,2}(T, f, \cdot) - f\|$ is of order no greater than $n^{-1} \sum_{k=1}^{n} \omega(1/k)$.

For $(0,1,2)$ interpolation, Szabados and Varma [8] proved that, as with Lagrange interpolation, for any system of nodes $X$ there exists $f \in C[-1, 1]$ such that $H_{n,3}(X, f, x)$ does not converge uniformly to $f(x)$. This result has recently been shown to extend to $(0,1,\ldots,m)$ interpolation for any even $m$ (see Szabados [7]). The first aim of this paper is to obtain an estimate for the error $|H_{n,3}(T, f, x) - f(x)|$ in approximating $f(x)$ by the $(0,1,2)$ interpolation polynomial based on the Chebyshev nodes. Our precise result, to be established in Section 3, is:

**Theorem 1.** There exist constants $k_1$, $k_2$ such that, for $f \in C[-1, 1]$, $n \geq 2$, and $-1 \leq x \leq 1$,

$$|H_{n,3}(T, f, x) - f(x)| \leq k_1 T_n(x)^2 \left[ \log n \omega\left(\frac{\sqrt{1 - x^2}}{n}\right) + \sum_{i=1}^{n} \frac{1}{i} \omega\left(\frac{i}{n^2}\right) \right]$$

$$+ k_2 \omega\left(\frac{|T_n(x)|}{n}\right).$$
We point out that (1.4) is similar to Kis’ estimate (1.2) for Lagrange interpolation. However, like the estimate (1.3) for (0, 1) interpolation, it also reflects the fact that the approximation error vanishes at the zeros of $T_n(x)$.

For (0, 1, 2, 3) interpolation, it is known (see, for example, Mills [6]) that if $f \in C[-1, 1]$, then $H_{n,4}(T, f, x)$ converges uniformly to $f(x)$, and that, as for (0, 1) interpolation, the approximation error $\|H_{n,4}(T, f, x) - f\|$ is of the order $n^{-1} \sum_{k=1}^{n} \omega(1/k)$.

In his paper [5], A. Meir developed a rational interpolatory process, based on (0, 1) and (0, 1, 2, 3) interpolation on the Chebyshev nodes, for which the approximation error is of order $\omega(1/n)$. This error is essentially less than the approximation error in the two convergent interpolation processes on which Meir’s technique is based. The second aim of this paper is to use Meir’s approach to develop a convergent rational interpolatory approximation method from the divergent (0) and (0, 1, 2) interpolation processes on the Chebyshev nodes. Before stating our main result, we introduce some terminology.

For any system of nodes $X = \{x_k, n\}$, and for $k = 1, 2, \ldots, n$, let

$$\ell_{k,n}(X, x) = \frac{\Omega(X, x)}{(x - x_{k,n}) \Omega(X, x_{k,n})},$$

where

$$\Omega(X, x) = \prod_{k=1}^{n} (x - x_{k,n}).$$

Then, as is well-known,

$$H_{n,1}(X, f, x) = \sum_{k=1}^{n} f(x_{k,n}) \ell_{k,n}(X, x).$$

Also (Szabados and Varma [8]),

$$H_{n,3}(X, f, x) = \sum_{k=1}^{n} f(x_{k,n}) A_{k,n}(X, x),$$

where

$$A_{k,n}(X, x) = \left\{ 1 - 3\ell_{k,n}(X, x_{k,n})(x - x_{k,n}) + \left[ 6(\ell_{k,n}(X, x_{k,n}))^2 - \frac{3}{2} \ell_{k,n}'(X, x_{k,n})(x - x_{k,n})^2 \right] (\ell_{k,n}(X, x))^3. \right\}$$

For the Chebyshev nodes $x_{k,n} = \cos((2k - 1)\pi/2n)$, it is readily verified that

$$\ell_{k,n}(T, x) = (-1)^{k-1} \frac{\sqrt{1 - x_{k,n}^2}}{n(x - x_{k,n})} T_n(x),$$

$$H_{n,4}(T, f, x) = \sum_{k=1}^{n} f(x_{k,n}) \ell_{k,n}(T, x).$$
and
\begin{align}
A_{k,n}(T, x) &= P_{k,n}(T, x) + Q_{k,n}(T, x) + R_{k,n}(T, x),
\end{align}
where
\begin{align}
P_{k,n}(T, x) &= P_k(x) = \frac{(-1)^{k-1} (1 - x z_{k,n}) \sqrt{1 - x_{k,n}^2}}{n^3 (x - x_{k,n})^3} T_n(x)^3, \\
Q_{k,n}(T, x) &= Q_k(x) = \frac{(-1)^k x_{k,n} \sqrt{1 - x_{k,n}^2}}{2n^3 (x - x_{k,n})^2} T_n(x)^3, \\
R_{k,n}(T, x) &= R_k(x) = \frac{(-1)^{k-1} (n^2 - 1) \sqrt{1 - x_{k,n}^2}}{2n^3 (x - x_{k,n})} T_n(x)^3.
\end{align}

Now define \( W_n(x) \) by
\begin{align}
W_n(x) &= 1 - \frac{n^2 - 1}{2n^2} T_n(x)^2
\end{align}
(so \( 1/2 < W_n(x) \leq 1 \) for \(-1 \leq x \leq 1 \)), and given \( f \in C[-1, 1] \), put
\begin{align}
(\Lambda_n f)(x) = \frac{1}{W_n(x)} \left( H_{n,3}(T, f, x) - \frac{n^2 - 1}{2n^2} T_n(x)^2 H_{n,1}(T, f, x) \right).
\end{align}

Note that \( \Lambda_n f \) is a rational function with numerator of degree at most \( 3n - 2 \), denominator of degree \( 2n \), and which agrees with \( f \) at the Chebyshev nodes. In Section 4 we prove the following result concerning the operator \( \Lambda_n \).

**Theorem 2.** There exist constants \( k_3, k_4 \) such that, for \( f \in C[-1, 1], n \geq 2, \) and \(-1 \leq x \leq 1, \)
\begin{align}
||\Lambda_n f - f|| \leq k_3 T_n(x)^2 \left[ \omega \left( \frac{\sqrt{1 - x^2}}{n} \right) + \frac{\log n}{n} \omega \left( \frac{1}{n} \right) \right] + k_4 \omega \left( \frac{|T_n(x)|}{n} \right).
\end{align}

From (1.13) it follows that \( ||\Lambda_n f - f|| \) is of order \( \omega(1/n) \), and so the divergent \((0)\) and \((0, 1, 2)\) interpolation processes have, indeed, been combined to give a convergent scheme of interpolation. Further, if \( x = x_{k,n} \), the right-hand side of (1.13) is zero, and so (1.13) reflects the fact that \( \Lambda_n \) is an interpolatory operator. We also point out that the order of magnitude \( \omega(1/n) \) for \( ||\Lambda_n f - f|| \) is the best possible, for if \( g(x) = |x| \) (so that \( \omega(1/n) = 1/n \)), it is not too hard to show that
\begin{align}
||\Lambda_{2m} g(0) - g(0)|| = \frac{8G}{n^2 m} + O(m^{-2})
\end{align}
as \( m \to \infty \). (Here \( G \) is Catalan's constant, \( G = \sum_{k=0}^{\infty} (-1)^k/(2k + 1)^2 \)).
REMARK. Since completing the above work, we have become aware of a result due to Yuan Xu [10, Theorem 4, p.149] which is similar to our Theorem 2.

2. PRELIMINARIES

In this section we introduce some preliminary results that will be required for the proofs of Theorems 1 and 2. Because the remainder of this paper is concerned solely with the Chebyshev nodes, we shall henceforth adopt the notation $H_{n,k}(f,x) = H_{n,k}(T,f,x)$, $\ell_k(x) = \ell_{k,n}(T,x)$, $A_k(x) = A_{k,n}(T,x)$, and $x_k = x_{k,n} = \cos((2k-1)\pi/2n)$. Also, define $t_k = (2k-1)\pi/2n$. For each $x \in [-1,1]$ write $x = \cos t$, where $0 \leq t \leq \pi$, and choose $j$ so that

$$\min\{|t_k - t| : k = 1,2,\ldots,n\} = |t_j - t|.$$ 

The results of the following lemma are due to Kis [4, Lemmas 1, 2, 3, 4].

**Lemma 1.** With notation as introduced in the above discussion, then

\begin{align*}
|f(x_k) - f(x)| &\leq 5\omega\left(\frac{i \sin t}{n}\right) + 13\omega\left(\frac{i^2}{n^2}\right), \text{ if } i = |k - j| \geq 1, \\
|f(x_{k+1}) - f(x_k)| &\leq 4\omega\left(\frac{\sin t}{n}\right) + 20\omega\left(\frac{i}{n^2}\right), \text{ if } i = k - j \geq 1, \\
|f(x_k) - f(x_{k-1})| &\leq 4\omega\left(\frac{\sin t}{n}\right) + 20\omega\left(\frac{i}{n^2}\right), \text{ if } i = j - k \geq 1, \\
|\ell_k(x)| &\leq \begin{cases} 2, & \text{if } k = j, \\ \frac{2}{i}, & \text{if } i = |k - j| \geq 1, \end{cases} \\
|\ell_k(x) + \ell_{k+1}(x)| &\leq \frac{7}{i^2}, \text{ if } i = k - j \geq 1, \\
|\ell_k(x) + \ell_{k-1}(x)| &\leq \frac{7}{i^2}, \text{ if } i = j - k \geq 1.
\end{align*}

The following elementary inequality will be used.

**Lemma 2.** If $0 \leq \alpha, \beta \leq \pi$, then

$$\sin \frac{1}{2}(\alpha + \beta) \geq \sin \frac{1}{2}|\alpha - \beta|.$$ 

Next, we obtain estimates for the magnitude of the component functions $P_k$, $Q_k$ and $R_k$ of the fundamental polynomials $A_k$ for $(0,1,2)$ interpolation.
Lemma 3. For $-1 \leq x \leq 1$, the functions $P_k(x)$, $Q_k(x)$ and $R_k(x)$, as defined by (1.9), (1.10) and (1.11), satisfy

$$(2.6) \quad |P_k(x)| \leq \begin{cases} 4, & \text{if } k = j, \\ \frac{4}{i^3} T_n(x)^2, & \text{if } i = |k - j| \geq 1, \\ \frac{1}{2n^2} T_n(x)^2 + 2, & \text{if } k = j, \end{cases}$$

$$(2.7) \quad |Q_k(x)| \leq \begin{cases} \frac{1}{2n^2} T_n(x)^2 + 2, & \text{if } k = j, \\ \frac{n^2 - 1}{n^2} T_n(x)^2, & \text{if } i = |k - j| \geq 1, \end{cases}$$

$$(2.8) \quad |R_k(x)| \leq \begin{cases} \frac{n^2 - 1}{n^2} T_n(x)^2, & \text{if } i = |k - j| \geq 1. \end{cases}$$

Proof: From (1.7) and (1.9) it follows that

$$(2.9) \quad P_k(x) = T_n(x)^2 \frac{\ell_k(x)}{n^2} \frac{1 - xz_k}{(x - z_k)^2}. $$

Now, as shown by Kis [4, p.32],

$$(2.10) \quad |T_n(x)| \leq 2n \sin \frac{1}{2} |t_j - t|. $$

Further,

$$(2.11) \quad \frac{1 - xz_k}{(x - z_k)^2} = \frac{1 - \cos t \cos t_k}{(\cos t - \cos t_k)^2}$$

$$= \frac{\sin^2 \frac{1}{2}(t_k + t) + \sin^2 \frac{1}{2}(t_k - t)}{4 \sin^2 \frac{1}{2}(t_k + t) \sin^2 \frac{1}{2}(t_k - t)}$$

$$= \frac{1}{4} \left( \frac{1}{\sin^2 \frac{1}{2}(t_k + t)} + \frac{1}{\sin^2 \frac{1}{2}(t_k - t)} \right)$$

$$\leq \frac{1}{2 \sin^2 \frac{1}{2}(t_k - t)}, $$

where the last inequality follows from (2.5). Also, if $i = |k - j| \geq 1$, then

$$(2.12) \quad \sin \frac{1}{2} |t_k - t| \geq \sin \left( \frac{2i - 1}{4n} \pi \right) \geq \frac{2i - 1}{2n} \geq \frac{i}{2n}. $$

Thus, by (2.9), (2.10), (2.11), (2.12),

$$|P_k(x)| \leq \begin{cases} 2 |\ell_j(x)|, & \text{if } k = j, \\ \frac{2}{i^2} |\ell_j(x)| T_n(x)^2, & \text{if } i = |k - j| \geq 1, \end{cases}$$
and then (2.6) results from (2.3).

For $Q_k(x)$, we note from (1.7) and (1.10) that

\begin{equation}
Q_k(x) = -T_n(x)^2 \, \ell_k(x) \frac{x_k}{x - x_k}.
\end{equation}

Consider

\begin{equation}
\left| \frac{x_k}{x - x_k} \right| = \left| \frac{\cos t_k}{\cos t - \cos t_k} \right| = \frac{1}{2} \left| \frac{\cos \frac{1}{2}(t_k + t) \cos \frac{1}{2}(t_k - t) - \sin \frac{1}{2}(t_k + t) \sin \frac{1}{2}(t_k - t)}{\sin \frac{1}{2}(t_k + t) \sin \frac{1}{2}(t_k - t)} \right|
\end{equation}

\begin{equation}
\leq \frac{1}{2} \left( 1 + \frac{1}{\sin \frac{1}{2}(t_k + t) \sin \frac{1}{2}|t_k - t|} \right),
\end{equation}

where (2.5) has been used to obtain the final inequality. Thus, by (2.10), (2.12), (2.13), (2.14),

\begin{equation}
|Q_k(x)| \leq \begin{cases} 
\left( \frac{T_n(x)^2}{4n^2} + 1 \right) |\ell_j(x)|, & \text{if } k = j, \\
\left( \frac{1}{4n^2} + \frac{1}{t^2} \right) |\ell_k(x)| \, T_n(x)^2, & \text{if } i = |k - j| \geq 1,
\end{cases}
\end{equation}

from which (2.7) follows by applying (2.3).

Finally, (2.8) is an immediate consequence of the identity

\begin{equation}
R_k(x) = T_n(x)^2 \frac{n^2 - 1}{2n^2} \ell_k(x),
\end{equation}

and the inequalities (2.3).

We shall also need a lemma of Goodenough and Mills [3, Lemma 3].

\textbf{Lemma 4.} For $x = \cos t$, $0 \leq t \leq \pi$, and with $j$ chosen so that $\min\{|t_k - t| : k = 1, 2, \ldots, n\} = |t_j - t|$, then

\begin{equation}
|t_j - t| \leq \frac{\pi}{2n} \left| \cos nt \right| = \frac{\pi}{2n} |T_n(x)|.
\end{equation}
3. Proof of Theorem 1

Because $\sum_{k=1}^{n} H_{n,k}(1,x) = 1$, it follows from (1.6) that

$$|H_{n,s}(f,x) - f(x)| = \left| \sum_{k=1}^{n} (f(x_k) - f(x))A_k(x) \right|,$$

where $A_k(x)$ is given by (1.8), (1.9), (1.10) and (1.11). On putting

$$U_k(x) = (f(x_k) - f(x))A_k(x),$$

we obtain

$$|H_{n,s}(f,x) - f(x)| \leq \left| \sum_{k=1}^{j-1} U_k(x) \right| + |U_j(x)| + \left| \sum_{k=j+1}^{n} U_k(x) \right|$$

$$= I_1 + I_2 + I_3, \quad \text{say.}$$

(Clearly if $j = 1$ or $n$, then one of these terms will not be present.)

First, we estimate $I_3$. If $n - j$ is odd, then

$$I_3 \leq |U_{j+1}(x) + U_{j+2}(x)| + |U_{j+3}(x) + U_{j+4}(x)| + \ldots$$

$$+ |U_{n-2}(x) + U_{n-1}(x)| + |U_n(x)|. $$

(If $n - j$ is even, the final term of this sum is $|U_{n-1}(x) + U_n(x)|$.) Now for $k = j + i$, $i = 1, 3, 5, \ldots$, $i < n - j$,

$$|U_k(x) + U_{k+1}(x)|$$

$$= |(f(x_k) - f(x))A_k(x) + (f(x_{k+1}) - f(x))A_{k+1}(x)|$$

$$\leq |A_k(x) + A_{k+1}(x)| \left| f(x_k) - f(x) \right| + |A_{k+1}(x)| \left| f(x_{k+1}) - f(x_k) \right|.$$

Consider

$$|A_k(x) + A_{k+1}(x)| \leq (|P_k(x)| + |P_{k+1}(x)|) + (|Q_k(x)| + |Q_{k+1}(x)|)$$

$$+ |R_k(x) + R_{k+1}(x)|$$

$$= (|P_k(x)| + |P_{k+1}(x)|) + (|Q_k(x)| + |Q_{k+1}(x)|)$$

$$+ \frac{n^2 - 1}{2n^2} T_n(x)^2 |\ell_k(x) + \ell_{k+1}(x)|$$

$$\leq T_n(x)^2 \left( \frac{8}{t^3} + \left( \frac{1}{n^2} + \frac{4}{t^2} \right) + \frac{7}{2t^2} \right), \quad \text{by (2.4), (2.6), (2.7)},$$

$$\leq \frac{33}{2} T_n(x)^2 \frac{1}{t^2}.$$
Also,

\[(3.5) \quad |A_k(x)| \leq |P_k(x)| + |Q_k(x)| + |R_k(x)| \leq T_n(x)^2 \left( \frac{4}{i^3} + \frac{1}{2n^2i} + \frac{2}{i^3} + \frac{n^2-1}{n^2i} \right), \text{ by (2.6), (2.7), (2.8)}, \]

\[\leq 7 \frac{T_n(x)^2}{i}.\]

Thus, by (2.1), (2.2), (3.3), (3.4) and (3.5), we deduce that

\[(3.6) \quad |U_k(x) + U_{k+1}(x)| \leq T_n(x)^2 \left[ \frac{33}{2i^2} \left( 5\omega \left( \frac{i \sin t}{n} \right) + 13\omega \left( \frac{i^2}{n^2} \right) \right) + \frac{7}{i} \left( 4\omega \left( \frac{\sin t}{n} \right) + 20\omega \left( \frac{i}{n^2} \right) \right) \right] \]

\[= O(1) \frac{T_n(x)^2}{i} \left[ \omega \left( \frac{\sin t}{n} \right) + \omega \left( \frac{i}{n^2} \right) \right].\]

(Here, and elsewhere, \(O(1)\) terms can be made independent of \(i, f\) and \(x\).)

If \(n - j\) is odd, we need an estimate for \(|U_n(x)|\). Now, by (2.1) and (3.5),

\[(3.7) \quad |U_n(x)| = |f(x_n) - f(x)| |A_n(x)| \leq 7 \frac{T_n(x)^2}{n-j} \left( 5\omega \left( \frac{(n-j)\sin t}{n} \right) + 13\omega \left( \frac{(n-j)^2}{n^2} \right) \right) \]

\[= O(1) T_n(x)^2 \left[ \omega \left( \frac{\sin t}{n} \right) + \omega \left( \frac{1}{n} \right) \right].\]

However,

\[(3.8) \quad \omega \left( \frac{1}{n} \right) = \sum_{i=1}^{n} \frac{1}{n} \omega \left( \frac{i}{n} \times \frac{i}{n^2} \right) \leq \sum_{i=1}^{n} \frac{1}{n} \left( \frac{n}{i} + 1 \right) \omega \left( \frac{i}{n^2} \right) = O(1) \sum_{i=1}^{n} \frac{1}{i} \omega \left( \frac{i}{n^2} \right).\]

Since \(\sin t = \sqrt{1 - x^2}\), we conclude from (3.2), (3.6), (3.7), (3.8),

\[(3.9) \quad I_3 = O(1) T_n(x)^2 \left[ \log n \omega \left( \frac{\sqrt{1-x^2}}{n} \right) + \sum_{i=1}^{n} \frac{1}{i} \omega \left( \frac{i}{n^2} \right) \right].\]
Similarly, we can write

\begin{equation}
I_1 = O(1) T_n(x)^2 \left[ \log n \omega \left( \frac{\sqrt{1 - x^2}}{n} \right) + \sum_{i=1}^{n} \frac{1}{i} \omega \left( \frac{i}{n^2} \right) \right],
\end{equation}

and so it remains to estimate \( I_2 \).

We have

\begin{equation}
|f(x_j) - f(x)| \leq \omega(|x_j - x|)
\leq \omega(|t_j - t|)
\leq \omega \left( \frac{\pi}{2n} \left| T_n(x) \right| \right), \text{ by (2.16)},
= O(1) \omega \left( \frac{\left| T_n(x) \right|}{n} \right).
\end{equation}

Also, by (2.6), (2.7), (2.8),

\[ |A_j(x)| \leq 6 + \frac{2n^2 - 1}{2n^2} T_n(x)^2 < 7. \]

Therefore

\begin{equation}
I_2 = |f(x_j) - f(x)| |A_j(x)| = O(1) \omega \left( \frac{|T_n(x)|}{n} \right).
\end{equation}

The proof of the theorem is now completed by substituting the results (3.9), (3.10) and (3.12) into (3.1).

4. PROOF OF THEOREM 2

We begin by noting that \( H_{n,3}(1, x) = H_{n,1}(1, x) = 1 \), and so \( \Lambda_n(1)(x) = 1 \). Also, by (1.5), (1.6), (1.8) and (2.15), the definition (1.12) of \( \Lambda_n \) can be rewritten as

\[ (\Lambda_n f)(x) = \frac{1}{W_n(x)} \left( \sum_{k=1}^{n} (P_k(x) + Q_k(x)) f(x_k) \right). \]

Therefore, since \( W_n(x) > 1/2 \), \(-1 \leq x \leq 1\), we obtain

\begin{equation}
| (\Lambda_n f)(x) - f(x) | < 2 \left| \sum_{k=1}^{n} (P_k(x) + Q_k(x)) (f(x_k) - f(x)) \right|
\end{equation}
\[
\begin{align*}
\leq & \ 2 \left( \sum_{k=1}^{i-1} |P_k(x) + Q_k(x)| |f(x_k) - f(x)| + \sum_{k=j+1}^{n} |P_k(x) + Q_k(x)| |f(x_k) - f(x)| \right) \\
= & \ 2(J_1 + J_2 + J_3), \quad \text{say.}
\end{align*}
\]

Consider \( J_3 \). If \( i = k - j, \ i \geq 1 \), then by (2.6) and (2.7) we have
\[
|P_k(x) + Q_k(x)| \leq \left( \frac{6}{i^3} + \frac{1}{2n^2i} \right) T_n(x)^2 = O(1) \frac{T_n(x)^2}{i^3}.
\]

From (2.1) it then follows that
\[
(4.2) \quad J_3 = O(1) T_n(x)^2 \left( \sum_{i=1}^{n} \frac{1}{i^3} \omega \left( \frac{i \sin t}{n} \right) + \sum_{i=1}^{n} \frac{1}{i^2} \omega \left( \frac{i^2}{n^2} \right) \right)
\]
\[
= O(1) T_n(x)^2 \left( \sum_{i=1}^{n} \frac{1}{i^2} \omega \left( \frac{\sin t}{n} \right) + \sum_{i=1}^{n} \frac{1}{i^3} \left( \frac{i^2}{n} + 1 \right) \omega \left( \frac{1}{n} \right) \right)
\]
\[
= O(1) T_n(x)^2 \left[ \omega \left( \frac{\sqrt{1 - x^2}}{n} \right) + \frac{\log n}{n} \omega \left( \frac{1}{n} \right) \right].
\]

Similarly,
\[
(4.3) \quad J_1 = O(1) T_n(x)^2 \left[ \omega \left( \frac{\sqrt{1 - x^2}}{n} \right) + \frac{\log n}{n} \omega \left( \frac{1}{n} \right) \right].
\]

Finally, by (2.6) and (2.7) we have
\[
|P_j(x) + Q_j(x)| \leq 6 + \frac{T_n(x)^2}{2n^2} \leq \frac{13}{2},
\]
and so (by (3.11)),
\[
(4.4) \quad J_2 = O(1) \omega \left( \frac{T_n(x)}{n} \right).
\]

Upon substituting (4.2), (4.3) and (4.4) in (4.1), the desired result (1.13) is obtained. ⊖

**REFERENCES**


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