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Monoidal Categories and Multiextensions

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Abstract. We associate to a group-like monoidal groupoid $\mathcal{C}$ a principal bundle E satisfying most of the axioms defining a biextension. The obstruction to the existence of a genuine biextension structure on E is exhibited. When this obstruction vanishes, the biextension E is alternating and a trivialization of E induces a trivialization of $\mathcal{C}$. The analogous theory for monoidal n-categories is also examined, as well as the appropriate generalization of these constructions in a sheaf-theoretic context. In the n-categorical situation, this produces a higher commutator calculus, in which some interesting generalizations of the notion of an alternating biextension occur. For $n = 2$, the corresponding cocycles are constructed explicitly, by a partial symmetrization process, from the cocycle describing the n-category.

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0. Introduction

Let A and B be a pair of Abelian groups. Central extensions

$$0 \to A \to E \to B \to 0$$

of B by A are classified up to equivalence by the A-valued cohomology group $H^2(B, A)$ (where A is viewed as a trivial B-module). To such an extension is associated the commutator map

$$\begin{aligned} \lambda : B \times B &\to A \\ (b_1, b_2) &\mapsto [s(b_1), s(b_2)] \end{aligned}$$

determined by the choice of an arbitrary set-theoretic section s of the projection from E to B. It is easily verified that this commutator map is independent of the choice of the section s, and that it is a bilinear alternating map from $B \times B$ to A. By construction, the map λ measures the lack of commutativity of the group law of E. In particular, the central extension E is actually commutative whenever the map λ vanishes, so that it then determines an element of the group $\mathrm{Ext}^1(B, A)$.

These facts, which are well known, may be interpreted as follows in cohomological terms. Since the group B is Abelian, its first integral homology group $H_1(B)$

is isomorphic to the group B itself. Furthermore, the Pontryagin product map $H_1(B) \times H_1(B) \to H_2(B)$ is bilinear, alternating, and therefore induces a map $\Lambda^2 B \to H_2(B)$ which is an isomorphism. The previous discussion now follows directly by considering the exact sequence

$$0 \to \mathrm{Ext}^1(B, A) \to H^2(B, A) \to \mathrm{Hom}(\Lambda^2 B, A) \tag{0.1}$$

provided by the universal coefficient theorem. The functor Λ^2 here denotes the second exterior power $\Lambda^2_{\mathbf{Z}}$, applied to the group B viewed as a **Z**-module. Unless explicitly stated the corresponding higher exterior power functors $\Lambda^j_{\mathbf{Z}}$ will in the sequel simply be denoted by Λ^j.

Our aim in the present paper is to analyze in a similar manner some of the higher cohomology groups $H^n(B, A)$. These have various geometric interpretations, analogous to the description of $H^2(B, A)$ in terms of central extensions, most of which are mentioned in [23]. The most general one of these interpretations of degree n cohomology groups provides a classification of n-monoidal categories. In the first case of interest, that in which $n = 3$, this was first worked out (for symmetric monoidal categories) in the barely accessible [27], where $H^3(B, A)$ was interpreted as the group of equivalence classes of group-like monoidal groupoids $\mathcal{C}$, whose group $\pi_0(\mathcal{C})$ of isomorphism classes of objects is isomorphic to B, and whose group $\mathrm{Aut}_{\mathcal{C}}(I)$ of self-arrows of the identity object I of $\mathcal{C}$ is isomorphic to the B-module A.

Our approach to the study of such monoidal category derives from the observation that there exists a natural filtration, determined by powers of the augmentation ideal, on the chains on a free Abelian simplicial resolution of $K(B, 1)$. We intend to study this in some detail in [9], where we will examine the effect of this filtration on the integral homology of the Abelian group B. Let us merely observe here that such a filtration on the chains of B determines a corresponding one on A-valued cochains, and therefore induces a filtration on the cohomology groups $H^n(B, A)$. The ith associated graded piece of this filtration is the group $\mathrm{Ext}^i(L\Lambda^j B, A)$, where $i + j = n$. Here $L\Lambda^j B$ is the object in the derived category of Abelian groups obtained by applying the exterior power functor Λ^j to a free Abelian simplicial resolution of B placed in degree zero, so that a more traditional notation for this derived object would be $L\Lambda^j(B, 0)$.

Let us begin by considering the case $n = 3$. The filtration on $H^3(B, A)$ determines three *a priori* nontrivial terms in the associated graded group. The first of these is the group $\mathrm{Hom}(L\Lambda^3 B, A)$. Since this group is isomorphic to $\mathrm{Hom}(\Lambda^3 B, A)$, a monoidal category $\mathcal{C}$ of the type described above determines a trilinear alternating map $\varphi \in \mathrm{Hom}(\Lambda^3 B, A)$. When this map φ vanishes, an element of the group $\mathrm{Ext}^1(L\Lambda^2 B, A)$ can be associated to the category $\mathcal{C}$. It was shown in [6] that this group classifies, up to equivalence, the set of alternating biextension E of $B \times B$ by A. This may be understood in the present context by considering the commutator of $\mathcal{C}$. This is a principal A-bundle $E_{\mathcal{C}}$ on $B \times B$, first introduced by P. Deligne in [15], whose fibre over an element $(x, y) \in B \times B$ is the set $E_{x,y}$ of all arrows* $YX \to XY$ in $\mathcal{C}$ (X and Y being chosen representative objects in $\mathcal{C}$ for the isomorphism classes x and y). Such a bundle may be endowed with a pair of partial composition laws determined by the multiplication law in $\mathcal{C}$, which are both associative, and compatible with each other. The obstruction to the commutativity of both of these partial group laws is described by the alternating map $\varphi: \Lambda^3 B \to A$ mentioned above. When φ is trivial, the commutator A-bundle $E_{\mathcal{C}}$ is therefore a genuine biextension of $B \times B$ by A, and in fact it is automatically an alternating one. Passing from the bundle $E_{\mathcal{C}}$ to its isomorphism class, we obtain in this manner the sought-after element of the group $\mathrm{Ext}^1(L\Lambda^2 B, A)$. When this element vanishes, the category $\mathcal{C}$ determines an element in the last component of the graded group associated to $H^3(B, A)$, in other words in the group $\mathrm{Ext}^2(B, A)$. In geometric terms, this may be interpreted as the assertion that a trivialization of $E_{\mathcal{C}}$ as an alternating biextension determines on $\mathcal{C}$ a strictly symmetric monoidal structure. By [14], we know that such strictly symmetric group-like monoidal groupoids are indeed classified by the sought-after group $\mathrm{Ext}^2(B, A)$. However, this group of extensions always vanishes in the category of Abelian groups, so that it does not provide a genuine invariant attached to $\mathcal{C}$. In geometric term, this is reflected in the assertion that such a strictly symmetric group-like monoidal groupoid is always equivalent to the trivial symmetric monoidal category associated to the pair of groups B and A.

It is instructive to carry out the previous discussion purely in terms of a given A-valued three-cocycle $f(x, y, z)$ on B. The alternating map φ which one then encounters is a very familiar one, being simply the map obtained by evaluating f on the decomposable elements of $H_3(B)$. In order to interpret the commutator biextension $E_{\mathcal{C}}$ directly in terms of $f(x, y, z)$, we have found it necessary to insert in our text a description of alternating biextensions in purely cocyclic terms. We believe that such a description, which was not carried out in [6], can be of independent interest. It turns out that the pair of cocycles $(g(x, y; z), h(x; y, z))$ which describe the commutator biextension E of $\mathcal{C}$ are obtained from the given three-cocycle $f(x, y, z)$ by a partial symmetrization process which already occurs (without the assumption that B is Abelian) in a computation by R. Dijkgraaf and E. Witten of the two-cocycle associated by Chern–Simons theory to a given three-cocycle [6] Section 6.6.

The rest of this text is devoted to various generalizations of the previous discussion. The first of these extends the theory from the study of monoidal categories to that of monoidal stacks. This level of generalization is analogous to that which occurs when one passes from the classification of central extensions of Abelian groups to that of topological Abelian groups [25] or of algebraic groups ([26], Chapter VII). The choice of objects or arrows in $\mathcal{C}$ required for a cocyclic description of the monoidal stack $\mathcal{C}$ can in general only be made locally. The cohomological obstruction to a global choice of objects is determined, as explained in [8],

* We will generally denote by XY, rather than by the more customary $X \otimes Y$, the product of two objects X and Y in a monoidal category $\mathcal{C}$.

by the underlying gerbe of $\mathcal{C}$. The obstruction to a corresponding global choice of arrows is reflected in the fact that the commutator biextension $E_{\mathcal{C}}$ of $\mathcal{C}$ no longer has, as in the category case, a global section above its base $B \times B$. $E_{\mathcal{C}}$ is now a genuine biextension of $B \times B$ by A in the sense of [19], rather than one which may be described, as in the category case, by a pair of cocycles (g, h).

Our next generalization consists in passing from the cohomology group $H^3(B, A)$ to the group $H^4(B, A)$. The latter classifies the monoidal two-groupoids $\mathcal{C}$ which satisfy the conditions $\pi_0(\mathcal{C}) = B$, $\pi_1(\mathcal{C}) = 0$ and $\pi_2(\mathcal{C}) = A$. The natural action of B on A is once more assumed to be trivial. A geometrical discussion of the associated graded pieces for the filtration on $H^4(B, A)$ requires a geometrical understanding of the corresponding groups $\mathrm{Ext}^i(L\Lambda^j B, A)$ for $i + j = 4$. We interpret these groups as the groups of equivalence classes of certain geometric objects which we call the (i, j)-extensions of B by A. When $i = 1$, these are simply, for an arbitrary j, the j-fold extensions of B by A introduced by A. Grothendieck in [19]. We will therefore use this concept here for $j = 3$, and we will call such objects triextensions of B by A. The next term in the filtration requires that we understand the notion of a $(2, 2)$-extension of B by A. This is an interesting new concept, consisting in a category (or more generally a stack) $\mathcal{E}$ for which $\pi_0(\mathcal{E}) = B \times B$ and $\pi_1(\mathcal{E}) = A$, and which is endowed with a pair of coherently associative and appropriately compatible partial group laws, which define on the restrictions of $\mathcal{E}$ to all subsets $x \times B$ and $B \times y$ the structure of a group-like symmetric monoidal category.

While these definitions of a triextension and of a (2,2)-extension present no great difficulty, there remains the question of imposing on each of these objects an alternating structure. In order to achieve this, we make use of Koszul complex techniques, and interpret the requisite groups $\mathrm{Ext}^i(L\Lambda^j B, A)$ in geometric terms. Once the appropriate definitions have been obtained, we can describe the geometric objects which our higher commutator calculus associates to a given monoidal 2-category. Part of this discussion is carried out in cocyclic terms, an efficient substitute in the present context for pasting diagrams in 2-categories. A pleasant feature of this discussion is the occurrence of a systematic partial symmetrization process, analogous to the one mentioned above in three-cocycle situation, and which points quite clearly to a general statement for the corresponding filtration on the cohomology groups of arbitrary degree. Certain of these symmetrized higher cocycles occur, for a non-Abelian group B, as the images of higher transgressions in recent work of J.-L. Brylinski and D. A. McLaughlin [11].

We have assumed throughout this text that the group B was Abelian, but the constructions carried out here remain for the most part valid without that hypothesis. Indeed, in Deligne's original construction [15] of the commutator $E_{\mathcal{C}}$ of a monoidal category $\mathcal{C}$, no such commutativity assumption on the group law of B was made, nor was it required in the previously mentioned texts [16] and [11]. Without such a commutativity hypothesis, the torsor $E_{\mathcal{C}}$ is only defined above that part of $B \times B$ which consists of pairs of commuting elements x, y of B. The partial group laws which are introduced here no longer yield in that case a biextension, but a weaker structure which deserves to be formalized. While we have not carried out this formalization here in order not to overburden this text, we intend to return to this question in the future. Let us simply observe for the present that those central extensions whose associated commutator maps are the most interesting are central extensions for which the quotient group B is not abelian. It is therefore to be expected that the same will be true for the higher constructions which we examine here.

While we have emphasized in this introduction the cohomological interpretation of our constructions, in terms of the derived functors of the exterior algebra functor, this will not be the case in the sequel. Indeed, the emphasis will henceforth be on the determination of the new higher alternating structures, rather than on the quest for an interpretation in geometric terms of the universal coefficient theorem. This text is therefore independent of the forthcoming [9]. Both approaches are, however, fully compatible, and shed light upon each other. In the present context, this is illustrated in [6], where alternating biextensions are analyzed *via* the universal coefficient theorem.

1. Cohomology and Categories

The most general interpretation of the cohomology group $H^3(B, A)$ is the one due to A. Grothendieck. It expresses degree three cohomology classes in terms of monoidal categories (see [27], [12] Section 2.1 and also, in a sheaf-theoretic context in which the three-cocycles do not appear explicitly, [14]). We begin by recalling this interpretation of $H^3(B, A)$, and refer to [10], IV Section 5, and to [23] and references therein for related descriptions of this cohomology group. Observe first of all that if $(\mathcal{C}, \otimes, a)$ is a monoidal group-like groupoid* with unit object I, then the monoidal structure on $\mathcal{C}$ determines, for each object $X \in \mathcal{C}$, a right multiplication isomorphism

$$A = \mathrm{Aut}(I) \xrightarrow{\otimes X} \mathrm{Aut}(X) \tag{1.1}$$

through which any group of automorphisms in $\mathcal{C}$ will henceforth be identified with A. The image in $\mathrm{Aut}(X)$ by the left multiplication isomorphism

$$A = \mathrm{Aut}(I) \xrightarrow{X\otimes} \mathrm{Aut}(X) \tag{1.2}$$

of an element $a \in A$ may be identified by (1.1) with an element ${}^{X}a \in A$ which actually only depends on the isomorphism class x of X in the group B of isomorphism classes of objects of $\mathcal{C}$, and which will therefore be denoted ${}^{x}a$. It is readily verified that this action of B on A endows A with a B-module structure.

* Also referred to as a *gr*-category [27] or a categorical group [21].

We now choose, for each $x \in B$, an object $X_x \in \mathcal{C}$ whose isomorphism class is x. For each pair of elements $x, y \in B$, the objects $X_x X_y$ and X_{xy} both live in the component of $\mathcal{C}$ described by the element $xy \in B$, so that there exist arrows between them. Choose such an arrow

$$c_{x,y}\colon X_x X_y \longrightarrow X_{xy} \tag{1.3}$$

for each $x, y \in B$. For every $x, y, z \in B$, the associativity isomorphism

$$a_{x,y,z}\colon X_x(X_y X_z) \longrightarrow (X_x X_y)X_z$$

determines an element $f(x, y, z) \in \mathrm{Aut}(X_{xyz}) = A$ such that the diagram

$$\begin{tikzcd}
X_x(X_y X_z) \arrow[r, "a_{x,y,z}"] \arrow[d, "X_x c_{y,z}"'] & (X_x X_y)X_z \arrow[d, "c_{x,y} X_z"] \\
X_x X_{yz} \arrow[d, "c_{x,yz}"'] & X_{xy} X_z \arrow[d, "c_{xy,z}"] \\
X_{xyz} \arrow[r, "f(x,y,z)"'] & X_{xyz}
\end{tikzcd} \tag{1.4}$$

commutes. The pentagon axiom in $\mathcal{C}$ then implies that $f(x, y, z)$ is a three-cocycle. We may even assume, by choosing the objects X_x and the arrows (1.3) carefully, that the three-cocycle f is normalized (as will be all those occurring from now on, unless explicitly stated). Other choices for these objects and arrows of $\mathcal{C}$ yield a cohomologous three-cocycle, so that the class of $\mathcal{C}$ in $H^3(B, A)$, for the B-module structure on A determined by (1.1)–(1.2), is well-defined.

Remark 1.1. (i) The previous construction may be interpreted as follows in topological terms. The nerve $\mathcal{G} = N\mathcal{C}$ of $\mathcal{C}$ is a two-stage Postnikov system with homotopy groups $\pi_0(\mathcal{G}) = B$ and $\pi_1(\mathcal{G}, I) = A$. The monoidal structure on $\mathcal{C}$ determines an A_∞ H-space structure on $\mathcal{G}$, so that $\mathcal{G}$ deloops to a connected space $\mathcal{X} = B\mathcal{G}$ whose homotopy groups B and A live respectively in degrees one and two. The k-invariant $k \in H^3(B, A)$ of the two-stage system $\mathcal{X}$ is the sought-for cohomology class associated to the monoidal category $\mathcal{C}$. Conversely, one can start from such 2-stage Posnikov system $\mathcal{X}$. The space $\mathcal{Y} = \Omega\mathcal{X}$ of loops on $\mathcal{X}$ is essentially the nerve of a groupoid $\mathcal{C}$, and the H-space structure on $\mathcal{Y}$ corresponds to the monoidal structure on $\mathcal{C}$.

(ii) When the group law in $\mathcal{C}$ is strict, the monoidal category $\mathcal{C}$ may be represented by a crossed module $N \longrightarrow E$ with E (resp. N) the group of objects (resp. the group of arrows sourced at the identity object) of $\mathcal{C}$. A comparison between the terms in the formula [10], IV (5.7) and the arrows in diagram (1.4) implies that the standard method [10] IV Section 5 for associating a three-cocycle to a crossed module is consistent with the one given above.

(iii) Suppose that the monoidal category $\mathcal{C}$ is endowed with a commutativity isomorphisms $s_{x,y}\colon X_y X_x \longrightarrow X_x X_y$. This determines, via the commutative diagram

$$\begin{tikzcd} X_y X_x \arrow[r, "s_{x,y}"] \arrow[d, "c_{y,x}"'] & X_x X_y \arrow[d, "c_{x,y}"] \\ X_{xy} \arrow[r, "g(x,y)"'] & X_{xy} \end{tikzcd}$$

a map $g\colon B \times B \to A$ for which the braiding axioms [21] imply that the cocycle conditions

$$\begin{aligned} &f(x,y,z) - f(x,z,y) + f(z,x,y) \\ &\quad = g(x+y,z) - g(x,z) - g(y,z), \\ &-f(x,y,z) + f(y,x,z) - f(y,z,x) \\ &\quad = g(x,y+z) - g(x,y) - g(x,z) \end{aligned} \tag{1.5}$$

are satisfied. Alternate choices yield a well-defined class in $H^4(K(B,2),A)$ [21], Proposition 3.1, [8] Section 7.8. The braiding axioms allow a double delooping of the nerve $\mathcal{G}$ of $\mathcal{C}$ to a two-stage space $\mathcal{Y}$, whose k-invariant is this cohomology class. When $\mathcal{C}$ is symmetric monoidal, the additional condition $g(x,y) = g(y,x)$ is satisfied. The two conditions (1.5) then coalesce and a class in the stable cohomology group $H^5(K(B,3),A)$, corresponding to the k-invariant of a further delooping of $\mathcal{G}$, is defined. It follows that $\mathcal{G}$ is in that case an infinite loop space. Finally, when the stronger condition

$$g(x,x) = 0 \tag{1.6}$$

is satisfied, the monoidal category $\mathcal{C}$ is strict Picard*. Prolonging by one step the canonical resolution [19] VII (3.5.1) of the Abelian group B, it is apparent that the pair (f,g) now describes a class in the group $\mathrm{Ext}^2(B,A)$. The nerve of $\mathcal{G}$ is now equivalent to a simplicial Abelian group, and the appropriate tunctation of its associated Moore complex determines the class in question. We observed earlier that such a group Ext^2 is always trivial in the category of Abelian groups. All such strict Picard categories are therefore equivalent to trivial ones. The cocycle (f,g) which describes such a strict Picard category $\mathcal{C}$ is therefore a coboundary, so that there exists a map $h\colon B^2 \longrightarrow A$ satisfying the following conditions:

$$f(x,y,z) = h(y,z) - h(x+y,z) + h(x,y+z) - h(x,y), \tag{1.7}$$

$$g(x,y) = h(x,y) - h(y,x). \tag{1.8}$$

* In other words a strictly symmetric group-like monoidal groupoid.

In particular, the underlying monoidal category of $\mathcal{C}$ is trivial, as reflected by the fact that h is a group cohomology coboundary for f.

We now investigate the conditions under which the given group law in the monoidal category $\mathcal{C}$ satisfies some form of commutativity. One possible approach would consist in expressing in geometric terms the obstructions to the surjectivity of the successive suspension maps

$$H^5(K(B,3),A) \to H^4(K(B,2),A) \to H^3(B,A)$$

which control the level of commutativity of the group law on $\mathcal{C}$. We will instead examine here, without passing through these intermediate steps, the conditions under which the monoidal category $\mathcal{C}$ can be endowed with a fully symmetric monoidal structure. Let us begin by making the following additional assumption.

HYPOTHESIS 1.2. The group B is Abelian, and the B-module structure on A is trivial.

Note that this is a very weak commutativity condition. Indeed, the requirement that there exists, for each pair of objects X, Y in $\mathcal{C}$, an isomorphism between YX and XY implies that the group B of isomorphism classes of objects of $\mathcal{C}$ is Abelian. If we also ask that this family of isomorphisms be natural in the objects X and Y, and compatible in the obvious sense with the identity object, then the B-module structure on A is trivial, the triviality being expressed by the commutativity of the following diagram

$$\begin{tikzcd}
 & X \arrow[dl] \arrow[dr] & \\
XI \arrow[rr] \arrow[d, "Xf"'] & & IX \arrow[d, "fX"'] \\
XI \arrow[rr] \arrow[dr] & & IX \arrow[dl] \\
 & X &
\end{tikzcd}$$

In particular, Hypothesis 1.2 is automatically satisfied whenever the category $\mathcal{C}$ is braided.

Hypothesis 1.2 allows us to apply the universal coefficient theorem to the computation of $H^3(B,A)$, and therefore to obtain an analog for H^3 of the exact sequence (0.1). Let us begin with the naive approach to this question. The terms of the universal coefficient exact sequence

$$0 \to \mathrm{Ext}^1(H_2(B),A) \to H^3(B,A) \to \mathrm{Hom}(H_3(B),A) \to 0 \tag{1.9}$$

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can be made explicit since the appropriate homology groups are known (see [9]). To the given class in $H^3(B, A)$ of a monoidal group-like groupoid $\mathcal{C}$ satisfying Hypothesis 1.2 is associated a trilinear alternating map $\varphi_{\mathcal{C}}: \Lambda^3 B \to A$. If φ vanishes, then a second map $\psi_{\mathcal{C}}: \Omega_2 B \to A$ may be associated to $\mathcal{C}$. The source $\Omega_2 B$ of this arrow is a group ΩB first defined by Eilenberg–Mac Lane [17], and which can be interpreted as the first (nonadditive) left derived functor $L_1\Lambda^2(B, 0)$ of the Abelian group B set in degree zero. A specific presentation of $\Omega_2 B$ expresses the map $\psi_{\mathcal{C}}$ in terms of a family of quadratic maps $\psi_n: {}_nB \to A$ for varying positive integers n, related to each other in an appropriate manner. Finally, if $\psi_{\mathcal{C}}$ is trivial, the class of $\mathcal{C}$ is determined by an element $\chi_{\mathcal{C}}$ in the extension group $\mathrm{Ext}^1(\Lambda^2 B, A)$.

While this is a complete discussion, there remains the question of interpreting it in geometric terms. The fact that the functor $\Omega_2 B$ is the first derived functor of the exterior algebra functor $\Lambda^2 B$ suggests that the real object of interest, encompassing both $\psi_{\mathcal{C}}$ and $\chi_{\mathcal{C}}$, lives in the group $\mathrm{Ext}^1(L\Lambda^2 B, A)$. Indeed it is shown in [6] Remark 3.6 that an element in this group determines by dévissage appropriate elements $\psi_{\mathcal{C}}$ and $\chi_{\mathcal{C}}$. The next three sections will provide a construction of the object $E_{\mathcal{C}}$ whose class determines the sought-after element in the group $\mathrm{Ext}^1(L\Lambda^2 B, A)$.

2. Alternating Biextensions

The group $\mathrm{Ext}^1(L\Lambda^2 B, A)$ was given a geometrical interpretation in [6], as the group of equivalence classes of alternating biextensions of B by A. Let us begin by reviewing the definition of an alternating biextension. Let E be an A-torsor above $B \times B$. Its fiber above a point $(x, y) \in B \times B$ will be denoted $E_{x,y}$. Recall first of all that an (ordinary) biextension of $B \times B$ by A is such an A-torsor E above $B \times B$, endowed with a pair of partial composition laws whose restrictions to the appropriate fibers may be depicted as morphisms of A-torsors

$$\overset{1}{+}: E_{x,y} \wedge E_{x',y} \longrightarrow E_{xx',y}, \tag{2.1}$$

$$\overset{2}{+}: E_{x,y} \wedge E_{x,y'} \longrightarrow E_{x,yy'}, \tag{2.2}$$

where $\wedge = \wedge^A$ denotes the contracted product of the corresponding A-torsors. These two composition laws are required to be associative, commutative and compatible with each other [24], [19] exposé VII Section 2. A torsor endowed with a pair of partial multiplication laws which are merely associative and compatible will be called a weak biextension*.

For the reader's convenience, we review the manner in which the structure on a biextension E whose underlying torsor is trivialized may be described in terms of cocycles. The triviality hypothesis asserts that the underlying torsor E may simply be defined by $E = A \times B \times B$. The first and second partial group laws

* In that case, a better notation for the two partial group laws would be $\overset{1}{\times}$ and $\overset{2}{\times}$.

are respectively determined by maps $g(b_1, b_2; b')$ and $h(b; b'_1, b'_2)$ from B^3 to A such that

$$\begin{aligned}(a, b_1, b') \overset{1}{+} (a, b_2, b') &= (a + g(b_1, b_2; b'),\ b_1 + b_2,\ b'),\\ (a, b, b'_1) \overset{2}{+} (a, b, b'_2) &= (a + h(b; b'_1, b'_2),\ b,\ b'_1 + b'_2)\end{aligned} \tag{2.3}$$

on E. The associativity conditions for these laws translate to the following cocycle conditions* on g and h [24].

$$\begin{aligned}\frac{g(b_2, b_3\,; b')\, g(b_1, b_2 + b_3\,; b')}{g(b_1 + b_2, b_3\,; b')\, g(b_1, b_2\,; b')} &= 1,\\ \frac{h(b\,; b'_2, b'_3)\, h(b\,; b'_1, b'_2 + b'_3)}{h(b\,; b'_1 + b'_2, b'_3)\, h(b\,; b'_1, b'_2)} &= 1,\end{aligned} \tag{2.4}$$

in other words to the standard two-cocycle condition for the maps $g(-, -\,;\ b')$ and $h(b\,;\ -, -)$ from B^2 to A, for all fixed $b, b' \in B$. Similarly, the commutativity conditions, when they are satisfied, translate to the standard symmetry cocycles with the last (resp. the first) variable fixed

$$g(b_1, b_2\,; b') = g(b_2, b_1\,; b'), \qquad h(b\,; b'_1, b'_2) = h(b\,; b'_2, b'_1).$$

Finally, the compatibility condition now becomes the rule

$$\frac{h(b_1 + b_2\,; b'_1, b'_2)}{h(b_1\,; b'_1, b'_2)\, h(b_2\,; b'_1, b'_2)} = \frac{g(b_1, b_2\,; b'_1 + b'_2)}{g(b_1, b_2\,; b'_1)\, g(b_1, b_2\,; b'_2)}. \tag{2.5}$$

A cocycle pair (g, h) is cohomologous to zero and, therefore, defines a trivial biextension structure, whenever there exists a map $k\colon B \times B \to A$ such that

$$\begin{aligned}g(b_1, b_2\,; b') &= \frac{k(b_1 + b_2, b')}{k(b_1, b')\, k(b_2, b')},\\ h(b\,; b'_1, b'_2) &= \frac{k(b, b'_1 + b'_2)}{k(b, b'_1)\, k(b, b'_2)}.\end{aligned} \tag{2.6}$$

Let us now pass from ordinary to alternating biextensions. Their description in [6] was modelled on the exact triangle derived from the Koszul sequence

$$0 \to \Gamma_2 B \to B \otimes B \to \Lambda^2 B \to 0. \tag{2.7}$$

One begins by considering an (ordinary) biextension E of $B \times B$ by A. The restriction ΔE of E to the diagonal in $B \times B$ is an A-torsor on B with the following additional properties.

* The group law of B will henceforth be written additively, and that of A multiplicatively.

(1) The torsor ΔE is symmetric, in other words there exist, for varying $x \in B$, a family of symmetry isomorphisms

$$\sigma_x : \Delta E_{-x} \to \Delta E_x. \tag{2.8}$$

(2) ΔE is endowed with a cube structure which is compatible, in a strong sense, with the symmetry isomorphism.

The precise sense in which the cube structure on the A-torsor $L = \Delta E$ on B is compatible with the symmetry isomorphism $\sigma\colon i^*L \longrightarrow L$ (2.8) is best explained as follows. It is essentially shown in [7] Section 5, though not made explicit there, that for any A-torsor L on B endowed with a cube structure the A-torsor $L \wedge i^*L^{-1}$ on B is canonically endowed with a composition law, which makes it into a (commutative) extension of B by A. The compatibility between cube structure and symmetry on L may be expressed as the requirement that the symmetry isomorphism σ, viewed as a section of $L \wedge i^*L^{-1}$ on B, splits it as a group extension. When this condition is satisfied, one says that the A-torsor L is endowed with a Σ-structure. An alternating structure on a biextension E is then defined as follows.

DEFINITION 2.1. An alternating biextension of $B \times B$ by A is a biextension E of $B \times B$ by A, together with a trivialization $t\colon B \to E$ of the restriction ΔE of E to the diagonal compatible with the Σ-structure of ΔE.

When the underlying torsor of E has a global section, this definition of an alternating biextension can be made explicit in terms of the pair of cocycles (g, h) attached to E. A trivialization t of ΔE compatible with the symmetry isomorphism (2.8) is expressed by a map $u\colon\ B \longrightarrow A$ (for which we may assume that $u(0) = 1$) such that

$$\frac{u(-b)}{u(b)} = \frac{g(b, -b\,; b)}{h(-b\,; b, -b)}, \tag{2.9}$$

for all $b \in B$. We now introduce a map $\lambda\colon B^2 \to A$ which may, in view of (2.5), be defined by either of the two following equations

$$\begin{aligned}\lambda(b_1, b_2) &= g(b_1, b_2\,; b_1 + b_2)\, h(b_1\,; b_1, b_2)\, h(b_2\,; b_1, b_2)\\ &= h(b_1 + b_2\,; b_1, b_2)\, g(b_1, b_2\,; b_1)\, g(b_1, b_2\,; b_2).\end{aligned} \tag{2.10}$$

The requisite compatibility between the trivialization u of ΔE and the cube structure on ΔE may now be expressed as the condition

$$\Theta(u)(b_1, b_2, b_3) = \frac{\lambda(b_1 + b_2, b_3)}{\lambda(b_1, b_3)\lambda(b_2, b_3)}\, g(b_1, b_2\,; b_3)\, h(b_3\,; b_1, b_2), \tag{2.11}$$

where $\Theta(u)$ is the second difference of the map u, defined by

$$\Theta(u)(b_1, b_2, b_3) = \frac{u(b_1+b_2+b_3)\,u(b_1)\,u(b_2)\,u(b_3)}{u(b_1+b_2)\,u(b_1+b_3)\,u(b_2+b_3)}.$$

Finally, an alternating biextension (g, h, u) is trivialized by a map $k\colon B \times B \to A$ satisfying the trivialization conditions (2.6), together with the additional condition

$$k(x, x) = u(x). \tag{2.12}$$

A somewhat more intuitive description of an alternating biextension is obtained by introducing first the simpler concept of an anti-symmetric biextension. Consider the functor $\mathrm{As}^2 B$ of anti-symmetric tensors on B, which fits into the following commutative diagram whose horizontal lines are exact when B is free.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Sym}^2 B \arrow[r] \arrow[d] & \otimes^2 B \arrow[r] \arrow[d] & \mathrm{As}^2 B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma_2 B \arrow[r] & \otimes^2 B \arrow[r] & \Lambda^2 B \arrow[r] & 0
\end{tikzcd}$$

By the snake lemma, this determines for every free Abelian group B a short exact sequence

$$0 \to B/2B \to \mathrm{As}^2 B \to \Lambda^2 B \to 0. \tag{2.13}$$

Anti-symmetric biextensions are to be thought of as those biextensions E of $B \times B$ by A which are classified up to isomorphism by the group $\mathrm{Ext}^1(L\mathrm{As}^2 B, A)$. Denoting by s the map from B^2 to itself which permutes the factors, this means that they are the biextensions E for which we are given a trivialization π of the induced biextension $F = E \wedge s^* E$ compatibly with the natural symmetric biextension structure on F. Such a trivialization π may also be described by the induced biextension isomorphism

$$\pi_{x,y}\colon E_{x,y}^{-1} \to E_{y,x} \tag{2.14}$$

between E^{-1} and the pullback s^*E of E. The symmetry condition on π then becomes the requirement that for each $(x, y) \in B \times B$, the map ${}^t\pi_{x,y}\colon E_{y,x}^{-1} \to E_{x,y}$ induced by π coincides with $\pi_{y,x}$. It is readily verified that any alternating biextension is anti-symmetric ([6] Proposition 1.4). The distinguished triangle associated to (2.13) gives us a new description of an alternating biextension E in terms of the underlying anti-symmetric one. Observe first of all that for any anti-symmetric biextension E of $B \times B$ by A, the pullback ΔE of E along the diagonal is actually a commutative extension* of B by A. Furthermore, the pullback of this extension by the 'multiplication by 2' map

$$2_B\colon B \to B \tag{2.15}$$

* The definition of the group law on ΔE given in (5.4) below for a particular biextension E is valid in the general case.

on B is canonically split as an extension. This is equivalent to the assertion that the square ΔE^2 of the extension ΔE (under Baer addition) is split. An alternating biextension may now be described in the following manner.

PROPOSITION 2.2. *An anti-symmetric biextension E is alternating if and only if its restriction ΔE to the diagonal is split as an extension, by a splitting which is compatible with the splitting of ΔE^2 determined by the anti-symmetry structure on E.*

Here is the cocyclic translation of this new description of an alternating biextension, when the underlying torsor of the biextension E is trivial. The biextension structure on such a trivial A-torsor is described, as before, by a pair of maps $g(b_1, b_2\,; b')$ and $h(b\,; b'_1, b'_2)$ (2.3). An anti-symmetry structure on E is determined by a map $\varphi\colon B^2 \to A$ which trivializes the induced biextension F, in other words a map φ such that the equations

$$\begin{aligned} \frac{\varphi(b_1+b_2, b')}{\varphi(b_1, b')\,\varphi(b_2, b')} &= g(b_1, b_2; b')\,h(b'; b_1, b_2), \\ \frac{\varphi(b,\, b'_1+b'_2)}{\varphi(b, b'_1)\,\varphi(b, b'_2)} &= h(b; b'_1, b'_2)\,g(b'_1, b'_2; b) \end{aligned} \tag{2.16}$$

are satisfied. Since the trivialization of F defined by φ must be compatible with the symmetry structure on F, the maps φ must satisfy the additional condition

$$\varphi(b, b') = \varphi(b', b), \tag{2.17}$$

for all $b, b' \in B$. The following assertion is proved by a rather elaborate cocycle computation, which we omit.

LEMMA 2.3. *The map $c\colon B^2 \to A$ defined by*

$$c(b, b') = \lambda(b, b')\,\varphi(b, b'), \tag{2.18}$$

(where $\lambda(b, b')$ is given by (2.10)*) is an A-valued two-cocycle on B.*

The commutativity condition for the partial group laws $\overset{1}{+}$ and $\overset{2}{+}$, together with Equation (2.17), imply that the two-cocycle c is symmetric, so that the triple (g, h, φ) determines a commutative extension of B by A, which in fact is one previously obtained by restricting the anti-symmetric biextension E above the diagonal in $B \times B$. Equations (2.4)–(2.5) imply that

$$\lambda(b, b')^2 = \frac{\varphi(b+b', b+b')}{\varphi(b, b)\,\varphi(b, b')\,\varphi(b', b)\,\varphi(b', b')}$$

so that the equation

$$c(b, b')^2 = \frac{\varphi(b+b', b+b')}{\varphi(b, b)\,\varphi(b', b')}$$

is satisfied. This shows that the 1-cochain $\psi(b)$ defined by $\psi(b) = \varphi(b, b)$ trivializes the two-cocycle $c(b, b')^2$ which describes ΔE^2. Taking into account the significance of a trivialization of E compatible with all this structure, we may now summarize the previous discussion in the following way.

PROPOSITION 2.4. *An alternating biextension of $B \times B$ by A, with trivial underlying torsor, is determined by a quadruple*

$$B^3 \xrightarrow{g} A \quad B^3 \xrightarrow{h} A \quad B^2 \xrightarrow{\varphi} A \quad B \xrightarrow{u} A. \tag{2.19}$$

The pair (g, h) satisfies Equations (2.4)–(2.5), *φ and u satisfy the conditions* (2.16) *and* (2.17), *together with the additional conditions*

$$c(b, b') = \frac{u(b+b')}{u(b)\,u(b')} \tag{2.20}$$

and

$$u(b)^2 = \varphi(b, b), \tag{2.21}$$

where c is defined, in terms of the triple (g, h, φ), by Equations (2.10) *and* (2.18). *A trivialization of the alternating biextension defined by (g, h, φ, u) is described by a map $k\colon B^2 \to A$ which satisfies Equations* (2.6) *and* (2.12), *and the additional condition*

$$k(x, y)\,k(y, x) \;=\; \varphi(x, y). \tag{2.22}$$

Remark 2.5. Here is the connection between this second cocyclic description of an alternating biextension and the original one in terms of a triple (g, h, u), where (g, h) again satisfies the biextension cocycle conditions (2.4)–(2.5) and u satisfies Equations (2.9) and (2.11). Starting from such a triple (g, h, u), one defines an anti-symmetry isomorphism map $\varphi\colon B^2 \to A$ by

$$\varphi(b, b') \;=\; \frac{u(b+b')}{u(b)\,u(b')\,\lambda(b, b')}. \tag{2.23}$$

It is immediate that $\varphi(b, b')$ also satisfies the symmetry condition (2.17), and Equations (2.16) are consequences of (2.11). In fact, the triple (g, h, φ) describes in cocyclic terms the anti-symmetric biextension determined by the alternating biextension (g, h, u). Furthermore, by definition of $\varphi(b, b')$, the cocycle $c(b, b')$ defined by (2.18) satisfies the trivialization condition (2.20). In order to verify that the quadruple (g, h, φ, u) associated to the triple (g, h, u) and to the map φ (2.23) satisfies the conditions of Proposition 2.4, it suffices to check condition (2.21), in other words that the equation

$$u(b)^2 = \frac{u(2b)}{u(b)^2\,\lambda(b,b)} \tag{2.24}$$

is satisfied. By specialization to the case $b_2 = -b_1$, $b_3 = b_1$, (2.11) yields the equation

$$\frac{u(b)^3 u(-b)}{u(2b)} = \frac{g(b,-b;b)\,h(b;b,-b)}{\lambda(b,b)\,\lambda(-b,b)}.$$

Substituting in this equation the values for $u(-b)$ and for λ given by (2.9) and (2.10) yields the requisite formula (2.24).

3. The Commutator of $\mathcal{C}$ as a Weak Biextension

We are now ready to describe the universal coefficient exact sequence (1.9) in geometric terms. Let $\mathcal{C}$ be a *gr*-category, as defined in Section 1, with invariants B and A satisfying Hypothesis 1.2. Suppose that we have chosen for each $x \in B$, as we did above, a representative object X_x of $\mathcal{C}$ in the isomorphism class of x. To $\mathcal{C}$ we associate the A-torsor E above $B \times B$, which Deligne [15] calls the commutator of $\mathcal{C}$, whose fibre above $(x, y) \in B^2$ is the set

$$E_{x,y} = \mathrm{Isom}_{\mathcal{C}}(X_y X_x, X_x X_y) \tag{3.1}$$

of arrows from $X_y X_x$ to $X_x X_y$. Composing the elements of $E_{x,y}$ on the right with automorphisms of $X_y X_x$, viewed as elements of A, makes E into a right A-torsor on $B \times B$. Alternate choices for the representative objects X'_x and X'_y of x and y yield an A-torsor E' on $B \times B$ isomorphic to E.

The main result of this section is the following proposition.

PROPOSITION 3.1. *The A-torsor E associated to the monoidal category $\mathcal{C}$ is endowed with a natural structure of a weak biextension of $B \times B$ by A.*

Proof. In order to simplify the notation, we will in the following discussion denote by X, Y, Z, etc ... the chosen representatives X_x, X_y, X_z in $\mathcal{C}$ of the elements $x, y, z \in B$. Let $u\colon YX \to XY$ and $v\colon YX' \to X'Y$ be given elements in $E_{x,y}$ and $E_{x',y}$. Their partial sum $u \overset{1}{+} v$ (2.1) is defined to be the section of $E_{xx',y}$ determined as follows. Consider the following composite arrow, in which the unlabelled arrows are the associativity isomorphisms.

$$\begin{aligned} Y(XX') &\to (YX)X' \xrightarrow{u} (XY)X' \to X(YX') \xrightarrow{v} X(X'Y) \\ &\to (XX')Y. \end{aligned} \tag{3.2}$$

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Reverting temporarily to our standard notation for objects, this is the middle arrow in the composite map

$$X_y X_{xx'} \to X_y(X_x X_{x'}) \to (X_x X_{x'})X_y \to X_{xx'}X_y \tag{3.3}$$

whose other arrows are defined by (1.3). If $w\colon Y'X \to XY'$ is another arrow in $\mathcal{C}$, the partial sum $u \overset{2}{+} w$ (2.2) is the composite map constructed in a similar way from the arrow

$$\begin{aligned}(YY')X &\to Y(Y'X) \xrightarrow{w} Y(XY') \to (YX)Y' \xrightarrow{u} (XY)Y' \\ &\to X(YY'). \end{aligned} \tag{3.4}$$

Observe that the composite arrows (3.2) and (3.4), which are built out of intertwining associativity and commutativity isomorphisms, are the familiar boundary arrows in the two hexagons occurring as axioms for braided monoidal categories* [21]. It is therefore not surprising that the diagrams describing the required associativity and compatibility conditions for our composition laws $\overset{1}{+}$ and $\overset{2}{+}$ are closely related to some of the higher braiding axioms embodied in the definition of a braided 2-category [22]. Specifically, for each set of elements $a\colon WX \to XW$, $b\colon WY \to YW$ and $c\colon WZ \to ZW$ of $E_{x,w}$, $E_{y,w}$ and $E_{z,w}$, we must consider a nonstrict version of the tetrahedral diagram analogous to the diagram of type $(\bullet \otimes (\bullet \otimes \bullet \otimes \bullet))$ associated in [22] Section 6 to the objects W, X, Y, Z of $\mathcal{C}$ (see also [2]). In order to take into account the associativities, this requires that we double certain edges of this diagram (in fact precisely those edges which are thickened in the diagram appearing in [22] Section 6).

$$\begin{tikzcd} & XYZW & \\ WXYZ \arrow[ur, shift left] \arrow[ur, shift right] \arrow[rr, dashed] \arrow[dr, shift left, "a(YZ)"] \arrow[dr, shift right, "(aY)Z"'] & & XYWZ \arrow[ul, shift left, "X(Yc)"'] \arrow[ul, shift right, "(XY)c"'] \\ & XWYZ \arrow[uu] \arrow[ur, shift left, "X(bZ)"] \arrow[ur, shift right, "(XY)c"'] & \end{tikzcd} \tag{3.5}$$

There are now five edges incident with each vertex. Replacing each of these vertices by the corresponding commutative pentagon in $\mathcal{C}$, we may now attach one of the incident edges to each of the five vertices of each pentagon. Taking into account the labels given to certain arrows, this can be done in a unique manner, if we require that exactly three edges be incident to each vertex of each pentagons. In our context, this diagram has the following interpretation. The axioms on $\mathcal{C}$ ensure that every face of the polyhedron is now commutative, except possibly for the face comprising the two arrows between $WXYZ$ and $XYZW$.

* We do not, however, assume that our monoidal category $\mathcal{C}$ is braided.

The latter face is the following square, whose vertical arrows are the associativity isomorphisms:

$$
\begin{tikzcd}
W(X(YZ)) \arrow[r, "a\overset{1}{+}(b\overset{1}{+}c)"] \arrow[d] & (X((YZ))W) \arrow[d] \\
W((XY)Z) \arrow[r, "(a\overset{1}{+}b)\overset{1}{+}c"] & ((XY)Z)W).
\end{tikzcd}
$$

Since all other faces of diagram (3.5) commute, so does this face. This finishes the proof that the composition law $\overset{1}{+}$ is indeed associative. The associativity of the law $\overset{2}{+}$ is obtained in a similar manner, by starting instead from the nonstrict version of diagram $((\bullet \otimes \bullet \otimes \bullet) \otimes \bullet)$ of [22] Section 6.

The compatibility between the composition laws $\overset{1}{+}$ and $\overset{2}{+}$ is proved in a similar manner, starting instead from a nonstrict version[⋆] of diagram $((\bullet \otimes \bullet) \otimes (\bullet \otimes \bullet))$ of [6] Section 6. This is the diagram

$$
\begin{tikzcd}
XYZW \arrow[rrrr, dashed] \arrow[rrrr, shift right] \arrow[rrd, dashed] \arrow[dd, shift right, "X(dW)"'] \arrow[dd, shift left, "(Xd)W"] \arrow[rrddd] & & & & ZWXY \\
& & XZWY \arrow[rru, dashed] \arrow[rrd, dashed, shift left, "(cW)Y"] \arrow[rrd, dashed, shift right, "c(WY)"'] & & \\
XZYW \arrow[rru, dashed, shift left, "X(Zb)"] \arrow[rru, dashed, shift right, "(XZ)b"'] \arrow[rrd, shift left, "(cY)W"] \arrow[rrd, shift right, "c(YW)"'] & & & & ZXWY \arrow[uu, shift right, "(Za)Y"] \arrow[uu, shift left, "Z(aY)"'] \\
& & ZXYW \arrow[rruuu] \arrow[rru, shift left, "(ZX)b"] \arrow[rru, shift right, "Z(Xb)"'] & &
\end{tikzcd}
\tag{3.6}
$$

associated to four given arrows $a: XW \to WX$, $b: YW \to WY$, $c: XZ \to ZX$, and $d: YZ \to ZY$. Every vertex of diagram $((\bullet\otimes\bullet)\otimes(\bullet\otimes\bullet))$ of [22] Section 6 has now been replaced by the corresponding associativity pentagon. The only *a priori* non commutative part of our diagram is the square involving the top two horizontal arrows between $XYZW$ and $ZWXY$ (and appropriate associativity arrows). In particular, the following commutative triangles which we extract from diagram (3.6) yield for us the appropriate labels for these horizontal arrows

⋆ I owe to E. Getzler the observation that the nonstrict versions of diagrams $(\bullet \otimes (\bullet \otimes \bullet \otimes \bullet))$ and $((\bullet \otimes \bullet) \otimes (\bullet \otimes \bullet))$ appear, respectively, as Figures 4 and 5 of [4].

$$\begin{tikzcd}[column sep=small] XYZW \arrow[rr, "(a\overset{1}{+}b)\overset{2}{+}(c\overset{1}{+}d)"] \arrow[dr, "(c\overset{1}{+}d)W"'] & & ZWXY & XYZW \arrow[rr, dashed, "(a\overset{2}{+}c)\overset{1}{+}(b\overset{2}{+}d)"] \arrow[dr, dashed, "X(b\overset{2}{+}d)"'] & & ZWXY \\ & ZXYW \arrow[ur, "(a\overset{1}{+}b)W"'] & & & XZWY \arrow[ur, dashed, "(a\overset{2}{+}c)Y"'] & \end{tikzcd} \tag{3.7}$$

The square involving the top two horizontal arrows of (3.6) may therefore be portrayed, with certain associativity morphisms neglected, as

$$\begin{tikzcd} XYZW \arrow[r, "(a\overset{1}{+}b)\overset{2}{+}(c\overset{1}{+}d)"] \arrow[d] & ZWXY \arrow[d] \\ XYZW \arrow[r, dashed, "(a\overset{2}{+}c)\overset{1}{+}(b\overset{2}{+}d)"] & ZWXY. \end{tikzcd} \tag{3.8}$$

This commutes, since all the other faces of diagram (3.6) do. This proves that the partial group laws $\overset{1}{+}$ and $\overset{2}{+}$ in E respectively defined by (3.2) and (3.4) are compatible and therefore finishes the proof of Proposition 3.1.

Remark 3.2. The chosen arrows $c_{x,y}$ (1.3) determines a section

$$d_{x,y}\colon X_y X_x \longrightarrow X_{yx} = X_{xy} \longrightarrow X_x X_y \tag{3.9}$$

of the torsor underlying the commutator E (3.1) of $\mathcal{C}$. The cocycles which express the partial group laws $\overset{1}{+}$ and $\overset{2}{+}$ of E in terms of this section may now be made explicit. Let the sections u and v of $E_{x,y}$ and $E_{x',y}$ be the chosen morphisms $d_{x,y}$ and $d_{x',y}$. The following commutative diagram, in which the unlabelled horizontal arrows are all identity maps, expresses the composite map (3.3) in terms of automorphism of $X = X_{xx'y}$.

$$\begin{tikzcd}[column sep=small] X_y(X_xX_{x'}) \arrow[r, "a_{y,x,x'}"] \arrow[d, dash] & (X_yX_x)X_{x'} \arrow[r, "d_{x,y}X_{x'}"] \arrow[d, dash] & (X_xX_y)X_{x'} \arrow[r, "a^{-1}_{x,y,x'}"] \arrow[d, dash] & X_x(X_yX_{x'}) \arrow[r, "X_xd_{x',y}"] \arrow[d, dash] & X_x(X_{x'}X_y) \arrow[r, "a_{x,x',y}"] \arrow[d, dash] & (X_xX_{x'})X_y \arrow[d, dash] \\ X_yX_{xx'} \arrow[d, dash] & X_{yx}X_{x'} \arrow[r, dash] \arrow[d, dash] & X_{xy}X_{x'} \arrow[d, dash] & X_xX_{yx'} \arrow[r, dash] \arrow[d, dash] & X_xX_{x'y} \arrow[d, dash] & X_{xx'}X_y \arrow[d, dash] \\ X \arrow[r, "f_{y,x,x'}"'] & X \arrow[r] & X \arrow[r, "f^{-1}_{x,y,x'}"'] & X \arrow[r] & X \arrow[r, "f_{x,x',y}"'] & X \end{tikzcd} \tag{3.10}$$

The cocycle $g(x, x'; y)$ which describes the partial sum $\overset{1}{+}$ may now be read off from the lower horizontal map of this diagram as the map $g\colon B^3 \to A$ obtained from the three-cocycle $f(x, y, z)$ by shuffling y through x, x', in other words by the formula

$$g(x, x'\,; y) = \frac{f(x, x', y)\ f(y, x, x')}{f(x, y, x')}. \tag{3.11}$$

Starting instead from Definition (3.4) for $\overset{2}{+}$, one sees that the second cocycle h occurring in the definition of the commutator biextension is obtained by shuffling x through y, y' in the opposite direction, in other words by the rule

$$h(x;\ y, y') = \frac{f(y, x, y')}{f(x, y, y')\ f(y, y', x)}. \tag{3.12}$$

That the pair $(g(x, y\,; z)\,,\ h(x;\ y, z))$ satisfies the cocycle conditions for a weak biextension follows from the previous discussion. It could also have been be proved directly by repeated use of the three-cocycle condition for f.

4. The Trilinear Map Associated to a Monoidal Category

We now examine the conditions under which the commutator weak biextension (3.1) is a genuine biextension. In view of Proposition 3.1, it suffices to check that both partial multiplication laws on E are commutative. In contrast to the associativity and compatibility conditions, the commutativity conditions are not automatically satisfied. At the cocycle level, it is immediate that each of the two commutativity axioms leads to the following condition on f

$$\frac{f(x, y, z)\ f(z, x, y)\ f(y, z, x)}{f(x, z, y)\ f(z, y, x)\ f(y, x, z)} = 1.$$

The expression $\varphi(x, y, z)$ defined by the left-hand side of this equation is simply the evaluation of the three-cocycle f on the triple Pontrjagin product cycle $x.y.z \in H_3(B)$ of classes $x, y, z \in H_1(B)$. It is well-known that φ is a trilinear alternating map. This proves the following proposition

PROPOSITION 4.1. *For any pair of Abelian groups A and B, the weak commutator A-biextension E on $B \times B$ (3.1) associated to a monoidal category $\mathcal{C}$ satisfying Hypothesis 1.2 described by a three-cocycle $f(x, y, z)$ is a genuine biextension of $B \times B$ by A if and only if the alternating map*

$$B \wedge B \wedge B \xrightarrow{\varphi_{\mathcal{C}}} A \tag{4.1}$$

defined by

$$\varphi_{\mathcal{C}}(x, y, z) = \frac{f(x, y, z)\ f(z, x, y)\ f(y, z, x)}{f(x, z, y)\ f(z, y, x)\ f(y, x, z)} \tag{4.2}$$

is trivial.

This statement may also be understood in geometric terms, without appealing to the preferred section d (3.9) of E, by considering the following diagram, which is built by pasting together two diagrams of type (3.2) associated to the partial sums $u \overset{1}{+} v$ and $v \overset{1}{+} u$ of a pair of arbitrary summable sections u, v of E. We no longer assume here that the vertical arrows are the specific morphisms $d_{x,y}$ and $d_{x',y}$, so that the automorphisms of X which u and v determine are not necessarily the identity arrows. The latter are denoted g_u and g_v.

$$
\begin{tikzcd}
X_y(X_{x'}X_x) \arrow[r, "a_{y,x',x}"] \arrow[ddd, no head] & (X_yX_{x'})X_x \arrow[r, "vX_x"] \arrow[d, no head] & (X_{x'}X_y)X_x \arrow[r, "a^{-1}_{x',y,x}"] \arrow[d, no head] & X_{x'}(X_yX_x) \arrow[r, "X_{x'}u"] \arrow[d, no head] & X_{x'}(X_xX_y) \arrow[r, "a_{x',x,y}"] \arrow[d, no head] & (X_{x'}X_x)X_y \arrow[ddd, no head] \\
 & X_{yx'}X_x \arrow[r, "g_vX_x"] \arrow[d, no head] & X_{x'y}X_x \arrow[d, no head] & X_{x'}X_{yx} \arrow[r, "X_{x'}g_u"] \arrow[d, no head] & X_{x'}X_{xy} \arrow[d, no head] & \\
 & X \arrow[r, "g_v"] & X \arrow[r, "f^{-1}_{x',y,x}"] & X \arrow[r, "{}^{x'}g_u"] & X \arrow[d, "f_{x',x,y}"] & \\
X_yX_{x'x} \arrow[r] \arrow[ddd, no head] & X \arrow[u, "f_{y,x',x}"] \arrow[d, "f_{y,x,x'}"'] & & & X \arrow[r, no head] & X_{x'x}X_y \arrow[ddd, no head] \\
 & X \arrow[r, "g_u"'] \arrow[d, no head] & X \arrow[r, "f^{-1}_{x,y,x'}"'] \arrow[d, no head] & X \arrow[r, "{}^{x}g_v"'] \arrow[d, no head] & X \arrow[u, "f_{x,x',y}"'] \arrow[d, no head] & \\
 & X_{yx}X_{x'} \arrow[r, "g_uX_{x'}"'] \arrow[d, no head] & X_{xy}X_{x'} \arrow[d, no head] & X_xX_{yx'} \arrow[r, "X_xg_v"'] \arrow[d, no head] & X_xX_{x'y} \arrow[d, no head] & \\
X_y(X_xX_{x'}) \arrow[r, "a_{y,x,x'}"'] & (X_yX_x)X_{x'} \arrow[r, "uX_{x'}"'] & (X_xX_y)X_{x'} \arrow[r, "a^{-1}_{x,y,x'}"'] & X_x(X_yX_{x'}) \arrow[r, "X_xv"'] & X_x(X_{x'}X_y) \arrow[r, "a_{x,x',y}"'] & (X_xX_{x'})X_y
\end{tikzcd}
\tag{4.3}
$$

All cells in this diagram are commutative, except possibly the large inner one, composed of automorphisms of the object $X = X_{xx'y}$. Since the group $A = \mathrm{Aut}(X)$ is Abelian, all the arrows lying on the boundary of this inner cell may be freely moved past one another. Furthermore, under Hypothesis 1.2, the elements ${}^{x'}g_u$ and g_u in the inner section coincide, so that they cancel each other out, and similar cancellation occurs between ${}^{x'}g_v$ and g_v. Keeping track of the orientations of the arrows, the obstruction to the commutativity of the inner region boils down to the triviality of the expected element $\varphi_{\mathcal{C}}(x, x', y)$ (4.2) of A.

Remark 4.2. The specific arrows u and v chosen in the construction of diagram (4) played no role in the definition of the map $\varphi_{\mathcal{C}}$ (4.1), which only depended on the three-cocycles $f_{x,x',y}$ determined by the associativity data $a_{x,x',y}$ in $\mathcal{C}$. Another set of choices for the vertical maps $X_xX_y \to X_{xy}$ will yield a three-cocycle f' cohomologous to f, and which therefore leaves unchanged the induced map (4.2). The axioms for $\mathcal{C}$ also ensure that alternate choices for the representative objects X_x of $x \in B$ have no effect in this construction.

5. The Alternating Structure on the Commutator Biextension

There remains yet one additional element of structure of E to be made explicit. We now assume that the map (4.1) associated to the given monoidal category $\mathcal{C}$ is trivial, so that by Proposition 4.1 the commutator torsor E (3.1) is a genuine biextension.

PROPOSITION 5.1. *Let $\mathcal{C}$ be a monoidal category with invariants the Abelian groups $\pi_0(\mathcal{C}) = B$ and $\pi_1(\mathcal{C}, I) = A$, satifying Hypothesis* 1.2 *and whose associated trilinear map* (4.2) *is trivial. The associated commutator biextensions E of $B \times B$ by A is alternating.*

Proof. We give here a geometric proof of this assertion, which one was in any case led to expect by the discussion in Section 1. We begin with a geometric proof of the following weaker assertion, which can also be deduced from the cocyclic description (3.11)–(3.12) of E.

LEMMA 5.2. *The commutator biextension E of $\mathcal{C}$ is anti-symmetric.*

Proof. Consider the section σ of $E \wedge s^*E$ above $B \times B$ (where $s: B \times B \to B \times B$ is the map which permutes the factors) defined by the rule which assigns to any pair of elements $x, y \in B \times B$ the element $\sigma(x, y) = t \wedge v$ in $\mathrm{Isom}(X_yX_x, X_x, X_y) \wedge \mathrm{Isom}(X_xX_y, X_yX_x)$ where $v: X_xX_y \longrightarrow X_yX_x$ is the inverse of the arrow $t: X_yX_x \longrightarrow X_x, X_y$. This section of $E \wedge s^*E$ does not in fact depend on the choice of a specific map $t: X_yX_x \longrightarrow X_x, X_y$. In order to check that σ trivializes $E \wedge s^*E$ as a biextension, it must be verified that it is multiplicative in each of its two variables. In the first variable, this boils down to the obvious assertion that for a given pair of sections $t: X_yX_x \longrightarrow X_xX_y$ and $t': X_yX_{x'} \longrightarrow X_{x'}X_y$ of $E_{x,y}$ and $E_{x',y}$ with inverses v and v', the inverse in $\mathcal{C}$ of the composite map (3.2) $t \overset{1}{+} t'$

$$X_yX_{x'}X_x \xrightarrow{t} X_xX_yX_{x'} \xrightarrow{t'} X_xX_{x'}X_y \tag{5.1}$$

is the map $v \overset{2}{+} v'$ (3.4)

$$X_xX_{x'}X_y \xrightarrow{v'} X_xX_yX_{x'} \xrightarrow{v} X_yX_xX_{x'}. \tag{5.2}$$

The multiplicativity of σ in the second variable is verified in a similar manner, so that the lemma is proved.

Since E is anti-symmetric, its restriction ΔE to the diagonal is a (commutative) extension of B by A, for which the group law $\star$ may be described explicitly by the following rule. Let u and v respectively be sections of $E_{x,x}$ and $E_{y,y}$, in other words arrows

$$u: X_xX_x \longrightarrow X_xX_x, \quad v: X_yX_y \longrightarrow X_yX_y \tag{5.3}$$

in $\mathcal{C}$. The arrow $u \star v\colon X_{xy}X_{xy} \longrightarrow X_{xy}X_{xy}$ is determined by the composite map

$$\begin{aligned} &X_xX_yX_xX_y \xrightarrow{X_x\alpha X_y} (X_xX_x)(X_yX_y) \\ &\xrightarrow{uv} (X_xX_x)(X_yX_y) \xrightarrow{X_x\alpha^{-1}X_y} X_xX_yX_xX_y \end{aligned} \tag{5.4}$$

for some arbitrarily chosen arrow $\alpha\colon X_yX_x \longrightarrow X_xX_y$. We know that $(\Delta E)^2$, being the restriction to the diagonal of $E \wedge s^*E$, is trivialized by the restriction $\Delta\sigma$ of σ to the diagonal. For any $x \in B$, it therefore follows that $\sigma(x)$ is the element $t \wedge t^{-1}$ in the set $\mathrm{Isom}(X_xX_x, X_xX_x) \wedge \mathrm{Isom}(X_xX_x, X_xX_x)$, for some arbitrarily chosen arrow $t\colon X_xX_x \longrightarrow X_xX_x$. Choosing for t the identity self-arrow $1_{X_xX_x}$, we may therefore set $\sigma(x) = 1_{X_xX_x} \wedge 1_{X_xX_x}$. Consider now the section τ of ΔE defined by setting $\tau(x) = 1_{X_xX_x}$. The equation $\tau(x) \star \tau(y) = \tau(x+y)$ follows from the definitions, so that the section τ splits ΔE as an extension. The formula $\sigma(x) = \tau(x) \wedge \tau(x)$ is also immediate. By Proposition 2.2, the section τ of ΔE therefore induces an alternating biextension structure on the anti-symmetric biextension E.

It is easily verified that a trivialization of E compatible with its alternating biextension structure determines a strict Picard structure on the monoidal category $\mathcal{C}$. As observed in the introduction, it follows from [14] Section 1.4 that such strict Picard categories are always trivial, since they are classified up to equivalence by the (trivial) group $\mathrm{Ext}^2(B, A)$. This may be spelt out as follows.

COROLLARY 5.3. *Let $\mathcal{C}$ be a monoidal category satisfying the conditions of Proposition* (5.1) *whose associated biextension E is trivial (as an alternating biextension). Then the monoidal structure on $\mathcal{C}$ is trivial.*

In more concrete terms, consider a monoidal category $\mathcal{C}$ determined by a crossed module $\delta\colon M \to N$, for which $\ker(\delta) = A$ and $\mathrm{coker}(\delta) = B$, and for which the B-module structure on A is trivial. The fibre $E_{x,y}$ of the commutator biextension $E_{\mathcal{C}}$ above $(x, y) \in B^2$ is the set $\delta^{-1}(k_{x,y})$, where $k_{x,y} \in K = \mathrm{im}(\delta)$ is the image of (x, y) under the commutator map associated to the central extension

$$0 \to K \to N \to B \to 0.$$

A trivialization of E determines in a map

$$\{\ ,\ \}\colon N \times N \to M \tag{5.5}$$

by associating to a pair of elements X and Y in N with projection x and y in B the element $m \in \delta^{-1}(k_{x,y}) \subset M$ determined by the trivialization. When the trivialization of E is compatible with the anti-symmetric biextension structure on E, the map (5.5) determines a stable crossed module structure [13] on $\delta\colon M \to N$. Compatibility of the trivialization of E with the alternating structure on E yields the additional relation $\{n, n\} = 0$ for all $n \in N$. In the present context, Deligne's result [14] asserts that the given crossed module is then equivalent to the crossed module $P \to Q$ determined by a (splittable) exact sequence of Abelian groups

$$0 \to A \to P \to Q \to B \to 0.$$

We end this section with a brief discussion in cocyclic terms of Proposition 5.1 and of its corollary. When the cocycle pair (g, h) has been defined in terms of a three-cocycle f by Equations (3.11) and (3.12), both the map $\lambda(b_1, b_2)$ (2.10) and the right-hand terms of Equation (2.16) are trivial. Taking into account (2.18), it follows that the quadruple $(g, h, 1, 1)$ satisfies the conditions of Proposition 2.4 so that it defines an alternating biextension structure on the biextension (g, h). The underlying triple $(g, h, 1)$ then satisfies the equivalent conditions (2.9) and (2.11), as asserted in Proposition 5.1. These two conditions on the triple $(g, h, 1)$ may also be verified directly without introducing explicitly the full quadruple $(g, h, 1, 1)$.

A discussion in similar terms of Corollary 5.3 goes as follows. Suppose that the quadruple $(g, h, 1, 1)$ associated to a cocycle f is trivial, so that there exists a map $k(b_1, b_2)$ satisfying conditions (2.6), (2.12) and (2.22). In that case the pair (f, k) defines an element in the trivial group $\mathrm{Ext}^2(B, A)$ so that, as observed at the end of Remark 1.1, there exists an A-valued 2-cochain $l(b_1, b_2)$ on B for which

$$k(b_1, b_2) = \frac{l(b_1,\ b_2)}{l(b_2,\ b_1)} \tag{5.6}$$

and such that the three-cocycle f is the coboundary of l. The latter assertion is the content of Corollary 5.3. We note in passing that the category $\mathcal{C}$ described by the three-cocycle f is braided if and only if there exists a map k which trivializes the pair (g, h) as a biextension (without taking into account the alternating structure), in other words which satisfies Equations (2.6) but not (2.12).

6. From Monoidal Categories to Monoidal Stacks

The previous analysis of monoidal categories *via* the universal coefficient theorem extends to a classification of group-like monoidal stacks in groupoids (also called *gr*-stacks) in a general topos T, as discussed in [8] Section 7, to which we refer for the requisite definitions. One is given a pair of Abelian groups B and A of T, with A viewed as a trivial B-module. The discussion in Section 1 generalizes to the assertion that monoidal stacks $\mathcal{C}$ of T with invariants $\pi_0(\mathcal{C})$ and $\pi_1(\mathcal{C})$, respectively, isomorphic to B and A and satisfying Hypothesis 1.2 are classified by the hypercohomology group $H^3(B, A)$. The difference between this hypercohomology group and the ordinary cohomology group $H^3(B, A)$ is analyzed by the first quadrant spectral sequence

$$E_1^{p,q} = H^q(X_p, A) \Longrightarrow H^{p+q}(B, A) \tag{6.1}$$

whose initial term is the A-valued cohomology of the degree p component B^p of the classifying space X_* of B. From the geometric point of view which concerns us here, the distinction between the hypercohomology group and the naive cohomology group $H^3(B, A)$ built from cochains $B^3 \to A$, and which classify *gr*-categories with invariants B and A, is reflected in the two sets of obstructions whose vanishing is necessary in order to carry out in the stack case the construction of the three-cocycle associated to the monoidal category $\mathcal{C}$. The first of these arises when one attempts to choose, for each section x of B above some object S of T, an object X_x in the fiber category C_S of $\mathcal{C}$ above S. Suppose that this obstruction vanishes, so that the requisite objects X_x exist for all sections x of B. Just as in (1.3), one may then attempt to choose, for every pair of sections x, y of B above a given object S of T, a morphism $c_{x,y}\colon X_x X_y \longrightarrow X_{xy}$ in $\mathcal{C}_S$. If the obstruction to achieving this also vanishes, then one can construct as in (1.4), an A-valued three-cocycle and therefore classify by the naive group $H^3(B, A)$ the stacks $\mathcal{C}$ for which both sets of obstructions vanish. These obstructions do not however vanish in general, but the stack axioms, and the definitions of the objects $\pi_i(\mathcal{C})$ in T ensure nevertheless that they both vanish locally*. The invariants which describe them are therefore of a cohomological nature. We refer to [8] Section 7 for a further discussion of these invariants, and simply observe here that they live respectively in the terms $E_1^{1,2}$ and $E_1^{2,1}$ of the spectral sequence (6.1), while the naive cohomology group $H^3(B, A)$ is its $E_2^{3,0}$ term. The comparison between the naive cohomology group and the associated hypercohomology group (in other words between the classification of *gr*-stacks and that of *gr*-categories) thus boils down to the analysis of the edge-homomorphism map $E_2^{3,0} \to H^3$ in the spectral sequence (6.1).

Another change occurs when one passes from categories to stacks. While the homology of the Abelian group B of a topos T is given by the same formulas as for an abstract group, the relation between the homology and the hypercohomology of B is now more complicated, since the universal coefficient theorem must now be replaced by the universal coefficient spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p(H_q(B), A) \Longrightarrow H^{p+q}(B, A). \tag{6.2}$$

In the abstract group situation, this spectral sequence reduces to the ordinary universal coefficient theorem, since in the category of Abelian groups the groups Ext^p vanish whenever $p > 1$ so that the spectral sequence degenerates. In a topos, no essential change occurs at the level of degree 2 cohomology, so that the analysis of central extensions of groups carried out by the exact sequence (0.1), together with its geometric interpretation, carries over to an arbitrary topos and therefore remains valid (except for the surjectivity of the right-hand arrow) when central extensions of topological groups, or of algebraic groups, are considered. The hypercohomology group $H^3(B, A)$, on the other hand, may no longer be described by a short exact sequence (1.9), since there now exists a new nontrivial initial term in the spectral sequence (6.2), provided by the group $\mathrm{Ext}^2(B, A)$. The latter group was given a geometric interpretation in [14] as the group of equivalence classes of strict Picard stacks. While this group vanishes in the category case, as we observed in Section 1, this is no longer true in the general stack context.

* In other words after base change.

The alternating biextension point of view for analyzing gr-categories carries over very satisfactorily to the gr-stack context. As we have already observed, the objects X_x, and arrows (1.3) on which the definition of the three-cocycle $f(x, y, z)$ depends no longer exist, but they do exist locally, so that the three-cocycle $f(x, y, z)$ is locally defined. Alternate choices for these objects and arrows yield cohomologous cocycles. Since the induced map $\varphi_{\mathcal{C}}$ (4.1) depends only on the cohomology class of f, its local representatives glue together to a globally defined arrow form $\Lambda^3 B$ to A. Similarly, the weak biextension E associated to a gr-stack $\mathcal{C}$ may be locally defined just as in (3.1), once representative objects X_x of $\mathcal{C}$ have been chosen, and the local biextensions obtained in this manner glue to a weak biextension E defined on all of $B \times B$. Its underlying torsor, however, is in general no longer endowed with a globally defined section d (3.9), so that the biextension E may no longer be readily described in terms of cocycles. Propositions 4.1 and 5.1 remain valid in the stack context, and assert that the weak biextension E is a genuine (alternating) biextension of $B \times B$ by A whenever the invariant $\varphi_{\mathcal{C}}$ vanishes. This biextension may be analyzed by the methods of [6]. As observed earlier, this yields a family of induced quadratic maps $\psi_n\colon {}_nB \to A$ whose vanishing implies by the universal coefficient exact sequence (3.19) of [6] that the biextension E descends to an ordinary extension of $\Lambda^2 B$ by A.

Suppose now that the alternating biextension E is trivial. We may then choose in a compatible manner, for each pair of sections x, y of B, an arrow

$$s(x, y)\colon X_y X_x \to X_x X_y \tag{6.3}$$

in $\mathcal{C}$. This actually determines, for an arbitrary pair of objects X, Y of $\mathcal{C}$ (with associated sections x, y in $B \times B$), a symmetry arrow by the rule

$$YX \xrightarrow{c_y c_x} X_y X_x \xrightarrow{s(x,y)} X_x X_y \xrightarrow{c_x^{-1} c_y^{-1}} XY. \tag{6.4}$$

This is independent of the (local) choice of arrows $c_x\colon X \longrightarrow X_x$ and $c_y\colon Y \longrightarrow X_y$ in $\mathcal{C}$. The compatibility of the section s of E with the partial multiplication laws (2.1) and (2.2) implies, as we have already observed, that the symmetry arrows (6.3) (and therefore more generally the corresponding symmetry arrows (6.4)) satisfy both hexagon conditions, so that $\mathcal{C}$ is a braided stack. Finally, compatibility of s with the section t (Definition 2.1) of E asserts that for $X = Y$ the composite map (6.4) is simply the identity map. This forces the braided category $\mathcal{C}$ to be Picard strict, and so provides a direct geometric interpretation of the degree 3 portion of the universal coefficient spectral sequence (6.2). We have therefore obtained in the stack context the following analog of Corollary 5.3.

COROLLARY 6.1. *Let $\mathcal{C}$ be a gr-stack of T satisfying the conditions of Proposition* 5.1, *and whose associated biextension E is trivial (as an alternating biextension). Then $\mathcal{C}$ is the underlying gr-stack of a strict Picard stack.*

Vanishing theorems for certain groups $\mathrm{Ext}^2(B\,,\ A)$ are proved in [5]. When this vanishing takes place, the corresponding strict Picard stack are trivial, so that the Corollary 6.1 takes on the form of a vanishing theorem for gr-stacks, along the same lines as in Corollary 5.3.

7. Higher Multiextensions

The cohomology groups $H^{n+1}(B, A)$ with $n > 2$ are the k-invariants of two-stage Postnikov systems with homotopy groups B and A concentrated in degrees 1 and n. From a categorical point of view, such a cohomology class may therefore be represented by a $(n-1)$-category (or rather $(n-1)$-groupoid) $\mathcal{C}$, endowed with a multiplication $\mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ satisfying the requisite higher associativity axiom. It is required that the group of isomorphism classes of objects of $\mathcal{C}$ be isomorphic to B and that the intermediate homotopy groups $\pi_i(\mathcal{C})$ vanish $(0 < i < n-1)$. The group $\pi_{n-1}(\mathcal{C})$, which is simply the group of self-$(n-1)$-maps of the identity $(n-2)$-arrow, is required to be isomorphic to A. Finally, the requirement that A is a trivial B-module can be translated into a weak commutativity condition, analogous to Hypothesis 1.2.

In the following discussion we will mainly be concerned with the case $n = 3$, where the definition of a monoidal 2-groupoid does not offer any difficulty. We note in passing that the analog in this 2-categorical context of a crossed module, which occurs when the associativity isomorphism in $\mathcal{C}$ is strict, has been worked out by D. Conduché in [13] Definition 2.2 under the name of 2-crossed modules. The requisite conditions on the homotopy groups now translates to the requirement that such a 2-crossed module $L \to M \to N$ lives in an exact sequence of groups

$$0 \to A \to L \to M \to N \to B \to 0.$$

A direct proof of the classification of such length 3 extensions by elements of the group $H^4(B, A)$ is given in [13] Theorem 4.7. Such a discussion can also be carried out from a categorical point of view by extending by one more step to a representation of pentagonal 2-arrows the geometric construction of the three-cocycle discussed in Section 1.

Let us now now examine the effect on cohomology of the filtration on the chains on $K(B, 1)$ by powers of the augmentation ideal mentioned in the introduction. Recall that the terms which occur in the analysis of $H^{n+1}(B, A)$ are the groups $\mathrm{Ext}^p(L\Lambda^q B, A)$, with $p + q = n + 1$. In particular, the filtration on the group $H^4(B, A)$ (which classifies 2-categories of the type envisaged above) yields successive geometric objects which respectively live in the groups $\mathrm{Hom}(\Lambda^4 B, A)$, $\mathrm{Ext}^1(L\Lambda^3 B, A)$, $\mathrm{Ext}^2(L\Lambda^2 B, A)$ and $\mathrm{Ext}^3(B, A)$. A prerequisite to a geometric discussion of this filtration of H^4 is the interpretation in geometric terms of the groups $\mathrm{Ext}^p(\overset{L}{\otimes}{}^q B, A)$ for varying integers p and q. We will call the objects whose isomorphism classes are classified by these groups (p, q)-extensions (or (p, q)-multi-extensions) of B by A. A $(p, 1)$-extension is simply a p-fold extension by A

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of the abelian group B, and is therefore geometrically described by classes of strict Picard $(p-1)$-categories with invariants B and A. Similarly, a $(1, 2)$-extension is an (ordinary) biextension of $B \times B$ by A and more generally a $(1, q)$-extension is, in the terminology of [19] VII 2.10.2, a q-extension of the q groups $B_1, \ldots, B_q$ by A, with $B_1 = \cdots = B_q = B$. These interpretations of $(p, 1)$ and $(1, q)$-extensions may be combined as follows. Choosing in the manner explained in [19] VII as representative for the object $L \otimes^q B$ of the derived category the q-fold tensor product of a canonical free resolution of B, it is apparent that a (p, q)-extension for a general pair of integers p and q may be thought of as an abelian $(p-1)$-gerbe $\mathcal{C}$ on B^q [8], together with a family of q partial group laws $\overset{1}{+}, \cdots, \overset{q}{+}$ on $\mathcal{C}$ living above the q composition laws on B^q determined by the group laws on each of the p factors. Each of these partial group laws is required to satisfy the requisite higher associativity and commutativity conditions, together with higher compatibily conditions between them. These higher conditions may be worked out by considering the cells and their boundaries in the chosen representative of $L \otimes^q B$. We simply spell this out in the case of (2, 2)-extensions, the only essentially new case required for an understanding of $H^4(B, A)$. As we have just asserted, this is an Abelian A-gerbe $\mathcal{C}$ above $B \times B$, together with a pair of partial group laws $\overset{1}{+}$ and $\overset{2}{+}$. The partial commutativity and associativity conditions assert that, for each section $x\colon S \to B$ of B, the groups laws $\overset{2}{+}$ and $\overset{1}{+}$ respectively endow the pullbacks $(x \times 1)^* \mathcal{C}$ and $(1 \times x)^* \mathcal{C}$ of $\mathcal{C}$ above the S-groups $S \times B$ and $B \times S$ with the structure of a strict Picard stacks [8]. The compatibility conditions between the two group laws are described as follows. We may choose functorial isomorphisms

$$c_{x_1, x_2\,;\, x_3, x_4} : (X \overset{1}{+} Y) \overset{2}{+} (Z \overset{1}{+} W) \longrightarrow (X \overset{2}{+} Z) \overset{1}{+} (Y \overset{2}{+} W), \tag{7.1}$$

where the projections π of the four objects X, Y, Z, W to the group of isomorphism classes of objects satisfy

$$\pi(X) = (x_1, x_3), \quad \pi(Y) = (x_2, x_3), \quad \pi(Z) = (x_1, x_4), \quad \pi(W) = (x_2, x_4)$$

for sections x_i of B, so that the source and target of (7.1) are well defined. These isomorphisms c are required to be compatible with the associativity and commutativity isomorphisms for $\overset{1}{+}$ and $\overset{2}{+}$. The compatibility of c with the commutativity isomorphisms asserts that the following diagram, in which the horizontal arrows are determined by the commutativity axiom for $\overset{2}{+}$, commutes for all allowable

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object. So must the corresponding one in which the role of $\overset{1}{+}$ and $\overset{2}{+}$ have been exchanged.

$$\begin{tikzcd}
(X \overset{1}{+} Y) \overset{2}{+} (Z \overset{1}{+} W) \arrow[r] \arrow[d, "c"'] & (Z \overset{1}{+} W) \overset{2}{+} (X \overset{1}{+} Y) \arrow[d, "c"] \\
(X \overset{2}{+} Z) \overset{1}{+} (Y \overset{2}{+} W) \arrow[r] & (Z \overset{2}{+} X) \overset{1}{+} (W \overset{2}{+} Y)
\end{tikzcd} \tag{7.2}$$

Similarly, the compatibility of the maps (7.1) with the associativity isomorphisms for the two partial group laws is described by the commutativity of the following diagram in which the vertical arrows are maps (7.1) and the horizontal diagrams are associativity isomorphisms for $\overset{2}{+}$, and by the corresponding one in which the role of $\overset{2}{+}$ and $\overset{1}{+}$ is interchanged.

$$\begin{tikzcd}
(X \overset{1}{+} Y) \overset{2}{+} ((Z \overset{1}{+} W) \overset{2}{+} (S \overset{1}{+} T)) \arrow[r] \arrow[d] & ((X \overset{1}{+} Y) \overset{2}{+} (Z \overset{1}{+} W)) \overset{2}{+} (S \overset{1}{+} T) \arrow[d] \\
(X \overset{1}{+} Y) \overset{2}{+} ((Z \overset{2}{+} S) \overset{1}{+} (W \overset{2}{+} T)) \arrow[d] & ((X \overset{2}{+} Z) \overset{1}{+} (Y \overset{2}{+} W)) \overset{2}{+} (S \overset{1}{+} T) \arrow[d] \\
(X \overset{2}{+} (Z \overset{2}{+} S)) \overset{1}{+} (Y \overset{2}{+} (W \overset{2}{+} T)) \arrow[r] & ((X \overset{2}{+} Z) \overset{2}{+} S) \overset{1}{+} ((Y \overset{2}{+} W) \overset{2}{+} T).
\end{tikzcd} \tag{7.3}$$

When associativity and compatibility constraints are given for the laws $\overset{1}{+}$ and $\overset{2}{+}$, but no commutativity constraint, (and *a fortiori* no commutative diagram (7.2) is introduced), we will say that $\mathcal{C}$ is a weak (2,2)-extension of $B \times B$ by A. Observe that, despite their somewhat abstract aspect, (2,2)-extensions are not hard to classify. Indeed, the group $\mathrm{Ext}^2(B \overset{L}{\otimes} B, A)$ of isomorphism classes of such (2,2)-extensions may be analyzed *via* the adjunction spectral sequence, whose low-degree terms are described in [19] VIII (1.1.4). In that context, when B is an Abelian variety over an algebraically closed field, and A is the multiplicative group G_m, the vanishing of both $\mathrm{Hom}(B, A)$ and $\mathrm{Ext}^2(B, A)$ (see [5]) ensures that such (2,2)-extensions are classified up to equivalence by the group $\mathrm{Ext}^1(B, B^t)$ of extensions of B by the dual Abelian variety B^t.

In order to describe alternating (2,2)-extensions, we first introduce the concept of an anti-symmetric (2,2)-extension. In the situation just examined, these have a very concrete interpretation, since they are described by extensions

$$0 \to B^t \to E \to B \to 0 \tag{7.4}$$

of B by B^t which are opposite (for the Baer sum) to the extension

$$0 \to B^t \to E^t \to B \to 0$$

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obtained by applying the contravariant duality functor $(\quad)^t$ to the exact sequence of Abelian varieties (7.4). Returning to the general situation, these anti-symmetric (2,2)-extensions may be described as follows. Let s once more denote the map which permutes the factors of B^2.

DEFINITION 7.1. A $(2,2)$-extension is anti-symmetric if it is endowed with a morphism of $(2,2)$-extensions

$$\pi\colon \mathcal{C}^{-1} \longrightarrow s^*\mathcal{C} \tag{7.5}$$

analogous to isomorphism (2.14), whose source is the A-gerbe $\mathcal{C}^{-1} = \mathrm{Hom}_A(\mathcal{C}, \text{Tors } A)$ of morphisms* of A-gerbes from $\mathcal{C}$ to the trivial A-gerbe. We further require that the composite morphism

$$s^*\pi_{\mathcal{C}} \circ \pi_{\mathcal{C}^{-1}}\colon \mathcal{C} \simeq (\mathcal{C}^{-1})^{-1} \longrightarrow s^*\mathcal{C}^{-1} \longrightarrow s^*s^*\mathcal{C} \simeq \mathcal{C} \tag{7.6}$$

be equivalent to the identity functor, by an equivalence which is unchanged when the factors of B^2 are permuted.

For any stack $\mathcal{C}$, let us denote by $\mathcal{C}^0$ the opposite stack of $\mathcal{C}$, whose fibers are the categories opposite to the fibers of $\mathcal{C}$. If $\mathcal{C}$ is a gerbe, then it is immediate that $\mathcal{C}^0$ also is one. Suppose further that $\mathcal{C}$ is an A-gerbe for some group A, so that we are given, for objects X of $\mathcal{C}$, a family of isomorphism $\lambda_X\colon A \longrightarrow \mathrm{Aut}_{\mathcal{C}}(X)$. Then these maps λ may also be viewed as isomorphisms $\lambda_X\colon A^0 \longrightarrow \mathrm{Aut}_{\mathcal{C}^0}(X)$ between the opposite groups, so that they define on $\mathcal{C}^0$ a natural A^0-gerbe structure. We believe that the following description of the inverse $\mathcal{C}^{-1}$ of an Abelian A-gerbe $\mathcal{C}$ may be of independent interest. It is a local statement, and may therefore be verified by supposing that $\mathcal{C}$ is the trivial A-gerbe, in which case it is immediate.

LEMMA 7.2. *Let A be an Abelian group of T, and $\mathcal{C}$ an Abelian A-gerbe. The Yoneda morphism*

$$\begin{array}{rcl} \mathcal{C}^0 & \to & \mathrm{Hom}_A(\mathcal{C}, \text{Tors } A) \\ X & \mapsto & h^X \end{array}$$

is an isomorphism of A-gerbes.

The morphism (7.5) which defines an anti-symmetry structure on the $(2,2)$-extension $\mathcal{C}$ may therefore be described by a morphism of A-gerbes

$$\pi_{\mathcal{C}}\colon \mathcal{C}^0 \longrightarrow s^*\mathcal{C}. \tag{7.7}$$

* A morphism $\Phi\colon \mathcal{C} \longrightarrow \mathcal{D}$ of A-gerbes is a morphism of gerbes for which the induced maps $\mathrm{Aut}(X) \to \mathrm{Aut}(\Phi X)$ are identified with the identity by the A-gerbe structure on the source and target gerbe. In particular, such a morphism Φ induces (in the terminology of Giraud [18]) the identity map on liens.

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In other words, by a 'contravariant morphism of A-gerbes' from $\mathcal{C}$ to its pullback $s^*\mathcal{C}$ which is compatible, in the obvious sense, with each of the two given partial group laws on $\mathcal{C}$. We further require that the additional anti-symmetry conditions on $\pi_{\mathcal{C}}$ analogous to those of Definition 7.1 be satisfied. The pullback $\Delta\mathcal{C}$ of $\mathcal{C}$ along the diagonal is then canonically endowed with the structure of a strict Picard category with invariants B an A, and in fact one whose pullback by the 'multiplication by 2' map (2.15) is a trivial strict Picard category. The full definition of an alternating $(2, 2)$-extension may now be given.

DEFINITION 7.3. A $(2, 2)$-extension $\mathcal{C}$ of $B \times B$ by A is alternating if it anti-symmetric, and if the induced strict Picard category $\Delta\mathcal{C}$ is trivial, by a trivialization whose pullback by the 'multiplication by 2' map (2.15) is the canonical one determined by the anti-symmetry structure*.

The other higher element of structure whose definition will be required is the concept of an alternating (1,3)-extension, an object classified up to equivalence by elements of the group $\mathrm{Ext}^1(L\Lambda^3 B, A)$. We begin with an ordinary (1,3)-extension (in other words a triextension) of B^3 by A. Recall that this is an A-torsor E on B^3, together with three partial multiplication laws

$$\begin{aligned}
\overset{1}{+}\colon\ E_{x,y,z}\,E_{x',y,z} &\;\to\; E_{x+x',y,z},\\
\overset{2}{+}\colon\ E_{x,y,z}\,E_{x,y',z} &\;\to\; E_{x,y+y',z},\\
\overset{3}{+}\colon\ E_{x,y,z}\,E_{x,y,z'} &\;\to\; E_{x,y,z+z'},
\end{aligned}$$

each of which is commutative and associative, and any two of which are compatible with each other. Just as alternating biextensions could be understood by considering the complex (2.7), information concerning an alternating structure on E may be inferred from the complex

$$0 \to \Gamma_3 B \to \Gamma_2 B \otimes B \to B \otimes \Lambda^2 B \to \Lambda^3 B \to 0 \tag{7.8}$$

which is simply a Koszul complex [20] Chapter 1 (4.3.1.3). By examining the right-hand arrow in this complex, we see that an alternating triextension E with general fibre $E_{x,y,z}$, when viewed for a fixed $x \in B$ as a biextension in y, z of B^2 by A, must be alternating in the variables y, z. Its restriction $E_{x,z,z}$ above the diagonal Δ_{23} is therefore provided with a section $t^2_{x,z} \in E_{x,z,z}$. It is required that this section be compatible, as in Definition 2.1, with the symmetry and cube structures in z on $E_{x,z,z}$ determined by the second and third group laws on E. This section $t^2_{x,z}$ must also be linear in x, in the sense that the equation

$$t^2_{x,z} \overset{1}{+} t^2_{x',z} = t^2_{x+x',z} \tag{7.9}$$

* It is equivalent to require that the square of the trivialization is that on $\mathcal{C}^2$ determined by the anti-symmetry structure.

must be satisfied in $E_{x+x',z}$. The middle map in the sequence (7.8) may be interpreted as determining a constraint on the triextension E which is to be quadratic in x and linear in z. Such a constraint is given for a fixed z by an alternating structure on $E_{x,y,z}$, viewed as a biextension in the variables x, y, in other words by a section $t^1_{x,z} \in E_{x,x,z}$ compatible with the Σ-structure in x and satisfying a relation analogous to (7.9) in the variable z.

We must now express the compatibility conditions between these sections t^1 and t^2 which follow from the left-hand arrow in (7.8). Since the group $\Gamma_3 B$ has two separate sorts of generators, those of the form $\gamma_3(b)$ and those of type $\gamma_2(b)\, b'$ for elements $b, b' \in B$, two distinct sorts of compatibilities will have to be verified between these sections. These compatibilities will however be related to each other by the conditions corresponding to the identities

$$3\gamma_3(b) = \gamma_2(b)b,$$

$$\gamma_3(b+b') - \gamma_3(b) - \gamma_3(b') = \gamma_2(b)\, b' + \gamma_2(b')\, b$$

in the group $\Gamma_3 B$. The first of these compatibilities, which is cubical in b and therefore corresponds to the generators $\gamma_3(b)$ of $\Gamma_3 B$, asserts that for all $x \in B$ the equation

$$t^1_{x,x} \;=\; t^2_{x,x} \tag{7.10}$$

is satisfied in $E_{x,x,x}$. In order to state the second compatibility condition between t^1 and t^2 as pleasantly as possible, we need the following lemma.

LEMMA 7.4. *Let $E_{x,y,z}$ be a triextension of B^3 by A, endowed as above with an alternating structure in x linear in z defined by a section $t^1_{x,z} \in E_{x,x,z}$, and with an alternating structure in z linear in x defined by a section $t^2_{x,z} \in E_{x,z,z}$. The restriction $E_{x,y,x}$ of E above the diagonal Δ_{13} is canonically endowed with an alternating structure $t^3_{x,y} \in E_{x,y,x}$ in x, which is linear in y.*

Proof. Since E is alternating with respect to x, it is *a fortiori* antisymmetric. This amounts to the assertion that the section $t^1_{x+y,z}\,(t^1_{x,z})^{-1}\,(t^1_{y,z})^{-1} \in E_{x+y,x+y,z}\, E^{-1}_{x,x,z}\, E^{-1}_{y,y,z}$ defines, *via* a canonical isomorphism determined by the biextension structure of E, a trivialization $s^1_{x,y,z}$ of the symmetric biextension $F^1_{x,y,z} = E_{x,y,z}\, E_{y,x,z}$. Since t^1 defines a Σ-structure in x on $E_{x,x,z}$, and is therefore quadratic in the variable x, the equation $t^1_{2x,z} \;=\; (t^1_{x,z})^4$ is satisfied up to canonical isomorphism, from which the equation

$$s^1_{x,x,z} = (t^1_{x,z})^2 \tag{7.11}$$

in $E_{x,x,z}$ follows immediately. The equation

$$s^2_{x,y,y} = (t^2_{x,y})^2, \tag{7.12}$$

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(where $s^2_{x,y,z} = t^2_{x,y+z}\,(t^2_{x,y})^{-1}\,(t^2_{x,z})^{-1}$) is proved in the same way, as a consequence of the quadraticity in z of $t^2_{x,z}$. Let us now set

$$t^3_{x,y} \;=\; s^1_{x,y,x}(t^2_{y,x})^{-1}. \tag{7.13}$$

This is, as required, an element of $(E_{x,y,x}\; E_{y,x,x})(E_{y,x,x})^{-1} \simeq E_{x,y,x}$. It is readily verified that the section $t^3_{x,y}$ of $E_{x,y,x}$ defined in this manner satisfies the requisite quadraticity condition in x and linearity condition in y, so that the lemma is proved.

Another section $\tilde{t}^{\,3}_{x,y}$ of $E_{x,y,x}$ with the same properties as $t^3_{x,y}$ could have been defined in terms of the alternating structure $t^1_{x,z}$ and the antisymmetry section $s^2_{x,y,z} \in E_{x,y,z}\; E_{x,z,y}$ determined by $t^2_{x,y}$ by setting

$$\tilde{t}^{\,3}_{x,y} \;=\; s^2_{x,y,x}(t^1_{x,y})^{-1}. \tag{7.14}$$

The second compatibility which the sections t^1 and t^2 must satisfy is the requirement that

$$t^3_{x,y} \;=\; \tilde{t}^{\,3}_{x,y}. \tag{7.15}$$

This may, of course, also be written as the condition

$$\frac{t^1_{y,x}}{t^2_{x,y}} \;=\; \frac{s^2_{y,x,y}}{s^1_{y,x,y}}. \tag{7.16}$$

Our definition of an alternating triextension is now complete. It can be summarized as follows, with the corresponding notion of a trivialization spelled out.

DEFINITION 7.5. A triextension E of B^3 by A is alternating if it is endowed with sections $t^1_{x,z} \in E_{x,x,z}$ and $t^2_{x,z} \in E_{x,z,z}$ such that t^1 defines a partial alternating structure on E with respect to x linear in z and t^2 defines a partial alternating structure on E with respect to z linear in x. The sections t^1 and t^2 must also satisfy the compatibility conditions (7.10) and (7.16) with s^1 and s^2 defined as in the proof of Lemma 7.4. A trivialization of E as an alternating triextension is determined by a section $\sigma_{x,y,z}$ of $E_{x,y,z}$ which trivializes E as a triextension (in other words compatibly with each of the three partial group laws), and such that $\sigma_{x,x,z} = t^1_{x,z}$, $\sigma_{x,z,z} = t^2_{x,z}$.

The section $t^3_{x,y}$ of such an alternating triextension defined by formula (7.13) determines as above a partial antisymmetry structure $s^3_{x,y,z} \in E_{x,y,z}\; E_{z,y,x}$, by setting, up to a canonical isomorphism

$$s^3_{x,y,z} = \frac{t^3_{x+z,y}}{t^3_{x,y}t^3_{z,y}}. \tag{7.17}$$

The formula

$$s^3_{x,y,z} = \frac{s^1_{x,y,z}s^1_{z,y,x}}{s^2_{y,x,z}} \tag{7.18}$$

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now follows from (7.14) and the definition of $s^2_{x,y,z}$. The equations

$$t^1_{x,z} = s^3_{x,x,z}(t^2_{z,x})^{-1}, \qquad t^2_{x,z} = s^3_{x,z,z}(t^1_{z,x})^{-1} \tag{7.19}$$

are consequences of (7.11), (7.12) and (7.16). The first of these equations shows that this new method for constructing t^1 out of t^2 and the anti-symmetry condition s^3 derived from t^3 yields the same result as the method (7.14) for constructing t^1 out of t^3 and the anti-symmetry condition s^2 derived from t^2. Similarly the second formula shows that the new method (7.19) for constructing t^2 out of t^1 and the anti-symmetry condition derived from t^3 yields the same result as the method (7.13) for constructing it out of t^3 and the anti-symmetry condition s^1 derived from t^1. Allowing ourselves a certain amount of redundancy, we may therefore give another description of alternating triextension which is entirely symmetric in the variables x, y, z, as befits an object associated to $\Lambda^3 B$.

PROPOSITION 7.6. *A triextension E of B^3 by A is alternating if and only if it is endowed with sections $t^1_{x,z} \in E_{x,x,z}$, $t^2_{x,z} \in E_{x,z,z}$ and $t^3_{x,y} \in E_{x,y,x}$ each of which defines a partial alternating structure with respect to the repeated variable which is linear with respect to the other variable and which satisfy the following compatibility conditions*

(1) *For each i, the two possible methods described above for constructing a section t^i in terms of the two other sections t^j and t^k yield the same result.*
(2) *For every $x \in B$, the equation $t^1_{x,x} = t^2_{x,x} = t^3_{x,x}$ is satisfied in $E_{x,x,x}$.*

Remark 7.7. This description of alternating triextensions may be obtained in a somewhat more symmetric manner by making use of the derived version of the sequence

$$0 \to \Gamma_3 B \to (\Gamma_2 B \otimes B) \oplus (B \otimes \Gamma_2 B) \to B^{\otimes 3} \to \Lambda^3 B \to 0$$

of [1], instead of the Koszul sequence (7.8). This sequence also makes it immediately clear that an alternating triextension whose underlying triextension is trivial, may be described in terms of pairs of compatible maps $f, g\colon B \times B \to A$, with f quadratic in the first variable and linear in the second one (resp. g linear in the first variable and quadratic in the second one). Note that in an algebro-geometric setting, this often implies that such alternating triextensions are trivial. For example, when B is an Abelian variety over an algebraically closed field, it is easily verified (see [19], VII 2.10.2) that any triextension of B by the multiplicative group G_m is trivial. The assertion is now immediate, since the only maps from $B \times B$ to G_m are the constant ones.

8. Picard Structures on Monoidal 2-Categories

Let $\mathcal{C}$ be a monoidal group-like 2-stack in groupoids, as defined under the name of 2-*gr*-stack in [8] Definition 8.4. The first invariant of $\mathcal{C}$ is the sheaf of groups $\pi_0(\mathcal{C})$

associated to the presheaf of isomorphism classes of objects of $\mathcal{C}$. In the category case which we will mostly consider, we will say that $\mathcal{C}$ is a monoidal group-like 2-groupoid. The group $\pi_0(\mathcal{C})$ is then simply the group of isomorphism classes of objects of $\mathcal{C}$. Our first assumption will be, as in the monoidal 1-category case, that this group is Abelian. Since the monoidal category $\mathrm{Aut}_{\mathcal{C}}(I)$ of self-arrows of the unit object I of $\mathcal{C}$ is automatically braided, the two other homotopy groups of $\mathcal{C}$, which may be defined by

$$\pi_i(\mathcal{C}) = \pi_{i-1}(\mathrm{Aut}_{\mathcal{C}}(I)) \tag{8.1}$$

for $i = 1, 2$, are both Abelian groups. It was explained in [8] that such monoidal 2-groupoids with $\pi_0(\mathcal{C}) \simeq B$ and $\mathrm{Aut}_{\mathcal{C}}(I)$ equivalent to a given braided category $\mathcal{A}$ are classified by an appropriately defined cohomology group $H^3(B, \mathcal{A})$. The group B acts by conjugation on the category $\mathcal{A}$, and we will assume that this action is equivalent to the trivial one*. Finally, we will assume in the sequel for simplicity that the Abelian group $\pi_1(\mathcal{C})$ is trivial, so that $\mathcal{A}$ is the category with a unique object whose arrows form an Abelian group A. The cohomology group $H^3(B, \mathcal{A})$ then reduces to the standard cohomology group $H^4(B, A)$ with values in the trivial B-module A. As we have said, the class of $\mathcal{C}$ may be viewed as the k-invariant of the two stage Postnikov system

$$\begin{tikzcd} K(A,3) \arrow[r] & X \arrow[d] \\ & K(B,1) \end{tikzcd}$$

defined by the the classifying space X of the nerve of the monoidal 2-category $\mathcal{C}$. In more explicit terms, one associates to $\mathcal{C}$ the A-valued four-cocycle $f(x, y, z, w)$ obtained as follows. Choose, as in the case of monoidal categories, representative objects X_x and arrows $c_{x,y}\colon X_x X_y \to X_{xy}$ in $\mathcal{C}$. Since it is assumed here that $\pi_1(\mathcal{C}) = 0$, we may also choose for every $x, y, z \in B$ a 2-arrow $\eta_{x,y,z}\colon 1_{X_{x,y,z}} \Longrightarrow f(x, y, z)$ between the identity 1-arrow, and the 1-arrow defined as in 1.4. The pentagon two-arrow associated to the four objects X_x, X_y, X_z, X_w then determines an element $f(x, y, z, w)$ in $\mathrm{Aut}_{\mathcal{C}}(X_{xyzw})$, i.e. a four-cochain $f\colon B^4 \to A$. Stasheff's K_5 relation [28] implies that f is a four-cocycle. Other choices of objects X_x, 1-arrows $c_{x,y}$ and 2-arrows $\eta_{x,y,z}$ determine cohomologous cocycles so that the class of f in $H^4(B, A)$ only depends on the equivalence class of $\mathcal{C}$.

As in our study of monoidal categories, we may analyze the monoidal two-category $\mathcal{C}$ by introducing, for each pair of elements $x, y \in B$, the commutator category

$$\mathcal{E}_{x,y} = \mathrm{Isom}_{\mathcal{C}}(X_y X_x, X_x X_y) \tag{8.2}$$

* In other words that the tensor functor $\phi\colon \underline{B} \to \mathrm{Aut}_{\mathcal{C}}(I)$ with source the discrete category defined by B is equivalent to the trivial functor.

This is a groupoid, on which the gr-category of self-arrows of $X_x X_y$ acts on the right fully and faithfully by composition of arrows. This category is equivalent to $\mathcal{A}$, and therefore, since $\pi_1(\mathcal{C}) = 0$, to the groupoid $A[1]$ with a single object defined by the Abelian group A. The categories $\mathcal{E}_{x,y}$ assemble, for varying x, y, to form an Abelian A-gerbe $\mathcal{E}$ on $B \times B$ which is a first element of structure associated to the monoidal 2-stack $\mathcal{C}$. In fact this gerbe is trivial, since one can choose a compatible family of sections $s(x, y) \in \mathcal{E}_{x,y}$, for example as in (3.9) those obtained by composing the chosen maps $X_x X_y \longrightarrow X_{xy}$ with an inverse of the maps $X_y X_x \longrightarrow X_{yx}$. There remains, however, some interesting structure on $\mathcal{E}$ to be explored. Indeed, the constructions (3.2) and (3.4) now define partial group laws $\overset{1}{+}$ and $\overset{2}{+}$ on the Abelian A-gerbe $\mathcal{E}$ on $B \times B$. Once more, no commutativity property for the partial laws $\overset{1}{+}$ and $\overset{2}{+}$ is asserted, so that $\mathcal{E}$ is in general a weak, rather than a genuine (2,2)-extension.

This analysis of $\mathcal{C}$, and of its associated commutator $\mathcal{E}$ carries over from monoidal 2-categories to monoidal 2-stacks, the only significant difference being that in that case the underlying A-gerbe of $\mathcal{E}$ is no longer trivial. The following higher analog of Proposition 3.1 is therefore true.

PROPOSITION 8.1. *Let $\mathcal{C}$ be a monoidal* 2*-stack with invariants B and A, satisfying the previous hypotheses. $\mathcal{C}$ is classified up to equivalence by an element of the (hyper)-cohomology group $H^4(B, A)$ (for A a trivial B-module). The constructions* (3.2) *and* (3.4) *define on the abelian A-gerbe $\mathcal{E}$* (8.2) *on $B \times B$ a weak* $(2, 2)$*-extension structure*.

In order to prove that each of the two group laws on $\mathcal{C}$ is coherently associative, one could simply examine the next higher versions of diagram (3.5), in other words the weak versions of the pair of 2-categorical diagrams which would, in the terminology of [22], be denoted by $(\bullet \otimes (\bullet \otimes \bullet \otimes \bullet \otimes \bullet))$ and $((\bullet \otimes \bullet \otimes \bullet \otimes \bullet) \otimes \bullet)$. The next higher version of the compatibility diagram (3.6) would then show that the two group laws are compatible with each other, in the sense made explicit for the map (7.1) by the two diagrams (7.3). Such an argument would certainly be sufficient in order to prove the proposition. However, if one wanted to fill in the details of a proof along these lines, one would be led to the consideration of a family of commuting 2-categorical diagrams, which cannot be represented here in an enlightening manner. We therefore prefer to give a proof of the proposition in cocyclic, rather than diagrammatic terms, even though this method of proof *a priori* only applies in the monoidal 2-category case, rather than the full monoidal 2-stack situation. The method of proof which we now propose will thus be analogous to the discussion in Remark 3.2, but at the next higher level.

Starting from an A-valued four-cocycle $f(x_1, x_2, x_3, x_4)$, we have seen that for a fixed $x \in B$ the group law $\overset{1}{+}$ is obtained by inserting 2-arrows derived from f into the pentagons by which the vertices of diagram (3.5) were replaced when we passed from this diagram to its nonstrict version. The three-cochain $\psi_{x_4}\colon B^3 \longrightarrow A$ which describes as in (1.4) the associativity morphism in the monoidal category $\mathcal{E}_{(\ ,x_4)}$ on B for the group law $\overset{1}{+}$ is therefore defined by composition of these two-arrows, in other words (once the sign has been taken into account) by the formula

$$\psi_{x_4}(x_1, x_2, x_3) = \prod_{\sigma(1)<\sigma(2)<\sigma(3)} f(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)})^{-\epsilon(\sigma)} \tag{8.3}$$

in which $\epsilon(\sigma)$ denotes the sign of the permutation σ. This is just the product of the signed permutations of $f(x, y, z, w)$ when the variable w is shuffled through (x, y, z). The group law $\overset{2}{+}$ on $\mathcal{E}_{(x_1,\ )}$ is similarly described by

$$\phi_{x_1}(x_2, x_3, x_4) = \prod_{\sigma(2)<\sigma(3)<\sigma(4)} f(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)})^{\epsilon(\sigma)} \tag{8.4}$$

i.e. by the product of the signed shuffles in f of x_1 through (x_2, x_3, x_4). That each of the two group laws satisfies the pentagon condition is equivalent to the assertion that the corresponding three-cochain (8.3),(8.4) is a three-cocycle, and this follows readily from the four-cocycle condition on f. In fact, it is unnecessary to perform this computation explicitly, in view of the following observation. Consider Eilenberg–Mac Lane's iterated bar-construction model $A(B, 2)$ [17] Section 14 for the complex of chains on the Eilenberg–Mac Lane space $K(B, 2)$. Since this is a chain complex, the square $\delta \circ \delta$ of the differential δ is trivial when applied to any cell c. Applying this respectively to the cells $[x_1 \ |_2\ x_2, x_3, x_4, x_5]$ and $[x_1, x_2, x_3, x_4 \ |_2\ x_5]$, and passing from chains to A-valued cochains on $K(B, 2)$ yields the sought-after assertion. With this in mind, we relabel the two previous associativity maps by setting

$$\begin{aligned} \phi(x_1, x_2, x_3 \,|_2\, x_4) &= \psi_{x_4}(x_1, x_2, x_3), \\ \phi(x_1 \,|_2\, x_2, x_3, x_4) &= \phi_{x_1}(x_2, x_3, x_4), \end{aligned}$$

even though the first of these definitions is only consistent with [17] up to a sign. Similarly, the compatibility isomorphism (7.1) between the two group laws on $\mathcal{E}$ is described by the cochain

$$\phi(x_1, x_2 \,|_2\, x_3, x_4) = \prod_{\sigma(1)<\sigma(2)\,;\ \sigma(3)<\sigma(4)} f(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)})^{-\epsilon(\sigma)}$$

obtained by shuffling (x_1, x_2) through (x_3, x_4). The vanishing of the image under $\delta \circ \delta$ of the cells $[x_1, x_2, x_3 \,|_2\, x_4, x_5]$ and $[x_1, x_2 \,|_2\, x_3, x_4, x_5]$ (or a direct computation) imply that the higher compatibility conditions (7.3) are satisfied in $\mathcal{E}$, so that Proposition 8.1 is proved.

In order to understand under which conditions the weak monoidal commutator (2,2)-extension obtained from Proposition 8.1 is a genuine (2,2)-extension (in other words one whose partial group laws are strictly commutative), we need only apply to the monoidal categories $\mathcal{E}_{(\ ,x_4)}$ and $\mathcal{E}_{(x_1,\ )}$ the theory developed in Sections 3 to 5. By (3.11)–(3.12), the weak biextension associated to the monoidal category $\mathcal{E}_{(\ ,x_4)}$ is described, for a fixed $x_4 \in B$, by the cochains

$$\begin{aligned} g(x_1, x_2; x_3 \mid_2 x_4) &= \prod_{\sigma(1)<\sigma(2)} \phi(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)} \mid_2 x_4)^{\epsilon(\sigma)} \\ &= \prod_{\sigma(1)<\sigma(2)} f(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)})^{-\epsilon(\sigma)} \end{aligned} \tag{8.5}$$

and

$$\begin{aligned} h(x_1; x_2, x_3 \mid_2 x_4) &= \prod_{\sigma(2)<\sigma(3)} \phi(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)} \mid_2 x_4)^{-\epsilon(\sigma)} \\ &= \prod_{\sigma(2)<\sigma(3)} f(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)})^{\epsilon(\sigma)}. \end{aligned} \tag{8.6}$$

Similarly, the weak biextension associated to the monoidal category $\mathcal{E}_{(x_1,\ )}$ is described, for a fixed $x_1 \in B$, by the pair

$$\begin{aligned} \gamma(x_1 \mid_2 x_2, x_3; x_4) &= \prod_{\sigma(2)<\sigma(3)} \phi(x_1 \mid_2 x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)})^{\epsilon(\sigma)} \\ \eta(x_1 \mid_2 x_2; x_3, x_4) &= \prod_{\sigma(3)<\sigma(4)} \phi(x_1 \mid_2 x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)})^{-\epsilon(\sigma)}, \end{aligned}$$

so that

$$\gamma(x_1 \mid_2 x_2, x_3; x_4) = h(x_1; x_2, x_3 \mid_2 x_4)$$

and

$$\eta(x_1 \mid_2 x_2; x_3, x_4) = \prod_{\sigma(3)<\sigma(4)} f(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)})^{-\epsilon(\sigma)}. \tag{8.7}$$

There is a unique condition under which the two pairs of partial group laws defining these two weak biextensions are commutative, thereby ensuring that both (g, h) and (γ, η) define genuine biextensions. This condition is given by the vanishing of the alternating map

$$l(x_1, x_2, x_3, x_4) = \prod_{\sigma \in \Sigma_4} f(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)})^{\epsilon(\sigma)}$$

determined by evaluating the four-cocycle f on the Pontryagin product $x_1 \cdot x_2 \cdot x_3 \cdot x_4$ of the four classes $x_i \in H_1(B) = B$. By Proposition 5.1, both biextensions (g, h)

and (γ, η) are then alternating. As explained in the discussion following Corollary 5.3, they may therefore be described in cocyclic terms by the triples $(g, h, 1)$ and $(\gamma, \eta, 1)$ (in which the term 1, which describes the alternating structure, is the trivial map $1\colon B \to A$ sending every element of B to the identity element of A). The compatibility condition (7.10) between the first and second alternating structure is automatically satisfied here. Since the sections $s^1_{x,y,z}$ and $s^2_{x,y,z}$ satisfying the corresponding relations (2.16) are described by trivial maps, this is also the case for the compatibility condition (7.16).

We summarize the previous discussion by the following

PROPOSITION 8.2. *Let $\mathcal{C}$ be a monoidal category defined by a four-cocycle $f(x_1, x_2, x_3, x_4)$, and for which the condition*

$$\prod_{\sigma \in \Sigma_4} (f(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}))^{\epsilon(\sigma)} = 1$$

is satisfied. The pair of triples $(g(x_1, x_2; x_3 \mid_2 x_4),\ h(x_1; x_2, x_3 \mid_2 w),\ 1)$ and $(\gamma(x_1 \mid_2 x_2, x_3; x_4), \eta(x_1 \mid_2 x_2; x_3, x_4),\ 1)$ defined by setting

$$g(x_1, x_2; x_3 \mid_2 x_4) = \prod_{\sigma(1) < \sigma(2)} f(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)})^{-\epsilon(\sigma)}, \tag{8.8}$$

$$h(x_1; x_2, x_3 \mid_2 x_4) = \prod_{\sigma(2) < \sigma(3)} f(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)})^{\epsilon(\sigma)},$$

$$\gamma(x_1 \mid_2 x_2, x_3; x_4) = h(x_1; x_2, x_3 \mid_2 x_4),$$

$$\eta(x_1 \mid_2 x_2; x_3, x_4) = \prod_{\sigma(3) < \sigma(4)} f(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)})^{-\epsilon(\sigma)},$$

respectively, determine, for every fixed $x_4 \in B$ and every fixed $x_1 \in B$, a biextension structure. Together they define an alternating triextension of $B \times B \times B$ by A.

Suppose now that this alternating triextension is trivial. The trivializing section $\sigma_{x,y,z}$ of $E_{x,y,z}$ (Definition 7.5) is then described by a map $\theta\colon B^3 \longrightarrow A$ such that the equations

$$g(x_1, x_2; x_3 \mid_2 x_4) = \frac{\theta(x_1 + x_2, x_3, x_4)}{\theta(x_1, x_3, x_4)\,\theta(x_2, x_3, x_4)},$$

$$h(x_1; x_2, x_3 \mid_2 x_4) = \frac{\theta(x_1, x_2 + x_3, x_4)}{\theta(x_1, x_2, x_4)\,\theta(x_1, x_3, x_4)},$$

$$\eta(x_1 \mid_2 x_2; x_3, x_4) = \frac{\theta(x_1, x_2, x_3 + x_4)}{\theta(x_1, x_2, x_3)\,\theta(x_1, x_2, x_4)},$$

$$\theta(x, x, z) = 1, \qquad \theta(x, z, z) = 1$$

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are satisfied. In that case the weak (2,2)-extension (8.2) determined by Proposition 8 is a genuine (2,2)-extension. We may now introduce additional cochains $\theta(x_1, x_2 \mid_2 x_3)$ and $\theta(x_1 \mid_2 x_2, x_3)$ defined by

$$\theta(x_1, x_2 \mid_2 x_3) = \theta(x_1, x_2, x_3), \qquad \theta(x_1 \mid_2 x_2, x_3) = \theta(x_1, x_2, x_3).$$

These respectively describe the commutativity isomorphisms in the strict Picard categories $\mathcal{E}_{(\, ,x_4)}$ and $\mathcal{E}_{(x_1, \,)}$, so that the pair $(\phi(x_1, x_2, x_3 \mid_2 x_4), \theta(x_1, x_2 \mid_2 x_4))$ for x_4 fixed and the pair $(\phi(x_1 \mid_2 x_2, x_3, x_4), \theta(x_1 \mid_2 x_2, x_4))$ for x_1 fixed each satisfy the cocycle conditions (1.5)–(1.6). This (2,2)-extension is automatically alternating, as may be verified by a discussion parallel to that of Section 5, or by a cocyclic argument.

While we could pursue this analysis in cocyclic terms of the (2,2)-extension $\mathcal{E}$, it is more expedient to return to a 2-categorical framework. A trivialization of $\mathcal{E}$ consists of a trivialization, for each x (resp. each w) in B of the Picard stack $\mathcal{E}_{(x, \,)}$ (resp. $\mathcal{E}_{(\, ,w)}$), together with a compatibility condition between these trivializations. Returning to the definition (8.2) of $\mathcal{E}$, we see that such a trivialization of $\mathcal{E}_{(x, \,)}$ (compatible with the Picard structure) consists, once a family of choices of one-arrows

$$R_{x,y} \colon X_y X_x \longrightarrow X_x X_y \tag{8.9}$$

have been made*, in a 'hexagon' 2-arrow

$$H_{[x|_2 y,z]} \colon R_{x,y} \circ R_{x,y'} \Longrightarrow R_{x,yy'}$$

in $\mathcal{C}$. The compatibility of this trivialization with the associativity isomorphism in $\mathcal{E}_{(x, \,)}$ implies that this hexagon 2-arrow satisfies the axiom denoted $(\bullet \otimes (\bullet \otimes \bullet \otimes \bullet))$ in [22], which we already encountered in a somewhat different context. A trivialization of $\mathcal{E}_{(\, ,w)}$ similarly defines the 'hexagon' 2-arrow between the 1-map defined as in (3.2) (for $Y = X_w$) from the 1-arrow obtained by composing $R_{x,w}$ and $R_{x',w}$ and the 1-arrow $R_{xx',w}$, and this then satisfies the corresponding axiom $((\bullet \otimes \bullet \otimes \bullet) \otimes \bullet))$. The compatibilities of these 2-arrows with the commutativity isomorphisms determined by the strict Picard structure yields the axioms $((\bullet \otimes \bullet) \otimes (\bullet \otimes \bullet))$ on the 2-category $\mathcal{C}$. Finally, the compatibility of this pair of hexagon 2-arrows with each other implies that the 2-category $\mathcal{C}$ is endowed with a slight variant of Kapranov–Voevodsky's two-braiding axioms, which we called a Z-braiding in [8], Ch. 8 (see also [2] for a discussion of this supplementary axiom).

If we now require that the chosen trivalization (8.9) of the (2,2)-extension $\mathcal{E}$ to be compatible with its anti-symmetry structure, defined as in Lemma 5.2, we must further require that there exists for all $x, y \in B$ a two-arrow

$$1_{X_y X_x} \overset{S_{x,y}}{\longmapsto} R_{y,x} \circ R_{x,y} \tag{8.10}$$

* After a preliminary choice of a representative object X_x of the fixed object x, and a family of representatives X_y of the varying $y \in Y$.

with source the identity one-arrow on $X_y X_x$. This 2-arrow automatically satisfies the two conditions which define on $\mathcal{C}$ the structure of a strongly braided two-category [8] (in other words what J. Baez calls a strongly involutory monoidal category [3]). Finally the compatibility of the trivialization of $\mathcal{E}$ with its alternating (rather than simply anti-symmetric) structure manifests itself in a trivialization of the Picard stack $\Delta\mathcal{E}$ obtained, as required in Definition 7.3, by restricting $\mathcal{E}$ above the diagonal. Since the alternating structure on $\mathcal{E}$ is determined by the identity arrow $1_{X_x X_x} \in (\Delta\mathcal{E})_x$, this compatibility may be interpreted as a 2-arrow $1_{X_x X_x} \xmapsto{S_x} R_{x,x}$. By compatibility of this 2-arrow with the group law on $\Delta\mathcal{E}$, S_x is additive in x. Furthermore, the required compatibility of its square with the trivialization of $\mathcal{E}^2$ determined by the anti-symmetry structure on $\mathcal{E}$ is the assertion that the composite 2-arrow in diagram (8.4.8) of [8] coincides with our 2-arrow $S_{x,x}$ (8.10). Observe also that the compatibility condition mentioned in [8] (8.4.6) is in fact a consequence of the required additivity in x of S_x, and therefore must not be imposed here as a supplementary condition. A trivialization of the alternating $(2, 2)$-extension $\mathcal{E}$ thus determines on $\mathcal{C}$ what we have called a strictly symmetric monoidal two-category structure. These strictly symmetric structures on monoidal two-categories with associated groups B and A are classified by the group $\mathrm{Ext}^3(B, A)$. In the 2-stack case, this is a genuine invariant, which classifies these structures up to equivalence. On the other hand, in the 2-category case, the vanishing of this group is automatic, since it is a higher Ext group in the category of Abelian groups. The 2-category $\mathcal{C}$ is therefore equivalent to the trivial one with invariant groups B and A. In particular, forgetting all the symmetry stucture, this implies that the underlying monoidal 2-category $\mathcal{C}$ is equivalent to the trivial one, and the original four-cocycle $f(x, y, z, w)$ which defined $\mathcal{C}$ is then cohomologous to zero.

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