THE SEMIGROUP OF ONE-TO-ONE TRANSFORMATIONS WITH FINITE DEFECTS

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Let $\mathscr{S}$ be the semigroup of all total one-to-one transformations of an infinite set X. For an $f \in \mathscr{S}$ let the *defect* of f, def f, be the cardinality of $X - R(f)$, where $R(f) = f(X)$ is the range of f. Then $\mathscr{S}$ is a disjoint union of the symmetric group $\mathscr{G}_X$ on X, the semigroup S of all transformations in $\mathscr{S}$ with finite non-zero defects and the semigroup $\bar{S}$ of all transformations in S with infinite defects, such that $S \cup \bar{S}$ and $\bar{S}$ are ideals of $\mathscr{S}$. The properties of $\mathscr{G}_X$ and $\bar{S}$ have been investigated by a number of authors (for the latter it was done via Baer–Levi semigroups, see [**2**], [**3**], [**5**], [**6**], [**7**], [**8**], [**9**], [**10**] and note that $\bar{S}$ decomposes into a union of Baer–Levi semigroups). Our aim here is to study the semigroup S. It is not difficult to see that S is left cancellative (we compose functions f, g in S as $fg(x) = f(g(x))$, for $x \in X$) and idempotent-free. All automorphisms of S are inner [4], that is of the form $f \mapsto hfh^{-1}$, $f \in S$, $h \in \mathscr{G}_X$.

In the present paper, we are concerned with congruences, Green's relations and ideals of S. A large variety of distinct types of congruences on S is present and the main results are the content of Theorems 4, 5 and 6. In the concluding remark we state some unsolved problems and conjectures on congruences on S.

For f, $g \in S$, let $D(f, g) = \{x : f(x) \neq g(x)\}$. The next lemma is easily verified.

LEMMA 1. *Let f, g, t be one-to-one transformations. Then*
(i) $\operatorname{def}(fg) = \operatorname{def} f + \operatorname{def} g$;
(ii) $D(tf, tg) = D(f, g)$;
(iii) $t(D(ft, gt)) = D(f, g) \cap R(t)$.

Let $\mathscr{L}$, $\mathscr{R}$, $\mathscr{H}$, $\mathscr{D}$, $\mathscr{J}$ be the Green's relations on S [**1**, p. 47–49] and ι be the diagonal congruence. For an $f \in S$ let $R_f[L_f, J_f]$ denote the principal right [left, two-sided] ideal of S generated by f. Denote by $\mathbb{N}$ the set of all natural numbers. Given $n \in \mathbb{N}$ let

$$C_n = \{f \in S : \operatorname{def} f = n\},\ I_n = \bigcup \{C_k : k \geq n\}.$$

It follows from Lemma 1 that for every $n \in \mathbb{N}$, I_n is an ideal of S.

PROPOSITION 2.(i) *$\mathscr{L} = \mathscr{R} = \mathscr{H} = \mathscr{D} = \mathscr{J} = \iota$ on S.*
For every $f \in S$,
(ii) *$L_f = J_f = \{f\} \cup I_n$, where $n = 1 + \operatorname{def} f$;*
(iii) *$R_f = f \cup T(A)$, where $A = R(f)$ and $T(A) = \{g \in S : R(g) \subsetneq A\}$.*
(iv) *A subset I of S is an ideal of S if and only if $I = B \cup I_n$ for some $n \geq 2$ and $B \subseteq C_{n-1}$.*

Proof. (i)–(iii) can be easily verified using Lemma 1, while (iv) follows from (ii) and an observation that every ideal is a union of principal ideals.

We remark that not every ideal of S is principal, for example, if B is a proper subset

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of C_1 consisting of more than one element, then the ideal $B \cup I_2$ is a non-principal ideal of S.

For an ideal I of a semigroup S the *Rees ideal congruence* I^* is such that $(f, g) \in I^*$ iff either $f = g$ or $f, g \in I$, where $f, g \in S$. As usual, we write S/I for S/I^*. Observe that Proposition 2 (iv) describes all Rees ideal congruences on S.

Another type of congruences on S is defined as follows. Let α be an infinite cardinal that does not exceed $|X|^+$, the cardinal successor of $|X|$. Then a relation Δ_α on S for which $(f, g) \in \Delta_\alpha$ iff $|D(f, g)| < \alpha$ is a congruence on S.

Let δ be a relation on S such that for $f, g \in S$, $(f, g) \in \delta$ iff $\operatorname{def} f = \operatorname{def} g$. Lemma 1 ensures that δ is a congruence on S.

PROPOSITION 3. (i) *For every infinite* $\alpha \leq |X|^+$, Δ_α *is a cancellative congruence on S.*
(ii) $\Delta_{\aleph_0} \subseteq \delta$.

Proof. While (i) follows from Lemma 1, to prove (ii) let $(f, g) \in \Delta_{\aleph_0}$, $D(f, g) = D$, $f(D) = A$, $g(D) = B$, and $(X - R(f)) \cap (X - R(g)) = C$. Then $X - R(f) = [(X - R(f)) \cap R(g)] \,\dot\cup\, [(X - R(f)) \cap (X - R(g))] = [(X - R(f)) \cap B] \,\dot\cup\, C = (B - A) \,\dot\cup\, C$. Similarly, $X - R(g) = (A - B) \,\dot\cup\, C$, and the result follows from the fact that $|A| = |D| = |B| < \aleph_0$.

Now we are in a position to present our main result on congruences on S. These are given in Theorems 4, 5 and 6 below. Theorem 4 describes all the congruences $\lambda \subseteq \Delta_{\aleph_0}$. It is shown that for every such λ there exists $n \in \mathbb{N}$ such that λ coincides with $\Delta_{\aleph_0}$ on I_n. Our description of $\lambda \subseteq \Delta_{\aleph_0}$ is given in terms of equivalence series defined below.

Let ρ_k be an equivalence on C_k, $k \geq 1$. We say that an equivalence ρ_l on C_l, $l > k$, is *derived* from ρ_k if whenever $(f, g) \in \rho_k$, $t \in S$ with $\operatorname{def} t = l - k$ then (ft, gt), $(tf, tg) \in \rho_l$. Every congruence λ on S induces in a natural way an equivalence $\tilde{\lambda}$ on C_k, $k \geq 1$. Given an equivalence ρ_k on C_k such that $\rho_k \subseteq \tilde{\Delta}_{\aleph_0}$ let Σ_k denote a set of equivalences ρ_l on C_l, $l \geq k$, such that for every $l > k$, $\rho_l \subsetneq \tilde{\Delta}_{\aleph_0}$ and ρ_l is derived from ρ_{l-1}. We refer to such Σ_k as an *equivalence series derived from* ρ_k. We show (Lemma 10) that every such series is finite and hence we can define a *maximal* equivalence series Σ_k derived from ρ_k such that if $m = \max\{l : \rho_l \in \Sigma_k\}$ then every equivalence derived from ρ_m coincides with $\Delta_{\aleph_0}$.

THEOREM 4. *Let* $n \in \mathbb{N}$, $\rho_n \subseteq \tilde{\Delta}_{\aleph_0}$ *be a non-trivial equivalence on* C_n *and* Σ_n *be an equivalence series derived from* ρ_n *with* $m = \max\{l : \rho_l \in \Sigma_n\}$. *Then*

$$\rho = \iota \cup \{\rho_k : \rho_k \in \Sigma_n\} \cup (\Delta_{\aleph_0} \cap (I_{m+1} \times I_{m+1})) \tag{1}$$

is a congruence on S contained in $\Delta_{\aleph_0}$.

Conversely, if $\lambda \subseteq \Delta_{\aleph_0}$ *is a non-diagonal congruence on S then there exists a non-trivial equivalence* $\rho_n \subseteq \tilde{\Delta}_{\aleph_0}$ *on* C_n *and a maximal equivalence series* Σ_n *derived from* ρ_n *such that* $\lambda = \rho$ *as defined above.*

In the following we describe congruences Δ_α, $\aleph_0 \leq \alpha \leq |X|^+$. We observe that every Δ_α can be extended in a natural way to a congruence Δ'_α on $\mathcal{G}_X \cup S$, (so that for $f, g \in \mathcal{G}_X \cup S$, $(f, g) \in \Delta_\alpha$ if $|D(f, g)| < \alpha$). Note that $\Delta_{|X|^+} = S \times S$, the universal congruence on S. Let Λ be the lattice of congruences on S, and if $\rho, \sigma \in \Lambda$, write

$$[\rho, \sigma] = \{\gamma \in \Lambda : \rho \subseteq \gamma \subseteq \sigma\}, \ [\rho) = \{\gamma \in \Lambda : \rho \subseteq \gamma\}, \ (\rho) = \{\gamma \in \Lambda : \rho \subsetneq \gamma\}.$$

THEOREM 5. (i) $\Delta_{\aleph_0}$ *is a minimal cancellative idempotent-free congruence on S.*
For every $\aleph_0 < \alpha \le |X|^+$ *the following holds*:
(ii) Δ_α *is a group congruence on S and* $(\mathcal{G}_X \cup S)/\Delta'_\alpha \cong S/\Delta_\alpha \cong \mathcal{G}_X/\Delta'_\alpha$;
(iii) $[\Delta_\alpha) = \{\Delta_\beta : \beta > \alpha\}$.

For $m, n \in \mathbb{N}$, let $\sigma(m, n)$ be a congruence on S such that $(f, g) \in \sigma(m, n)$ if either $f = g$ or $\operatorname{def} f$, $\operatorname{def} g \ge m$ and $\operatorname{def} f \equiv \operatorname{def} g \bmod n$. Evidently $\delta \subseteq \sigma(m, n)$. Moreover, if a congruence γ contains δ, then S/γ is isomorphic to a homomorphic image of $S/\delta \cong (\mathbb{N}, +)$. Hence S/γ is isomorphic to a monogenic subsemigroup of type (m, n), for some $m, n \in \mathbb{N}$. It follows that $\gamma = \sigma(m, n)$. This proves the first part of Theorem 6 below. Now let $\mathbb{N}_|$ be the lattice of natural numbers ordered by divisibility, $\mathbb{N}_|^*$ be the dual lattice. Let ω be the first infinite ordinal, ω^* be the dual order type. For a lattice A let A^0 denote the lattice obtained from A by adjoining a (new) least element 0.

THEOREM 6. (i) $[\delta) = \{\sigma(m, n) : m, n \in \mathbb{N}\}$.
(ii) $[\delta) \cong (\omega^* \times \mathbb{N}_|^*)^0 (\cong ((\omega \times \mathbb{N}_|)^*)^0)$.
(iii) *For every* $n > 1$, $\sigma(1, n)$ *is a group congruence on S and* $S/\sigma(1, n) \cong Z_n$, *a cyclic group of order n.*
(iv) *There is no least group congruence on S.*
For every $\alpha > \aleph_0$,
(v) $S/\delta \cap \Delta_\alpha \cong (\mathcal{G}_X/\Delta'_\alpha) \times (\mathbb{N}, +)$;
(vi) $S/\sigma(m, n) \cap \Delta_\alpha \cong (\mathcal{G}_X/\Delta'_\alpha) \times (\mathbb{N}, +)/\eta(m, n)$,
where $\eta(m, n)$ *is a congruence on the semigroup* $(\mathbb{N}, +)$ *such that* $(k, l) \in \eta(m, n)$ *if either* $k = l$ *or* $k, l \ge m$ *and* $k \equiv l \bmod n$;
(vii) *for* $m, n \in \mathbb{N}$, $\mathcal{A} \subseteq C_m/\Delta_\alpha$, $\mathcal{A} \ne \emptyset$, $A = \bigcup \mathcal{A}$, *and* $\bar{A} = \left(\bigcup_{i=1}^{m} C_i\right) - A$

$$\theta(\alpha, m, n, \mathcal{A}) = (\delta \cap \Delta_\alpha \cap (\bar{A} \times \bar{A})) \cup (\sigma(m, n) \cap \Delta_\alpha \cap ((I_{m+1} \cup A) \times (I_{m+1} \cup A))$$

is a congruence on S.
(viii) *every congruence* $\gamma \in (\delta \cap \Delta_\alpha, \Delta_\alpha)$, $\gamma \nsubseteq \delta$, *has the form* $\theta(\alpha, m, n, \mathcal{A})$ *as described in* (vii).

Proofs of Theorems 4, 5 and 6 constitute the remainder of the paper.

LEMMA 7. *Given* $f, g, t, s \in S$ *such that* $D(f, g) = D(t, s) = \{x\}$, *where* $x \in X$, *and* $\operatorname{def} f = \operatorname{def} g > \operatorname{def} s = \operatorname{def} t$, *there exists* $l \in S$ *with* $ls = f$ *and* $lt = g$.

Proof. Observe that $g(x) \notin R(f)$, $t(x) \notin R(s)$ and choose a 1–1 function

$$l_1 : X - R(s) - \{t(x)\} \to X - R(f) - \{g(x)\},$$

(note that since $\operatorname{def} s < \operatorname{def} f$, l_1 has a finite non-zero defect in $X - R(f) - \{g(x)\}$). Define a function $l : X \to X$ as follows. For a $y \in X$, let

$$l(y) = \begin{cases} f(z) \text{ if } y = s(z) & \text{for some } z \in X, \\ g(x) \text{ if } y = t(x), \\ l_1(y) \text{ if } y \in X - R(s) - \{t(x)\}. \end{cases}$$

Clearly, $l \in S$ and $ls = f$, also $lt(x) = g(x)$ and for $u \ne x$, $lt(u) = ls(u) = f(u)$, so that $lt = f$.

LEMMA 8. *Let* $(f, g) \in \Delta_{\chi_0}$, $f \neq g$. *Then there exist* $f_1, \ldots, f_n \in S$ *such that* $|D(f_i, f_{i+1})| = 1$, *for* $i = 1, \ldots, n-1$, *and* $f_1 = f$, $f_n = g$.

Proof. Let $|D(f, g)| = m$. We prove the result by induction on m. Assume that the statement is true for every pair of functions in S that differ on at most $m-1$ points. Let $D(f, g) = D$, $f(D) = A$ and $g(D) = B$. Consider two cases:

(i) $A \neq B$.

Choose $b \in B - A$, and let $g(d) = b$, where $d \in D$. Let $f_1 = f$ and define f_2 as follows: $f_2|_{X-\{d\}} = f_1|_{X-\{d\}}$ and $f_2(d) = b$. Then $D(f_1, f_2) = \{d\}$ and $D(f_2, g) = D - \{d\}$. By induction supposition there are $f_3, \ldots, f_n \in S$ such that $f_n = g$ and $|D(f_i, f_{i+1})| = 1$ for every $i = 2, 3, \ldots, n-1$. Thus $f_1, f_2, f_3, \ldots, f_n$ is the required sequence.

(ii) $A = B$.

Using an observation that $A = B$ implies that $X - (R(f) \cup R(g)) \neq \varnothing$, choose $x \in X - (R(f) \cup R(g))$. Choose $d \in D$, let $f_1 = f$ and define f_2 so that $f_2|_{X-\{d\}} = f_1|_{X-\{d\}}$ and $f_2(d) = x$. Let g_1 be such that $g_1|_{X-\{d\}} = g|_{X-\{d\}}$ and $g_1(d) = x$.

Then $D(f_1, f_2) = D(g_1, g) = \{d\}$ and $D(f_2, g_1) = D - \{d\}$, so there exist $f_3, \ldots, f_n \in S$ with $f_n = g_1$ and $|D(f_i, f_{i+1})| = 1$ for every $i = 2, \ldots, n-1$. The result follows.

LEMMA 9. *Let* λ *be a congruence on* S, $\lambda \subseteq \Delta_{\aleph_0}$. *Let*

$N = \{\operatorname{def} f : f \in S, |[f]| \neq 1, [f] \in S/\lambda\}$, *and for every* $n \in N$ *let*
$L(n) = \{|D(f, g)| : f, g \in S, f \neq g, (f, g) \in \lambda, \operatorname{def} f = n\}$,

$\Psi_n : L(n) \to \mathbb{N}$ *be defined by* $\Psi_n(l) = n + l + 1$, $l \in L(n)$. *Let* m_n *be a minimum value of* Ψ_n *and*

$$m = \min\{m_n : n \in N\}.$$

Then $\lambda \cap (I_m \times I_m) = \Delta_{\aleph_0} \cap (I_m \times I_m)$.

Proof. Let $m = m_n = n + l + 1$, for some $n \in N$ and $l \in L(n)$. Let $f, g \in S$ such that $(f, g) \in \lambda$, $f \neq g$, $\operatorname{def} f = \operatorname{def} g = n$ and $|D(f, g)| = l$. Let $(t, s) \in \Delta_{\aleph_0}$ with $\operatorname{def} t = \operatorname{def} s = k \geq m$. In view of Lemma 8 we can assume that $|D(t, s)| = 1$. Let $D(t, s) = \{x\}$, $D(f, g) = D$. Choose $d \in D$, $a \in X - D$ and $q \in S$ with $R(q) = (X - D) \cup \{d\} - \{a\}$ and $q(x) = d$. Then $D(fq, gq) = \{x\}$, $\operatorname{def} fq = \operatorname{def} gq = n + |D| = m - 1 < k$, and so Lemma 7 implies there exists $l \in S$ such that $lfq = t$, $lgq = s$, so that $(t, s) \in \lambda$.

LEMMA 10. *Every equivalence series* Σ_k *derived from* $\rho_k \subseteq \bar{\Delta}_{\aleph_0}$ *is finite.*

Proof. The result follows from the proof of Lemma 9 and an observation that the relation *derived from* is transitive, that is if $l_i \in \mathbb{N}$, $i = 1, 2, 3$ and ρ_{l_i} is an equivalence on C_{l_i} such that $\rho_{l_{i+1}}$ is derived from ρ_{l_i}, $i = 1, 2$, then ρ_{l_3} is derived from ρ_{l_1}.

Proof of Theorem 4. That ρ defined in (1) is a congruence on S follows from the definition of a congruence series derived from ρ_n and Lemma 10.

To show the converse let $\lambda < \Delta_{\aleph_0}$ be a non-diagonal congruence on S and let n be the minimal integer such that $\lambda \cap (C_n \times C_n) \neq i \cap (C_n \times C_n)$, m be the maximal integer with

$\lambda \cap (C_m \times C_m) \neq \Delta_{\aleph_0} \cap (C_m \times C_m)$ if $\lambda \cap (C_n \times C_n) \neq \Delta_{\aleph_0} \cap (C_n \times C_n)$ and $m = n$ otherwise. Then for every k, $n \leq k \leq m$, λ induces in a natural way equivalences ρ_k on C_k. The set of all these equivalences forms a maximal congruence series Σ_n derived from ρ_n and the result follows.

Corollary 11. (i) *Every right [left] cancellative congruence η on S contains $\Delta_{\aleph_0}$.*
(ii) *Every group congruence on S contains $\Delta_{\aleph_0}$.*

Proof. (i) Assume $\eta \supsetneqq \Delta_{\aleph_0}$ is right cancellative, and let $(f, g) \in \Delta_{\aleph_0}$. Let m be such that $(\Delta_{\aleph_0} \cap \rho) \cap (I_m \times I_m) = \Delta_{\aleph_0} \cap (I_m \times I_m)$ (Lemma 9) and $t \in C_m$. Then $(ft, gt) \in \Delta_{\aleph_0} \cap \eta$, so that $(f, g) \in \eta$ since η is right cancellative.
(ii) Follows from (i).
The next lemma is self-evident.

Lemma 12. *For any $f \in \mathcal{G}_X \cup S$, $n \geq 0$ there exists $g \in \mathcal{G}_X \cup S$ with* def $g \dot{=} n$ *and $(f, g) \in \Delta'_{\aleph_0}$, where $\aleph_1 = \aleph_0^+$.*

Proof of Theorem 5. (i) It is not difficult to verify that $\Delta_{\aleph_0}$ is a cancellative congruence. Hence, in view of Corollary 11 (i) it is sufficient to show that $\Delta_{\aleph_0}$ is idempotent-free. This follows from an observation that for any $f \in S$, $D(f, f^2) = D(f\iota_X, f^2) = D(\iota_X, f)$ (where ι_X is the identity mapping on X), so that $|D(f, f^2)| \geq \aleph_0$ and $(f, f^2) \notin \Delta_{\aleph_0}$.

(ii) It is sufficient to show the existence of the indicated isomorphisms. For that let $f \in S$, $h \in \mathcal{G}_X$, $[f]$ and $[h]$ be Δ'_α-classes of f in S and h in $\mathcal{G}_X$ respectively. We show that $[f] \cup [h]$ is the Δ'_α-class of f (or, equivalently, h) in $\mathcal{G}_X \cup S$ if any only if $(f, h) \in \Delta'_\alpha$. While the necessity is clear the sufficiently follows from the observation that if $(f, h) \in \Delta'_\alpha$ and A is the Δ'_α class of f in $\mathcal{G}_X \cup S$ then $A \cap \mathcal{G}_X = [h]$ and $A \cap S = [f]$.

(iii) Let γ be a congruence on S containing Δ_α. Since $S/\Delta_\alpha \cong \mathcal{G}_X/\Delta_\alpha$ there exists a homomorphism from $\mathcal{G}_X/\Delta'_\alpha$ onto S/γ, so that there is a congruence γ' on $\mathcal{G}_X$ such that $\mathcal{G}_X/\gamma' \cong S/\gamma$. But then $\gamma' = \Delta_\beta$ for some $\beta \geq \alpha$ and so

$$S/\gamma \cong \mathcal{G}_X/\Delta'_\beta \cong S/\Delta_\beta, \quad \text{and} \quad \gamma = \Delta_\beta.$$

The next result describes some properties of the congruence δ. Recall that $(f, g) \in \delta$ if $\operatorname{def} f = \operatorname{def} g$.

Proposition 13. (i) $\delta \vee \Delta_{\aleph_1} = S \times S$.
(ii) *δ is the unique congruence on S such that $S/\delta \cong (\mathbb{N}, +)$, where $(\mathbb{N}, +)$ denotes the semigroup of positive integers under addition.*

Proof. (i) Let f, $g \in S$. According to Lemma 12, there exists $k \in S$ such that $(f, k) \in \delta$, $(k, g)\Delta_{\aleph_1}$, so that $(f, g) \in \Delta_{\aleph_1} \circ \delta \subseteq \delta \vee \Delta_{\aleph_1}$.

(ii) We show that if θ is a homomorphism from S onto $(\mathbb{N}, +)$ then $\theta^{-1} \circ \theta = \delta$. Firstly observe that if $f \in C_m$ and $g \in C_n$ with $m < n$, then $\theta(f) < \theta(g)$, for there exists $q \in S$ such that $qf = g$ and $\theta(q) + \theta(f) = \theta(g)$. Since θ is onto, $\theta(C_1) = 1$ and so $\theta(C_m) = m$, for every $m \in \mathbb{N}$ (indeed, if $f \in C_m$ then there exist $f_1, f_2, \ldots, f_m \in C_1$ such that $f = f_1 f_2 \ldots f_m$, so that $\theta(f) = \theta(f_1) + \theta(f_2) + \ldots + \theta(f_m) = m$).

Proof of Theorem 6. Let $\sigma(m, n) \to (m, n)$ be a mapping of (δ) onto $\mathbb{N} \times \mathbb{N}$. The result then follows from an observation that $\sigma(m_1, n_1) \subseteq (m_2, n_2)$ if and only if $m_2 \leq m_1$ and n_2 divides n_1.

(iii) Follows from the fact that a monogenic semigroup of type (m, n) is a group if $m = 0$ and Proposition 13.

(iv) Observe firstly that $\bigcap \{\sigma(1, n) : n \in \mathbb{N}\} = \delta$, indeed, if $(f, g) \in \bigcap \{\sigma(1, n) : n \in \mathbb{N}\}$, then $\operatorname{def} f \equiv \operatorname{def} g \bmod n$ for every $n \in \mathbb{N}$, so that $\operatorname{def} f = \operatorname{def} g$, or $(f, g) \in \delta$. If there is the least group congruence τ on S, then $\tau \subseteq \sigma(1, n)$ for every n, so that $\tau \subseteq \bigcap \{\sigma(1, n) : n \in \mathbb{N}\} = \delta$. It follows that δ is a group congruence (for $S/\delta \cong (S/\tau)/(\delta/\tau)$, which is a homomorphic image of the group S/τ, a contradiction to Proposition 13).

The verification of (v)–(vii) is trivial.

(viii) Let $\gamma \in (\delta \cap \Delta_\alpha, \Delta_\alpha)$, $\gamma \nsubseteq \delta$, m be the minimal integer for which there exists $f \in S$ with $\operatorname{def} = m$ such that the γ-class of f does not coincide with the $(\delta \cap \Delta_\alpha)$-class of f. Let $\mathcal{B}$ be the set of all classes of γ that contain an element g of C_m and that do not coincide with the g-class of $\delta \cap \Delta_\alpha$ and $\mathcal{A} = \{C_m \cap B : B \in \mathcal{B}\}$. Let $k = \min\{\operatorname{def} g : (f, g) \in \gamma,\ (f, g) \notin \delta\}$ and $n = k - m$. Clearly, γ coincides with $\delta \cap \Delta_\alpha$ on $\bigcup \{C_l : 1 \leq l \leq m - 1\}$. We show firstly that

$$\gamma \cap (I_{m+1} \times I_{m+1}) \supseteq \sigma(m, n) \cap \Delta_\alpha \cap (I_{m+1} \times I_{m+1}). \tag{2}$$

For $f \in S$ let $S(f) = \{x \in X : f(x) \neq x\}$ and shift $f = |S(f)|$.

Let $(f, g) \in \gamma$ with $\operatorname{def} f = m$, $\operatorname{def} g = m + n$. We observe that for every $t \in C_1$ with shift $t = \aleph_0$, $(t^n f, g) \in \delta \cap \Delta_\alpha \subseteq \gamma$, so that $(f, t^n f) \in \gamma$. Moreover, for any integer $l > 0$,

$$(f, t^{ln} f) \in \gamma, \tag{3}$$

since $(f, t^n f) \in \gamma$ implies $(t^n f, t^{2n} f) \in \gamma$, so that $(f, t^{2n} f) \in \gamma$ etc.

Take $(p, q) \in \sigma(m, n) \cap \Delta_\alpha \cap (I_{m+1} \times I_{m+1})$, and let f be as above. Since $\operatorname{def} p > m = \operatorname{def} f$ there exists an s in S such that $p = sf$. Without loss of generality assume that $\operatorname{def} q$-$\operatorname{def} p = an$, $a \in \mathbb{N}$, $a > 0$. Now (3) implies that $(sf, st^{an} f) \in \gamma$, or $(p, st^{an} f) \in \gamma$. Also, $\operatorname{def}(st^{an} f) = \operatorname{def} q$ and $D(st^{an} f, q) \subseteq D(p, q) \cup f^{-1}(S(t^{an}))$, so that $|D(st^{an} f, q)| < \alpha$ and $(st^{an} f, q) \in \delta \cap \Delta_\alpha \leq \gamma$. We conclude that $(p, q) \in \gamma$.

To complete the proof it suffices to show that if $(u, v) \in \gamma \cap (I_{m+1} \times I_{m+1})$ then $(u, v) \in \sigma(m, n)$. Assume $m < \operatorname{def} u < \operatorname{def} v$ and let $(\operatorname{def} v)$—$(\operatorname{def} u) = bn + r$, where $b, r \in \mathbb{N}$, $0 \leq r < n$. Let f and t be chosen as above. We show that if $r > 0$ then $(f, t^r f) \in \gamma$, a contradiction to the choice of n that assures that $r = 0$ and so $(u, v) \in \sigma(m, n)$. Observe that there exists $w \in S$ and $l \in \mathbb{N}$, $l > 0$ such that $t^{ln} f = wu$. But then $(wu, wv) \in \gamma$ implies that $(t^{ln} f, wv) \in \gamma$, so that using (3) we conclude that $(f, wv) \in \gamma$. Note that $\operatorname{def} wv = (l + b)n + m + r$ and since $\gamma \supseteq \delta \cap \Delta_\alpha$ we have $(f, t^{n(l+b)+r} f) \in \gamma$. By (3), we have that $(t^r f, t^{n(l+b)+r} f) \in \gamma$, so that $(f, t^r f) \in \gamma$.

REMARK. In this paper we described certain large classes of congruences and parts of the lattice of congruences on S. A description of *all* congruences and the lattice of congruences of S is as yet an open problem. In particular, we conjecture that for every infinite cardinal $\alpha \leq |X|$, every congruence in the interval $(\Delta_\alpha \cap \delta, \Delta_{\alpha^+} \cap \delta)$ can be described in terms of a finite equivalence series derived from a given equivalence λ on C_k for some $k \geq 1$ such that $\tilde{\Delta}_\alpha \leq \lambda \leq \tilde{\Delta}_{\alpha^+}$ as was done in Theorem 4 for congruences $\rho \subseteq \Delta_{\aleph_0}$.

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