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FREE ALGEBRAS IN THE VARIETY OF THREE-VALUED CLOSURE ALGEBRAS

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Abstract

In this paper, the variety of three-valued closure algebras, that is, closure algebras with the property that the open elements form a three-valued Heyting algebra, is investigated. Particularly, the structure of the finitely generated free objects in this variety is determined.

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1. Introduction and preliminaries

In a paper of paramount importance titled 'The algebra of topology', McKinsey and Tarski [12] started the investigation of a class of algebraic structures which they named closure algebras. A *closure algebra* is an algebra $(A; \vee, \wedge, -, \nabla, 0, 1)$ such that $(A; \vee, \wedge, -, 0, 1)$ is a Boolean algebra and ∇ is an *additive closure operator*, that is, ∇ is a unary operator on A that satisfies the 'Kuratowski axioms', for all $x, y \in A$:

$$\nabla(0) = 0, \tag{1}$$

$$x \leq \nabla(x), \tag{2}$$

$$\nabla(\nabla(x)) = \nabla(x), \tag{3}$$

$$\nabla(x \vee y) = \nabla(x) \vee \nabla(y). \tag{4}$$

Closure algebras have been extensively studied by several authors. Particularly, Blok in an exhaustive and very deep work, developed in [4] the general properties of the lattice of subvarieties of the variety of closure algebras.

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An important feature in the structure of a closure algebra is the set of open elements. In a continuation of their work on closure algebras, McKinsey and Tarski showed in [13] and [14] that the set of open elements of a closure algebra is a Heyting algebra. Conversely, any Heyting algebra can be embedded as the lattice of open elements of a closure algebra.

The present paper is devoted to a deeper investigation of a subvariety of the variety of closure algebras, namely, the variety of three-valued closure algebras $\mathscr{C}_T$ (see [7]). This is the subvariety of those closure algebras such that the set of open elements form a three-valued Heyting algebra.

The variety of monadic algebras is the largest variety of closure algebras whose associated variety of Heyting algebras consists of Boolean algebras; the variety of 'three-valued closure algebras' studied in this paper is the variety of closure algebras whose associated Heyting algebras of open elements belong to the variety generated by the three-element chain. This last variety is the unique cover in the lattice of varieties of Heyting algebras of the variety of Boolean Heyting algebras.

When investigating the structure of algebras in a given variety it is of particular interest to find out what the finitely generated members are. In [4], Blok devotes a large part of his work to obtain the closure algebra with one free generator, which shows the difficulty of the problem (see also [5] and [6]). The main result in Section 3 is the determination of the free finitely generated objects in the variety $\mathscr{C}_T$. To this end, a study of the variety $\mathscr{C}_T$ is carried out, paying particular attention to the determination of simple and subdirectly irreducible algebras, as well as the characterization of maximal subalgebras of subdirectly irreducible algebras. We also study the finitely generated subdirectly irreducible algebras (Section 2).

Throughout this paper, $\mathscr{D}_{01}$, $\mathscr{B}$, $\mathscr{H}$ and $\mathscr{M}$ will denote the equational classes of all distributive lattices with 0 and 1, all Boolean algebras, all Heyting algebras and all monadic Boolean algebras, respectively. If $\mathscr{K}$ is a class of similar algebras, the lattice of congruences of an algebra $A \in \mathscr{K}$ is denoted by $\mathbf{Con}(A)$. In general, for a variety $\mathscr{V}$ and $A, B \in \mathscr{V}$, $A \lhd_{\mathscr{V}} B$ means that A is a $\mathscr{V}$-subalgebra of B. The subalgebra generated by a part X of $A \in \mathscr{V}$ is denoted by $[X]_{\mathscr{V}}$. Finally, the free algebra over a finite poset G in $\mathscr{V}$ is denoted $\mathbf{F}_{\mathscr{V}}(G)$.

With the operators ∇ and $-$ we can define a new unary operator Q (*interior operator*) by means of $Q(x) = -\nabla(-x)$, for all $x \in A$. This operator satisfies the following conditions:

$$Q(1) = 1, \tag{5}$$

$$x \geq Q(x), \tag{6}$$

$$Q(Q(x)) = Q(x), \tag{7}$$

$$Q(x \wedge y) = Q(x) \wedge Q(y). \tag{8}$$

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In addition, it is readily verifiable that the following properties hold:

$$Q(0) = 0, \tag{9}$$

$$Q(Q(x) \vee Q(y)) = Q(x) \vee Q(y), \tag{10}$$

$$\text{if } x \leq y \text{ then } Q(x) \leq Q(y). \tag{11}$$

Closure algebras can be defined by means of the equations (5) to (8) and in that case, by defining $\nabla(x) = -Q(-x)$ we obtain the closure operator satisfying equations (1) to (4).

The equational class of closure algebras will be denoted by $\mathscr{C}$. These algebras were named interior algebras by Blok in [4]. Other authors called them topological Boolean algebras, but they were named Lewis algebras by Monteiro after the founder of the S4 logic [11]. We will use the most traditional name of closure algebras, though we will make use of the interior operator Q.

It is known that $\mathscr{C}$ and $\mathscr{H}$ are generated by their finite members, but neither of these two varieties is locally finite ([1, 12, 13]).

If $A \in \mathscr{C}$, then $Q(A)$ is a (0, 1)-sublattice of A, and it is a Heyting algebra if we define $a \to b = Q(-a \vee b)$, for every $a, b \in A$. If $b \in Q(A)$, b is said to be open.

Conversely, if $A \in \mathscr{B}$ and L is a (0, 1)-sublattice of A, then there exists a unique interior operator Q on A satisfying properties (5) to (8) and such that $L = Q(A)$ if and only if for every $a \in A$, the set $(a] \cap L$ has a greatest element. In this case, $Q(a) = \text{Max}((a] \cap L) = \bigvee\{x \in L : x \leq a\}$.

It is known that if $L \in \mathscr{H}$, then the lattice $\mathbf{Con}(L)$ of congruences of L is isomorphic to the lattice $\mathfrak{F}(L)$ of all filters of L. If $F \in \mathfrak{F}(L)$, then the congruence θ associated with F is defined by $(a, b) \in \theta \Leftrightarrow a \wedge u = b \wedge u$ for some $u \in F$.

If $A \in \mathscr{C}$ and F is a filter in A, F is said to be an *open filter* if $Q(x) \in F$ whenever $x \in F$. It is known ([13]) that $\mathbf{Con}(A)$ is isomorphic to the lattice $\mathfrak{F}(A)$ of all open filters of A, and it is not difficult to see that $\mathfrak{F}(A)$ and $\mathfrak{F}(Q(A))$ are isomorphic. So we have:

THEOREM 1.1 ([6]). *Let $A \in \mathscr{C}$. Then* $\mathbf{Con}(A)$ *and* $\mathbf{Con}(Q(A))$ *are isomorphic.*

Recall (see [1]) that a Heyting algebra L is subdirectly irreducible if and only if $L = L_1 \oplus 1$, with $L_1 \in \mathscr{H}$ and $L_1 \oplus 1$ is the lattice obtained by adjoining a new 1 to L_1.

The following corollary follows immediately from Theorem 1.1 and the above remark.

COROLLARY 1.2. *Let $A \in \mathscr{C}$. Then A is subdirectly irreducible if and only if $Q(A)$ is subdirectly irreducible as a Heyting algebra, and hence, A is subdirectly irreducible if and only if $Q(A) = L_1 \oplus 1$, for some $L_1 \in \mathscr{H}$.*

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2. Three-valued closure algebras

In this section we investigate subdirectly irreducible algebras and maximal subalgebras of subdirectly irreducible algebras in the variety $\mathscr{C}_T$. Recall that a closure algebra A is said to be three-valued if $Q(A)$ is a three-valued Heyting algebra, and a three-valued Heyting algebra is a Heyting algebra $(A, \wedge, \vee, \rightarrow, 0, 1)$ such that $b = (\neg a \rightarrow b) \wedge ((b \rightarrow a) \rightarrow b)$, for every $a, b \in A$, where $\neg a = a \rightarrow 0$ [16].

THEOREM 2.1 ([1, 2, 9]). *The variety of three-valued Heyting algebras is generated by the three-element chain.*

The following theorem gives us an equational characterization of three-valued closure algebras.

THEOREM 2.2. *Let $A \in \mathscr{C}$. Then, $A \in \mathscr{C}_T$ if and only if for every $b, a \in A$, the following identity holds:*

$$Q(b) = (Q(-Q(a)) \rightarrow Q(b)) \wedge ((Q(b) \rightarrow Q(a)) \rightarrow Q(b)). \tag{12}$$

By a simple application of Jónsson's Lemma (see [10]) we see from Theorem 2.1 that the only subdirectly irreducibles in the variety of three-valued Heyting algebras are, up to isomorphism, **2**, the 2-element chain, and **3**, the 3-element chain (see also [16]). Then we can conclude:

THEOREM 2.3. *Let $A \in \mathscr{C}_T$. A is subdirectly irreducible if and only if either $Q(A) = \{0, 1\}$ or $Q(A) = \{0, a, 1\}$.*

Observe that the simple algebras in $\mathscr{C}$ are the simple monadic Boolean algebras [17].

As a consequence of Theorem 2.1 we have the following characterization of the ordered set of prime filters of an algebra in the variety of three-valued Heyting algebras.

THEOREM 2.4 ([16]). *Let A be a Heyting algebra. Then the following are equivalent*:

(a) *A is a three-valued Heyting algebra.*

(b) *Every prime filter of A is either maximal or minimal, and every prime filter is contained in at most one maximal prime filter.*

A variety $\mathscr{V}$ has the *Fraser-Horn Property* if there are no skew congruences on any direct product of a finite number of algebras in $\mathscr{V}$, that is, for all $A_1, A_2 \in \mathscr{V}$, every $\theta \in \mathbf{Con}(A_1 \times A_2)$ is a product congruence $\theta_1 \times \theta_2$, $\theta_i \in \mathbf{Con}(A_i)$, $i = 1, 2$. Every congruence-distributive variety has the Fraser-Horn Property. In particular, the variety $\mathscr{C}_T$ has the Fraser-Horn Property.

If the congruence lattice of an algebra A has a unique coatom, then A is directly indecomposable. A variety $\mathscr{V}$ has the *Apple Property* if the converse holds as well for all finite algebras; that is, if the finite directly indecomposables in $\mathscr{V}$ are precisely the finite algebras whose congruence lattices have a unique coatom. If A is a finite directly indecomposable algebra in $\mathscr{C}_T$, then, from Theorem 1.1, $Q(A)$ is directly indecomposable as a three-valued Heyting algebra. So, from Theorem 2.4, $Q(A)$ is of the form $0 \oplus B$ where B is a finite Boolean algebra. Then $\mathfrak{F}(Q(A))$ has a unique coatom and so $\mathbf{Con}(Q(A))$, and consequently $\mathbf{Con}(A)$, has a unique coatom. Hence the variety $\mathscr{C}_T$ has the Apple Property.

The Fraser-Horn and Apple Properties, extensively studied in [3], will play an important role in the determination of the free algebra over a finite poset in the variety $\mathscr{C}_T$.

2.1. Maximal subalgebras In this subsection we determine the maximal subalgebras of the finite subdirectly irreducible algebras.

In the rest of the paper, a will denote the only non-trivial open element of any non-simple subdirectly irreducible algebra. The finite non-simple subdirectly irreducible algebra with $k+l$ atoms, where there are k atoms preceding a and l atoms preceding $-a$, will be denoted by $\mathbf{B}_{k,l}$. Similarly, the simple monadic Boolean algebra with k atoms will be denoted by $\mathbf{B}_k$ (or $\mathbf{B}_{k,0}$). A_a and A_{-a} will be the sets of atoms preceding a and $-a$, respectively. So, the set $\mathrm{At}(\mathbf{B}_{k,l})$ of atoms of $\mathbf{B}_{k,l}$, can be written $\mathrm{At}(\mathbf{B}_{k,l}) = A_a \dot{\cup} A_{-a}$.

Recall that if $A \lhd_{\mathscr{B}} B$, A, B finite, then A is determined by a partition P of the set $\mathrm{At}(B)$ of atoms of B. If $s \leq k$ and $t \leq l$ then, identifying isomorphic algebras, $\mathbf{B}_{s,t} \lhd_{\mathscr{C}} \mathbf{B}_{k,l}$.

Next, we characterize maximal subalgebras of finite non-simple subdirectly irreducible three-valued closure algebras.

Let $M \lhd_{\mathscr{C}} \mathbf{B}_{k,l}$. Consider two cases.

Case 1: $a \in M$.

LEMMA 2.5. *Let $M \lhd_{\mathscr{C}} \mathbf{B}_{k,l}$ and $a \in M$. M is maximal if and only if M is a maximal Boolean subalgebra of* $\mathbf{B}_{k,l}$.

PROOF. If M is not maximal as a Boolean subalgebra of $\mathbf{B}_{k,l}$, then there exists $M' \lhd_{\mathscr{B}} \mathbf{B}_{k,l}$ such that $M \subsetneq M'$ and $M' \neq \mathbf{B}_{k,l}$. Let $x \in M'$; then $Q(x) \in \{0, a, 1\}$. But $\{0, a, 1\} \subseteq M'$, so $Q(x) \in M'$. Hence $M' \lhd_{\mathscr{C}} \mathbf{B}_{k,l}$, a contradiction.

The converse is trivial. □

Case 2: $a \notin M$.

Let P_M be the partition of $\mathrm{At}(\mathbf{B}_{k,l})$ associated to M, $P_M = \dot{\bigcup}_{i=1}^{s} P_M^i$, $(s \leq k+l)$, where P_M^i are the blocks of P_M.

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LEMMA 2.6. $P_M^i \cap A_a \neq \emptyset$.

PROOF. If there would exist a block P_M^i such that $P_M^i \cap A_a = \emptyset$, and x_i is the atom of M associated to the block P_M^i, then $P_M^i \subseteq A_{-a}$ and hence $x_i \leq -a$. So $-x_i \geq a$ and $-x_i \neq 1$. Consequently, $Q(-x_i) = a \in M$, a contradiction. □

Observe that if M is a maximal subalgebra of $\mathbf{B}_{k,l}$, with $a \notin M$, then $[M \cup \{a\}]_{\mathscr{B}} = [M \cup \{a\}]_{\mathscr{C}} = \mathbf{B}_{k,l}$. In addition, $[a]_{\mathscr{C}} = \{0, a, -a, 1\}$ is a $\mathscr{C}$-subalgebra with associated partition $P_a = \{A_a, A_{-a}\}$, and $[M \cup \{a\}]_{\mathscr{C}} = [M \cup [a]_{\mathscr{C}}]_{\mathscr{C}} = [M \cup [a]_{\mathscr{B}}]_{\mathscr{B}} = \mathbf{B}_{k,l}$. Since the partition associated to $[M \cup [a]_{\mathscr{B}}]_{\mathscr{B}}$ is the intersection of P_M and P_a, then for $x \in \mathrm{At}(\mathbf{B}_{k,l})$ we have that either $\{x\} = P_M^i \cap A_a$ or $\{x\} = P_M^i \cap A_{-a}$. From this we conclude that $|P_M^i \cap A_a| \leq 1$ and that $|P_M^i \cap A_{-a}| \leq 1$, thus $|P_M^i| \leq 2$. But if $|P_M^i| = 2$, then by the previous lemma, $|P_M^i \cap A_a| = 1$ and $|P_M^i \cap A_{-a}| = 1$. If $|P_M^i| = 1$, then $|P_M^i \cap A_a| = 1$ and $|P_M^i \cap A_{-a}| = 0$.

Let $M \lhd_{\mathscr{C}} \mathbf{B}_{k,l}$ and $a \notin M$. Then we have:

LEMMA 2.7. *M is maximal if and only if for each block P_M^i the following conditions are satisfied*:

(i) $|P_M^i \cap A_a| = 1$.
(ii) $|P_M^i| = 1$ *or* $|P_M^i| = 2$.
(iii) *If* $|P_M^i| = 2$ *then* $|P_M^i \cap A_{-a}| = 1$.

PROOF. If M is maximal, the conclusion follows from Lemma 2.6 and the previous remark.

For the converse, let M' a subalgebra such that $M \subsetneq M'$. Then P_M properly contains $P_{M'}$. From the hypotheses there exists a block P_M^i such that $P_M^i = P_{M'}^{i1} \cup P_{M'}^{i2}$, with $P_{M'}^{i2} \subseteq A_{-a}$. Let x_{i2} the atom of M' associated to $P_{M'}^{i2}$. Then, $x_{i2} \leq -a$, so $-x_{i2} \geq a$ and $x_{i2} \neq 1$. Therefore $Q(-x_{i2}) = a \in M'$. So $\mathbf{B}_{k,l} = [M \cup \{a\}]_{\mathscr{C}} \subseteq M'$. Hence $\mathbf{B}_{k,l} = M'$, and M is maximal. □

The following theorem gives us the number of maximal subalgebras of a non-simple subdirectly irreducible algebra.

THEOREM 2.8. *Let $k \geq 1$, $l \geq 1$ be arbitrary. Then, in $\mathbf{B}_{k,l}$, there exist $\binom{k}{2}$ maximal subalgebras isomorphic to $\mathbf{B}_{k-1,l}$, $\binom{l}{2}$ maximal subalgebras isomorphic to $\mathbf{B}_{k,l-1}$. If $k \geq l$ there exist $V_{k,l} = k!/(k-l)!$ maximal subalgebras isomorphic to $\mathbf{B}_k$.*

PROOF. Let k, l be arbitrary and suppose that $a \in M$. Then, by Lemma 2.5, the partition P_M has $k + l - 1$ blocks. That is, there exists a unique block P_M^i containing two elements. We claim that $P_M^i \subseteq A_a$ or $P_M^i \subseteq A_{-a}$. To see this, suppose that $X = P_M^i \cap A_a \neq \emptyset$ and $Y = P_M^i \cap A_{-a} \neq \emptyset$. Since $a \in M$, $[M \cup \{a\}]_{\mathscr{C}} = M$.

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X and Y are blocks of the partition associated to $[M \cup \{a\}]_{\mathscr{C}}$. Then X and Y are blocks of P_M, a contradiction. In these conditions it is easy to see that there exist $\binom{k}{2}$ partitions determining a maximal subalgebra isomorphic to $\mathbf{B}_{k-1,l}$, and $\binom{l}{2}$ partitions determining a maximal subalgebra isomorphic to $\mathbf{B}_{k,l-1}$. For $k < l$, observe that from Lemma 2.7, there are no maximal subalgebras M such that $a \notin M$. Suppose that $k \geq l$. From Lemma 2.7, if P_M is the partition associated to a maximal subalgebra M such that $a \notin M$, then each block has either a single element (necessarily in A_a), or two elements, one of them in A_a and the other one in A_{-a}. Then each partition defines a one-to-one mapping g from A_{-a} to A_a in the following way: for $x \in A_{-a}$, $g(x)$ is the element $y \in A_a$ such that y belongs to the same block P_M^i as x. Since it is clear that there are k blocks in P_M, it follows that there are $V_{k,l}$ maximal subalgebras isomorphic to $\mathbf{B}_k$. □

2.2. Finitely generated subdirectly irreducible algebras Now we are going to determine the largest simple algebra and the largest non-simple subdirectly irreducible algebra which are homomorphic images of the free three-valued closure algebra $\mathbf{F}(G)$ over a given poset G. We need the following results about the free Boolean algebra over a poset.

DEFINITION 2.9. For a poset G, the Boolean algebra $\mathbf{B}(G)$ is said to be free over G provided the following conditions hold:

($B1$) $G \subseteq \mathbf{B}(G)$ and $[G]_{\mathscr{B}} = \mathbf{B}(G)$.

($B2$) Given an order-preserving mapping f from G to D, with D a Boolean algebra, there exists a homomorphism h from $\mathbf{B}(G)$ to D such that $h\restriction_G = f$.

The following is a construction of $\mathbf{B}(G)$ (see [15]).

For a poset G, consider the set $E = \mathbf{2}^{[G]}$ of all order–preserving mappings from G into $\mathbf{2}$. For $i \in G$, let $G_i = \{f \in E$ such that $f(i) = 1\}$. Let B be the field of sets generated by $\mathscr{G} = \{G_i : i \in G\}$ in $\mathscr{P}(E) \cong \mathbf{2}^E$ ($\mathscr{P}(E) =$ the set of subsets of E). It can be proved that B is the free Boolean algebra over the poset $\mathscr{G} = \{G_i : i \in G\} \cong G$.

THEOREM 2.10. *Let G be a finite poset. Then* $\mathbf{B}(G) \cong \mathbf{2}^{|\mathbf{2}^{[G]}|} \cong \mathscr{P}\left(\mathbf{2}^{[G]}\right)$.

Let $B^* \in \mathscr{C}_T$ be a subdirectly irreducible algebra generated by a poset G^* which is a homomorphic image of G. Let $f : G \to G^*$ be an onto order-preserving mapping. Then f can be extended to a homomorphism of closure algebras $\overline{f} : \mathbf{F}(G) \to B^*$. Then

$$\overline{f}(\mathbf{F}(G)) = \overline{f}([G]_{\mathscr{C}}) = [G^*]_{\mathscr{C}} = B^*.$$

If B^* is simple, $B^* = [G^*]_{\mathscr{B}}$. Thus the following theorem holds.

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THEOREM 2.11. *Let* $B^* \in \mathscr{C}_T$ *a simple algebra that is a homomorphic image of* $\mathbf{F}(G)$. *Then* $|B^*| \leq 2^{|2^{|G|}|} = |\mathbf{B}(G)|$.

If B^* is not simple, then $B^* = [G^* \cup \{a\}]_{\mathscr{B}}$. Indeed, $Q(B^*) = \{0, a, 1\} \subseteq [\{a\} \cup G^*]_{\mathscr{B}}$. Hence $B^* = [[\{a\} \cup G^*]_{\mathscr{B}}]_{\mathscr{C}} = [\{a\} \cup G^*]_{\mathscr{B}}$. So we can conclude that

$$|B^*| \leq 2^{|2^{|G^*+1|}|} = |\mathbf{B}(G^* + 1)|,$$

where $G^* + 1$ is the cardinal sum of the poset G^* and the 1-element poset.

We show that a non-trivial relation must hold among a and the elements of G^*, that is, $G^* \cup \{a\}$ is not a free generating set for B^*, if we just consider the Boolean structure of B^*. In other words, $B^* \neq \mathbf{B}(G^* + \{a\})$, and therefore $|B^*| \lneq |\mathbf{B}(G^* + \{a\})|$. To see this, first observe that $a \in Q([G^*]_{\mathscr{B}})$. Indeed, suppose that $a \notin Q([G^*]_{\mathscr{B}})$, then $Q([G^*]_{\mathscr{B}}) = \{0, 1\}$ and consequently $[G^*]_{\mathscr{B}} = [[G^*]_{\mathscr{B}}]_{\mathscr{C}} = [G^*]_{\mathscr{C}} = B^*$. Thus $a \notin B^*$, a contradiction. Therefore, $Q([G^*]_{\mathscr{B}}) = \{0, a, 1\}$.

Let $\overline{G^*} = \{-g : g \in G^*\}$. It is known that $[G^*]_{\mathscr{B}} = [G^* \cup \overline{G^*}]_{\mathscr{D}_{0,1}}$. If we put $\mathscr{G}^* = G^* \cup \overline{G^*}$, then every element $y \in [G^*]_{\mathscr{B}}$ can be written

$$y = \bigwedge_{i=1}^{n} \bigvee_{z \in H_i} z,$$

where $H_i \subseteq \mathscr{G}^*$.

In particular, since $a \in Q([G^*]_{\mathscr{B}})$ it follows that there exists $x \in [G^*]_{\mathscr{B}}$, $x \neq 1$, such that $Q(x) = a$. Then

$$a = Q(x) = Q\left(\bigwedge_{i=1}^{n} \bigvee_{y \in H_i} y\right) = \bigwedge_{i=1}^{n} Q\left(\bigvee_{y \in H_i} y\right).$$

But a is open meet-irreducible, so $a = Q\left(\bigvee_{x \in H} x\right)$, for some $H \subseteq \mathscr{G}^*$. Then

$$a \leq \bigvee_{x \in H} x \neq 1. \tag{13}$$

Let $H^* = H \cup \{g_i \in G^* : g_i \notin H \text{ and } -g_i \notin H\}$. Then

$$a \leq \bigvee_{x \in H^*} x. \tag{14}$$

It is clear that if $g_i \in G^*$ then either $g_i \in H^*$ or $-g_i \in H^*$, and, in addition, if $x \in H^*$ then $-x \notin H^*$. Hence H^* is a generating set for $[G^*]_{\mathscr{B}}$.

The above inequality (14), will allow us to prove that $G^* \cup \{a\}$ cannot be a free generating set for B^*, as a Boolean algebra. Indeed, let $h : G^* \cup \{a\} \to \mathbf{B}(G^*) \times \mathbf{2}$

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be given by $h(g) = (g, 0)$, $g \in G^*$, and $h(a) = (0, 1)$. Then $h(a) \not\leq \bigvee h(G)$, so h cannot be extended over $\mathbf{B}(G^* + \{a\})$.

Hence we have that $|B^*| \neq 2^{|2^{[G+1]}|} = |\mathbf{B}(G+1)|$, and consequently, we obtain the following theorem.

THEOREM 2.12. *If B^* is a non-simple subdirectly irreducible algebra which is a homomorphic image of the free algebra $\mathbf{F}(G)$, then $|B^*| \leq 2^{|2^{[G+1]}|-1}$.*

Let us see that the upper bound given in this theorem is the best, that is, that there exists a non-simple subdirectly irreducible algebra which is generated as closure algebra by a homomorphic image of G and whose cardinality is the number given in Theorem 2.12.

Let $G + 1 = \{g_1, g_2, \dots, g_n\} \cup \{g_{n+1}\}$ be a free generating set for $\mathbf{B}(G+1)$. Let $x = \bigvee_{i=1}^{n+1} g_i$. It is clear that x is a dual atom. Let $B^* = \mathbf{B}(G+1)/F_x$, where F_x is the principal filter generated by x. Let $k : \mathbf{B}(G+1) \to \mathbf{B}(G+1)/F_x$, the natural homomorphism. Then $1 = k(x) = k\left(\bigvee_{i=1}^{n+1} g_i\right) = \bigvee_{i=1}^{n+1} k(g_i)$. Thus $\bigvee_{i=1}^{n} k(g_i) \geq -k(g_{n+1}) = k(-g_{n+1})$.

In B^* we consider as a nontrivial open element $a = k(-g_{n+1})$. Let $G_n = \{k(g_1), k(g_2), \dots, k(g_n)\}$. Since $\bigvee_{i=1}^{n} g_i \lneq x$, it follows that $\bigvee_{i=1}^{n} k(g_i) \lneq 1$. So $1 \gneq Q\left(\bigvee_{i=1}^{n} k(g_i)\right) \geq Q(k(-g_{n+1})) = a$. Therefore, $Q\left(\bigvee_{i=1}^{n} k(g_i)\right) = a$.

Hence, we can conclude that $a \in [G_n]_{\mathscr{C}}$. Then

$$B^* = [k(G)]_{\mathscr{B}} = [G_n \cup \{a\}]_{\mathscr{B}} = [G_n]_{\mathscr{C}}.$$

So B^* is a non-simple subdirectly irreducible algebra with a generating set G_n which is a homomorphic image of G, and $|B^*| = 2^{|2^{[G+1]}|}/2 = 2^{|2^{[G+1]}|-1}$. Then we have proved the following theorem.

THEOREM 2.13. *There exists a non-simple subdirectly irreducible algebra B^* which is a homomorphic image of $\mathbf{F}(G)$ and $|B^*| = 2^{|2^{[G+1]}|-1}$.*

Now, let F_a be the open filter generated by $a \in B^*$. Then B^*/F_a is simple and is a homomorphic image of $\mathbf{F}(G)$. Then, by Theorem 2.11, $|B^*/F_a| \leq 2^{|2^{[G]}|}$, that is, there are at most $|2^{[G]}|$ atoms preceding the open element a in B^*. If $h : B^* \to B^*/F_a$ is the natural homomorphism, $h(G^*)$ is a generating set for B^*/F_a as a Boolean algebra. In the same way as for non-simple subdirectly irreducible algebras, it can be seen that $h(G^*)$ is not a free generating set for B^*/F_a, that is, B^*/F_a cannot be isomorphic to the Boolean algebra $\mathbf{B}(G)$. Therefore, $|B^*/F_a| \leq 2^{|2^{[G]}|-1}$, that is, the number of atoms preceding the element a in a non-simple subdirectly irreducible algebra is at most $|2^{[G]}| - 1$.

If I_a is the principal ideal generated by a in B^* and $q : B^* \to B^*/I_a$ is the natural Boolean homomorphism, then $q(B^*) = [q(G^*)]_{\mathscr{B}}$. Hence $|B^*/I_a| \leq 2^{|2^{[G]}|}$. That is, there exist at most $|2^{[G]}|$ atoms not preceding the open element a (preceding the element $-a$).

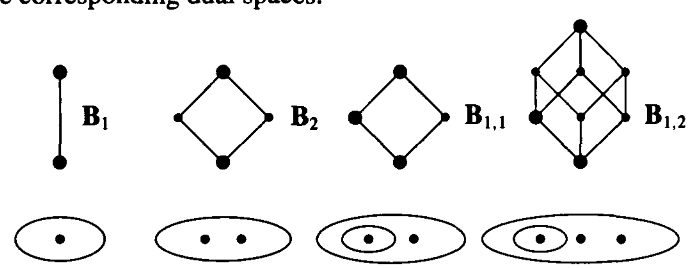
THEOREM 2.14. *Let B^* be a non-simple subdirectly irreducible algebra which is a homomorphic image of* $\mathbf{F}(G)$. *Then* $B^* \cong \mathbf{B}_{k,l}$ *with* $1 \leq k \leq |2^{[G]}| - 1$ *and* $1 \leq l \leq |2^{[G]}|$. *In addition, for every* $1 \leq k \leq |2^{[G]}| - 1$ *and* $1 \leq l \leq |2^{[G]}|$, *every algebra* $B^* \cong \mathbf{B}_{k,l}$ *is a homomorphic image of* $\mathbf{F}(G)$.

PROOF. From the construction of $B(G+1)/F_x$ (x a dual atom), from the previous theorems and remarks, and since $|2^{[G+1]}| - 1 = |2^{[G]} \times 2| - 1 = |2^{[G]}| - 1 + |2^{[G]}|$, it follows that

$$B(G+1)/F_x \cong \mathbf{B}_{|2^{[G]}|-1,|2^{[G]}|}.$$

The second part of the theorem is immediate, since every algebra $\mathbf{B}_{k,l}$, with $1 \leq k \leq |2^{[G]}| - 1$, $1 \leq l \leq |2^{[G]}|$, is a $\mathscr{C}$-subalgebra of $\mathbf{B}_{|2^{[G]}|-1,|2^{[G]}|}$. □

For example (see [4, page 129, Lemma 6.1]), if $G = \{g_1\}$, the subdirectly irreducible algebras which are homomorphic images of $\mathbf{F}(G)$ are, up to isomorphism, the algebras listed in the following figure, where the open elements are highlighted. We also represent the corresponding dual spaces.



3. Free algebras over a poset

The aim of this section is to give explicitly the structure of $\mathbf{F}(G) = \mathbf{F}_{\mathscr{C}_T}(G)$, the free algebra over a finite poset G in the variety $\mathscr{C}_T$.

Since the finitely generated subdirectly irreducible algebras in $\mathscr{C}_T$ are finite and there are only finitely many n-generated subdirectly irreducibles for every natural number n, it follows that $\mathscr{C}_T$ is a locally finite variety. Then the algebra $\mathbf{F}(G)$ is finite, and consequently, every meet-irreducible open filter M_p of $\mathbf{F}(G)$ is generated by a join-irreducible open element p.

If $\mathscr{V}$ is a variety, the variety $\mathscr{V}_0$ generated by the finite simple algebras in $\mathscr{V}$ is the *prime variety* associated with $\mathscr{V}$ (see [3]).

In [3, Corollary 7.2], Berman and Blok showed that if $\mathscr{V}$ is a locally finite variety with the Fraser-Horn and Apple Properties, which, in addition, has the property that every subalgebra of a finite simple algebra is a product of simple algebras, then the number of directly indecomposable factors of $\mathbf{F}_{\mathscr{V}_0}(G)$ equals that of $\mathbf{F}_{\mathscr{V}}(G)$. They also proved ([3, Theorem 7.3]) that if a given finite simple algebra A is a direct factor of the free algebra in $\mathscr{V}_0$, there exists a directly indecomposable factor of $\mathbf{F}_{\mathscr{V}}(G)$ having A as homomorphic image. These results can be applied to the variety $\mathscr{C}_T$, as this variety has the Fraser-Horn and Apple Properties, and, in addition, every subalgebra of a finite simple algebra is simple.

The prime variety $\mathscr{C}_{T0}$ is the variety $\mathscr{M}$ of monadic Boolean algebras. It is known ([17] and [8]) that the free monadic Boolean algebra $\mathbf{F}_{\mathscr{M}}(G)$ is given by

$$\mathbf{F}_{\mathscr{M}}(G) \cong \prod_{k=1}^{|2^{[G]}|} \mathbf{B}_k^{\binom{|2^{[G]}|}{k}},$$

where $\binom{|2^{[G]}|}{k}$ is the number of (monadic) epimorphisms from $\mathbf{F}_{\mathscr{M}}(G)$ onto $\mathbf{B}_k$, that is, the number of (Boolean) epimorphisms from $\mathbf{B}(G)$ onto $\mathbf{B}_k$, and $|2^{[G]}|$ is the cardinal number of the greatest simple monadic algebra generated by a copy of G.

So, from [3, Corollary 7.5], the algebra $\mathbf{F}(G)$ has a factorization as

$$\mathbf{F}(G) \cong \prod_{k=1}^{|2^{[G]}|} \mathbf{A}_k^{\binom{|2^{[G]}|}{k}},$$

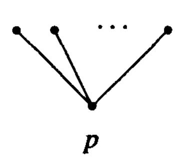
where each $\mathbf{A}_k$ has as homomorphic image a factor of the free monadic Boolean algebra $\mathbf{F}_{\mathscr{M}}(G)$.

We will now determine the structure of the factors $\mathbf{A}_k$ of $\mathbf{F}(G)$.

Let $\mathscr{J}(Q(\mathbf{F}(G)))$ be the set of join-irreducible elements of $Q(\mathbf{F}(G))$. Observe that M_p is maximal (minimal) if and only if p is minimal (maximal) in $\mathscr{J}(Q(\mathbf{F}(G)))$. Let $\mathfrak{m}$, $\mathfrak{M}$ respectively denote the set of minimal and maximal elements in $\mathscr{J}(Q(\mathbf{F}(G)))$. Then

$$\mathscr{J}(Q(\mathbf{F}(G))) = \sum_{p\in\mathfrak{m}} C_p,$$

where $C_p = \{q \in \mathscr{J}(Q(\mathbf{F}(G))) : q \geq p\}$. Every C_p looks like the diagram in the following figure (Theorem 2.4):



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Then

$$Q(\mathbf{F}(G)) \cong_{\mathscr{H}} \prod_{p \in \mathfrak{m}} D_p,$$

where D_p is the distributive lattice such that $\mathscr{J}(D_p) \cong C_p$. Thus the elements $p^* = \bigvee_{q \in C_p} q$ are complemented, the complement coincides with the complement in F(G) and is given by

$$-p^* = \bigvee_{q \in \mathscr{J}(Q(\mathbf{F}(G))) \setminus C_p} q.$$

In particular, $-p^*$ is open.

We establish the following simple but useful lemma.

LEMMA 3.1. *Let* $x \in \mathrm{At}(\mathbf{F}(G))$. *Then there exists* $p \in \mathscr{J}(Q(\mathbf{F}(G)))$ *such that* $x \leq p$.

PROOF. Let $p \in \mathfrak{m}$. If $x \leq q$ for some $q \in C_p$, then the lemma holds. Suppose that $x \not\leq q$, for every $q \in C_p$. In particular, $x \not\leq p^*$. Then $x \leq -p^* = \bigvee_{q \in \mathscr{J}(Q(\mathbf{F}(G))) \setminus C_p} q$. Since x is an atom it follows that $x \leq q$ for some $q \in \mathscr{J}(Q(\mathbf{F}(G))) \setminus C_p$. □

The above lemma shows that the set $P = \{\mathrm{At}(p^*)\}_{p \in \mathfrak{m}}$, where $\mathrm{At}(p^*) = \{x \in \mathrm{At}(\mathbf{F}(G)) : x \leq p^*\}$, is a partition of the set $\mathrm{At}(\mathbf{F}(G))$.

Let F_{p^*} and I_{p^*} respectively denote the principal filter and principal ideal generated by p^*. Then we have the following theorem.

THEOREM 3.2. $\mathbf{F}(G) \cong_{\mathscr{C}} \prod_{p \in \mathfrak{m}} \mathbf{F}(G)/F_{p^*} \cong_{\mathscr{C}} \prod_{p \in \mathfrak{m}} I_{p^*}$.

If $p, q \in \mathfrak{m}$ are such that $I_p \cong I_q \cong \mathbf{B}_k$, then there exists an automorphism α of $\mathbf{F}(G)$ such that $\alpha(p) = q$. Then $\alpha(C_p) = C_q$, that is, $\alpha(p^*) = q^*$, and consequently, $I_{p^*} \cong I_{q^*}$. It is not difficult to see that the algebras $I_{p_k^*}$, $1 \leq k \leq |2^{[G]}|$, are the directly indecomposable factors $\mathbf{A}_k$ of $\mathbf{F}(G)$. Then

THEOREM 3.3. $\mathbf{F}(G) \cong \prod_{k=1}^{|2^{[G]}|} I_{p_k^*}^{\binom{|2^{[G]}|}{k}}$.

Our next objective is to determine the number of elements of $\mathbf{F}(G)$.

Let $p \in \mathscr{J}(Q(\mathbf{F}(G)))$. If $p \in \mathfrak{m}$, then $\mathbf{F}(G)/M_p \cong \mathbf{B}_k$ and thus, there exist k atoms preceding p. If $p \in \mathfrak{M}$, then $\mathbf{F}(G)/M_p \cong \mathbf{B}_{k,l}$. Thus there are $k + l$ atoms preceding p. In addition, k of these atoms precede the only element $q \in \mathfrak{m}$ such that $q \leq p$.

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If we put $\mathfrak{m}_k = \{p \in \mathfrak{m} : \mathbf{F}(G)/M_p \cong \mathbf{B}_k\}$ and $\mathfrak{M}_{k,l} = \{p \in \mathfrak{M} : \mathbf{F}(G)/M_p \cong \mathbf{B}_{k,l}\}$, then the number of atoms of the free algebra is

$$|\mathrm{At}(\mathbf{F}(G))| = \sum_{1\le k\le |2^{[G]}|} |\mathfrak{m}_k| k + \sum_{1\le k\le |2^{[G]}-1|, 1\le l\le |2^{[G]}|} |\mathfrak{M}_{k,l}| l.$$

From what we have seen above,

$$|\mathfrak{m}_k| = \binom{|2^{[G]}|}{k}, \quad 1 \le k \le \left|2^{[G]}\right|.$$

Now, given k, l, $1 \le k \le \left|2^{[G]}\right| - 1$, $1 \le l \le \left|2^{[G]}\right|$, (see Theorem 2.14), let $\mathrm{Ep}(\mathbf{F}(G), \mathbf{B}_{k,l})$ be the set of all epimorphisms from $\mathbf{F}(G)$ onto $\mathbf{B}_{k,l}$, and let $\mathrm{Aut}(\mathbf{B}_{k,l})$ be the set of all automorphisms of $\mathbf{B}_{k,l}$. Then it is readily verifiable that

$$|\mathfrak{M}_{k,l}| = \frac{|\mathrm{Ep}(\mathbf{F}(G), \mathbf{B}_{k,l})|}{|\mathrm{Aut}(\mathbf{B}_{k,l})|}. \tag{15}$$

If $h \in \mathrm{Aut}(\mathbf{B}_{k,l})$, then h is a Boolean automorphism such that $h(a) = a$, and thus h is characterized by the bijections $h\restriction_{A_a} : A_a \to A_a$ and $h\restriction_{A_{-a}} : A_{-a} \to A_{-a}$. Hence

$$|\mathrm{Aut}(\mathbf{B}_{k,l})| = k!l!.$$

Let us compute now the numerator of (15). Let $G = \{g_1, \dots, g_n\}$. Let $G^* = \{g_1^*, \dots, g_n^*\}$ be an isomorphic copy of G, and $G^* + 1 = G^* + \{g_{n+1}^*\}$. Let $\mathrm{Ep}^*(\mathbf{B}(G^* + 1), \mathbf{B}_{k,l})$ denote the set of all Boolean epimorphisms H from the free Boolean algebra $\mathbf{B}(G^* + 1)$ onto $\mathbf{B}_{k,l}$, such that $H(g_{n+1}^*) = a$.

Consider two cases:

I. $k < l$.

In this case the open element a of $\mathbf{B}_{k,l}$ belongs to every maximal subalgebra of $\mathbf{B}_{k,l}$ (see the proof of Theorem 2.8). Let

$$\psi : \mathrm{Ep}(\mathbf{F}(G), \mathbf{B}_{k,l}) \to \mathrm{Ep}^*(\mathbf{B}(G^* + 1), \mathbf{B}_{k,l})$$

be the mapping defined by $\psi(F) = H$, where H is the extension of the mapping h such that $h(g_i^*) = F(g_i)$ for every $i \ne n+1$ and $h(g_{n+1}^*) = a$. We have that $H(\mathbf{B}(G^* + 1)) = [H(G^* + 1)]_{\mathscr{B}} = [F(G) \cup \{a\}]_{\mathscr{B}} = \mathbf{B}_{k,l}$, so H is onto and consequently, $H \in \mathrm{Ep}^*(\mathbf{B}(G^* + 1), \mathbf{B}_{k,l})$. It is clear that ψ is one-to-one. To see that ψ is onto, let $H \in \mathrm{Ep}^*(\mathbf{B}(G^* + 1), \mathbf{B}_{k,l})$. Observe that $H(G^*) \not\subseteq B$ for any maximal subalgebra B of $\mathbf{B}_{k,l}$. Indeed, if we suppose that $H(G^*) \subseteq B$ for B a maximal subalgebra of $\mathbf{B}_{k,l}$, then, since $a \in B$, $[H(G^*) \cup \{a\}]_{\mathscr{B}} \subseteq B$. But $\mathbf{B}_{k,l} = H(\mathbf{B}(G^* + 1)) = [H(G^*) \cup \{H(g_{n+1}^*)\}]_{\mathscr{B}} = [H(G^*) \cup \{a\}]_{\mathscr{B}}$, which is a contradiction.

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As a consequence we have that $[H(G^*)]_{\mathscr{C}} = \mathbf{B}_{k,l}$.

Now, let F be the extension of the mapping f such that $f(g_i) = H(g_i^*)$, $1 \le i \le n$. Then $F \in \text{Ep}(\mathbf{F}(G), \mathbf{B}_{k,l})$ since $[H(G^*)]_{\mathscr{C}} = [F(G)]_{\mathscr{C}} = \mathbf{B}_{k,l}$, and it is clear that $\psi(F) = H$. Therefore

$$|\text{Ep}(\mathbf{F}(G), \mathbf{B}_{k,l})| = |\text{Ep}^*(\mathbf{B}(G^* + 1), \mathbf{B}_{k,l})|,$$

and this is the number of injective functions $f : \text{At}(\mathbf{B}_{k,l}) \to \text{At}(\mathbf{B}(G^* + 1))$ such that $f(A_k) \subseteq \text{At}(g_{n+1})$ and $f(A_{-a}) \subseteq \text{At}(-g_{n+1})$, and since $\text{At}(g_{n+1}) = \mathbf{2}^{[G]}$, it follows that

$$|\text{Ep}(\mathbf{F}(G), \mathbf{B}_{k,l})| = V_{|2^{[G]}|,l} V_{|2^{[G]}|,k} = \frac{|\mathbf{2}^{[G]}|!}{(|\mathbf{2}^{[G]}| - k)!} \frac{|\mathbf{2}^{[G]}|!}{(|\mathbf{2}^{[G]}| - l)!}.$$

II. $k \ge l$.

In this case (see Theorem 2.8) there exist maximal subalgebras in $\mathbf{B}_{k,l}$ which do not contain the open element a. Let $H \in \text{Ep}^*(\mathbf{B}(G^* + 1))$. If B is a maximal subalgebra of $\mathbf{B}_{k,l}$ such that $a \in B$, then from case I, $H(G^*) \not\subseteq B$.

If B is a maximal subalgebra of $\mathbf{B}_{k,l}$ such that $a \notin B$ and $H(G^*) \subseteq B$, we claim that $[H(G^*)]_{\mathscr{B}} = B$. To see this, suppose that $[H(G^*)]_{\mathscr{B}} \neq B$. Then, in the partition P associated with $[H(G^*)]_{\mathscr{B}}$ there exists a block P_i such that $|P_i \cap A_a| = 2$ or $|P_i \cap A_{-a}| = 2$. Since the partition associated with $[[H(G^*)]_{\mathscr{B}} \cup \{a\}]_{\mathscr{B}}$ is $P \cap \{A_a, A_{-a}\}$, it follows that $\mathbf{B}_{k,l} \neq [[H(G^*)]_{\mathscr{B}} \cup \{a\}]_{\mathscr{B}} = [H(G^*) \cup \{h(g_{n+1}^*)\}]_{\mathscr{B}}$, a contradiction.

From this we conclude that there exist $H \in \text{Ep}^*(\mathbf{B}(G^* + 1))$ which do not satisfy the condition $[H(G^*)]_{\mathscr{C}} = \mathbf{B}_{k,l}$. If H is such a homomorphism, then $H\restriction_{[G^*]_{\mathscr{B}}} : [G^*]_{\mathscr{B}} \to B$ is an epimorphism, with B maximal subalgebra and $a \notin B$. Since $[G^*]_{\mathscr{B}} \cong \mathbf{B}(G)$ and $B \cong \mathbf{B}_k$, then for each maximal subalgebra B with $a \notin B$, there exist $V_{|2^{[G]}|,k}$ homomorphisms H such that $[H(G^*)]_{\mathscr{C}} \neq \mathbf{B}_{k,l}$. Hence

$$\begin{aligned} |\text{Ep}(\mathbf{F}(G), \mathbf{B}_{k,l})| &= |\text{Ep}^*(\mathbf{B}(G^* + 1), \mathbf{B}_{k,l})| \\ &\quad - |\{H \in \text{Ep}^*(\mathbf{B}(G^* + 1)) : [H(G^*)]_{\mathscr{C}} \neq \mathbf{B}_{k,l}\}| \\ &= V_{|2^{[G]}|,l} V_{|2^{[G]}|,k} - V_{k,l} V_{|2^{[G]}|,k} = V_{|2^{[G]}|,k}(V_{|2^{[G]}|,l} - V_{k,l}) \\ &= \frac{|\mathbf{2}^{[G]}|!}{(|\mathbf{2}^{[G]}| - k)!} \left(\frac{|\mathbf{2}^{[G]}|!}{(|\mathbf{2}^{[G]}| - l)!} - \frac{k!}{(k - l)!} \right). \end{aligned}$$

Consequently, if we put $\binom{k}{l} = 0$, whenever $l > k$, $\binom{k}{l} = k!/l!\,(k-l)!$, whenever $l \le k$, and $M = |\mathbf{2}^{[G]}|$, then $|\mathfrak{M}_{k,l}| = \binom{M}{k} \left(\binom{M}{l} - \binom{k}{l} \right)$.

The following theorem gives the cardinality of $\mathbf{F}(G)$.

THEOREM 3.4. $|\mathbf{F}(G)| = 2^{M(2^{2M-1} - 3^{M-1})}$.

PROOF. From the previous considerations it follows that

$$(16)\qquad |\operatorname{At}(\mathbf{F}(G))| = \sum_{1\le k\le M} k\binom{M}{k} + \sum_{1\le l\le M, 1\le k\le M-1} l\binom{M}{k}\left(\binom{M}{l}-\binom{k}{l}\right).$$

In addition,

$$\sum_{1\le k\le M} k\binom{M}{k} = M2^{M-1} \quad\text{and}\quad \sum_{1\le k\le M-1} k\binom{M}{k}2^{k-1} = M(3^{M-1}-2^{M-1}).$$

Thus

$$\begin{aligned}|\operatorname{At}(\mathbf{F}(G))| &= M2^{M-1} + \sum_{1\le k\le M-1}\binom{M}{k}\left(\sum_{1\le l\le M} l\binom{M}{l} - \sum_{1\le l\le M} l\binom{k}{l}\right)\\ &= M2^{M-1} + \sum_{1\le k\le M-1}\binom{M}{k}M2^{M-1} - \sum_{1\le k\le M-1}\binom{M}{k}k2^{k-1}\\ &= M2^{M-1} + (2^M-2)M2^{M-1} - M(3^{M-1}-2^{M-1})\\ &= M(2^{2M-1}-3^{M-1}).\end{aligned}$$

□

From this theorem and the previous remarks, it is possible to evaluate the number of join-irreducible elements $p_{k,l} \in \mathfrak{M}_{k,l}$, for k and l given. Nevertheless, for $p_k \in \mathfrak{m}_k$ it remains to evaluate how many covers it has in $\mathscr{J}(Q(\mathbf{B}(G)))$, since this will allow us to determine the algebraic structure of $\mathbf{F}(G)$.

From Theorem 3.3 and (16) the closure algebra $I_{p_k^*}$ has

$$k + \sum_{1\le l\le M} l\left(\binom{M}{l}-\binom{k}{l}\right) = k + M2^{M-1} - 2^{k-1}k = M2^{M-1} + k\left(1-2^{k-1}\right)$$

atoms, with $M = \left|2^{[G]}\right|$. In addition, $Q\left(I_{p_k^*}\right) \cong 1 \oplus B_{S_k}$, where B_{S_k} is the Boolean algebra with $S_k = \sum_{1\le l\le M}\left(\binom{M}{l}-\binom{k}{l}\right) = 2^M - 2^k$ atoms. From this we conclude

COROLLARY 3.5. $Q(\mathbf{F}(G)) \cong \prod_{k=1}^{M}\left(1\oplus B_{S_k}\right)^{\binom{M}{k}}$.

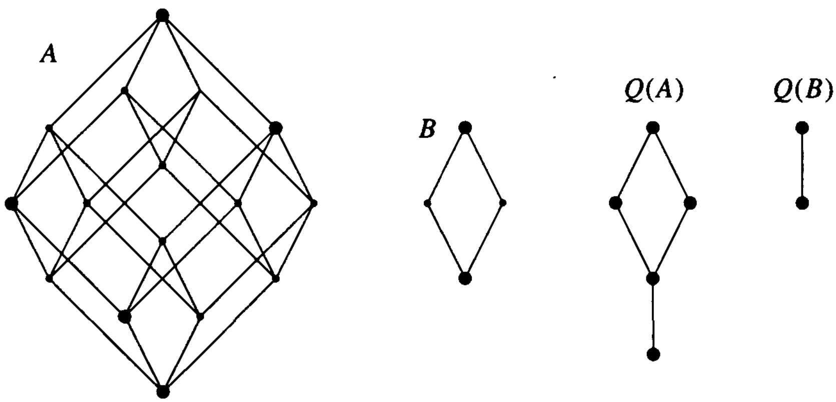
Let $\mathbf{F}(r)$ be the three-valued closure algebra with r free generators. This a special case of the free algebra over a poset, where the poset is an antichain. Then

THEOREM 3.6. $\mathbf{F}(r) \cong \prod_{k=1}^{2^r} I_{p_k^*}^{\binom{2^r}{k}}$, $|\mathbf{F}(r)| = 2^{2^r(2^{2^{r+1}-1}-3^{2^r-1})}$ *and* $Q(\mathbf{F}(r)) \cong \prod_{k=1}^{2^r}(1\oplus B_{S_k})^{\binom{2^r}{k}}$, *with* $S_k = 2^{2^r} - 2^k$.

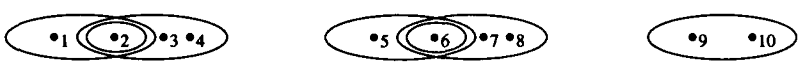
PROOF. It is an immediate consequence of Corollary 3.5, Theorem 3.4 and Theorem 3.3. □

The following example was also worked out in [4].

EXAMPLE 3.7. Let $\mathbf{F}(1)$ the free algebra with one generator, and let



Then $\mathbf{F}(1) \cong A^2 \times B$ and $Q(\mathbf{F}(1)) \cong Q(A^2) \times Q(B) \cong (1 \oplus B_2)^2 \times B_1$. The dual space of $\mathbf{F}(1)$ looks like the following diagram:



A generator is given by $g = \{2, 3, 5, 8, 9\}$, and the atoms can be obtained from g in the following way:

- $\{1\} = (\nabla(Q(g)) \wedge -g) \wedge -((\nabla(Q(g)) \wedge -g) \wedge \nabla(g \wedge \nabla(Q(g)) \wedge \nabla(-g)))$;
- $\{2\} = Q(g)$;
- $\{3\} = g \wedge \nabla(Q(g)) \wedge \nabla(-g)$;
- $\{4\} = (\nabla(Q(g)) \wedge -g) \wedge \nabla(g \wedge \nabla(Q(g) \wedge \nabla(-g))$;
- $\{5\} = (\nabla(g) \wedge \nabla(-g)) \wedge -\nabla(\nabla(g) \wedge -g))$;
- $\{6\} = Q(-g)$;
- $\{7\} = (\nabla(g) \wedge \nabla(Q(-g))) \wedge (-g \wedge \nabla(Q(-g)))$;
- $\{8\} = (g \wedge \nabla(Q(-g))) \wedge -((\nabla(g) \wedge \nabla(-g)) \wedge -\nabla(\nabla(g) \wedge -g)))$;
- $\{9\} = g \wedge Q(\nabla(g)) \wedge Q(\nabla(-g))$;
- $\{10\} = (Q(\nabla(g)) \wedge -g) \wedge Q(\nabla(-g))$.

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References

[1] R. Balbes and P. Dwinger, *Distributive lattices* (University of Missouri Press, Columbia, 1974).

[2] J. Berman, 'Free spectra of 3-element algebras', in: *Universal Algebra and Lattice Theory, Proceedings, Puebla, 1982* (eds. R. S. Freese and O. C. García), Lecture Notes in Math. 1004 (Springer, New York, 1983) pp. 10–16.

[3] J. Berman and W. J. Blok, 'The Fraser-Horn and Apple Properties', *Trans. Amer. Math. Soc.* **302** (1987), 427–465.

[4] W. J. Blok, *Varieties of interior algebras* (Ph.D. Thesis, University of Amsterdam, 1976).

[5] ———, 'The free closure algebra on finitely many generators', *Indag. Math.* **39** (1977), 362–379.

[6] W. J. Blok and Ph. Dwinger, 'Equational classes of closure algebras I', *Indag. Math.* **37** (1975), 189–198.

[7] J. P. Díaz Varela, *Algebras de Clausura y su Estructura Simétrica* (Ph.D. Thesis, Universidad Nacional del Sur, 1997).

[8] P. Halmos, 'Algebraic logic I. Monadic Boolean algebras', *Compositio Math.* **12** (1955), 217–249.

[9] T. Hecht and T. Katrinák, 'Equational classes of relative Stone algebras', *Notre Dame J. Formal Logic* **13** (1972), 248–254.

[10] B. Jónsson, 'Algebras whose congruence lattices are distributive', *Math. Scand.* **5** (1967), 110–121.

[11] C. I. Lewis and C. H. Langford, *Symbolic logic* (Century Co., New York, 1932).

[12] J. C. C. McKinsey and A. Tarski, 'The algebra of topology', *Ann. of Math. (2)* **45** (1944), 141–191.

[13] ———, 'On closed elements in closure algebras', *Ann. of Math. (2)* **47** (1946), 122–162.

[14] ———, 'Some theorems about the sentential calculi of Lewis and Heyting', *J. Symbolic Logic* **13** (1948), 1–15.

[15] L. Monteiro, 'Une construction du réticulé distributif libre sur un ensemble ordonné', *Colloq. Math.* **17** (1967), 23–27.

[16] ———, 'Algèbre du calcul propositionel trivalent de Heyting', *Fund. Math.* **74** (1972), 99–109.

[17] ———, 'Algèbres de Boole monadiques libres', *Algebra Universalis* **8** (1978), 374–380.

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