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A NOTE ON FREE ACTIONS OF GROUPS ON PRODUCTS OF SPHERES

JANG HYUN JO ✉ and JONG BUM LEE

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Abstract

It has been conjectured that if $G = (\mathbb{Z}_p)^r$ acts freely on a finite CW-complex X which is homotopy equivalent to a product of spheres $S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k}$, then $r \le k$. We address this question with the relaxation that X is finite-dimensional, and show that, to answer the question, it suffices to consider the case where the dimensions of the spheres are greater than or equal to 2.

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1. Introduction

The purpose of this note is to give a contribution to the following long-standing conjecture in algebraic topology, which remains unanswered despite considerable effort by many mathematicians.

Conjecture 1.1. If $G = (\mathbb{Z}_p)^r$ acts freely on a finite CW-complex which is homotopy equivalent to a product of spheres $S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k}$, then $r \le k$.

In fact, the converse of Conjecture 1.1 is also open, that is, it is not known that any finite group G with $\operatorname{rk} G = r$ can act freely on a finite (or finite-dimensional) CW-complex which is homotopy equivalent to a product of spheres $S^{n_1} \times S^{n_2} \times \cdots \times S^{n_r}$, where $\operatorname{rk} G$ is the maximum of p-ranks $\operatorname{rk}_p G$ as p runs over the primes dividing $|G|$. The case of a single sphere for Conjecture 1.1 and its converse were solved by Smith [13], Milnor [11], Swan [14] and Madsen *et al.* [10]. Moreover, a generalisation of this result has been established by Adem and Smith [2] and Mislin and Talelli [12]. The affirmative answer to Conjecture 1.1 for the case of free $(\mathbb{Z}/p)^r$-actions on $(S^n)^k$ with trivial action on homology was first given by Carlsson [5]

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using a spectral sequence argument. Browder [4] and Benson and Carlson [3] gave other beautiful proofs of this result via different approaches. Adem and Browder [1] settled the homologically nontrivial case except in the cases $p = 2$ and $n = 1, 3$ or 7. Subsequently, Yalçin [15] solved the case for $p = 2$ and $n = 1$, leaving the other two cases open. The nonequidimensional case is known for the product of two spheres [8]. Recently, Hanke [7] gave the most general result using the subject of tame homotopy theory for Conjecture 1.1.

To settle the case for $p = 2$ and $n = 1$, Yalçin used the following algebraic result to prove the case of $p = 2$ and $n = 1$ for Conjecture 1.1.

Theorem 1.2 [15, Theorem 3.2]. *Let G be a finite group, and let $0 \to \mathbb{Z}^k \to \Gamma \to G \to 1$ be an extension of G. If Γ is torsion-free, then* $\mathrm{rk}(G) \leq k$.

Recently, we generalised Yalçin's result from free abelian groups to strongly polycyclic groups.

Theorem 1.3 [9, Theorem 3.2]. *Let Γ be a strongly polycyclic group and G be a finite group. Let $1 \to \Gamma \to \Pi \to G \to 1$ be an extension. Then $\mathrm{rk}_p(G) - \mathrm{rk}_p(\Pi) \leq h(\Gamma)$ for all primes p with $p \mid |G|$, where $h(\Gamma)$ is the Hirsch rank of Γ.*

Note that Yalçin's topological application hinges on the fact that the k-torus is a finite dimensional $K(\mathbb{Z}^k, 1)$-space. In particular, $\pi_1(G \setminus X)$ is torsion-free, where X is a free G-complex with the homotopy type of the k-torus. Thus Theorem 1.3 sheds some light on the following conjecture which is relaxed with the condition of finite-dimensionality.

Conjecture 1.4. If $G = (\mathbb{Z}_p)^r$ acts freely on a finite-dimensional CW-complex which is homotopy equivalent to a product of spheres $S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k}$, then $r \leq k$.

The purpose of this note is to show that, to prove Conjecture 1.4, it suffices to consider the case where the dimensions of the spheres are greater than or equal to 2. We will consider this in a more general setting in Theorem 2.2.

Conjecture 1.5. If $G = (\mathbb{Z}_p)^r$ acts freely on a finite-dimensional CW-complex which is homotopy equivalent to a product of spheres $S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k}$ and all $n_i \geq 2$, then $r \leq k$.

2. Main results

Recall that for a finite-dimensional CW-complex X and p a prime, the free p-rank of X, denoted by $\mathrm{frk}_p(X)$, is the largest r such that $(\mathbb{Z}_p)^r$ acts freely on X [16, Definition 1.1]. Similarly, we can give the following definition to handle Conjectures 1.4 and 1.5.

Definition 2.1. Let X be a finite-dimensional CW-complex and p a prime. The homotopy free p-rank of X, denoted by $\mathrm{hfrk}_p(X)$, is defined as the largest r such that $(\mathbb{Z}_p)^r$ acts freely on Y, where Y is a finite-dimensional CW-complex homotopy equivalent to X.

By definition, $\mathrm{frk}_p(X) \le \mathrm{hfrk}_p(X)$ for any finite-dimensional CW-complex X. It is well known that the following statements hold:

(1) $\mathrm{frk}_p(S^n) = \mathrm{hfrk}_p(S^n) = 1$ for n odd;
(2) $\mathrm{frk}_p(S^n \times S^m) \le \mathrm{hfrk}_p(S^n \times S^m) \le 2$ [8].

Note also that if Conjecture 1.4 is true, then $\mathrm{hfrk}_p(S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k}) \le k$. From [16] it is known that $\mathrm{hfrk}_p(\prod^k L_p^{2n-1}) = k$, where L_p^{2n-1} is the lens space. We do not know of an example of a finite-dimensional CW-complex satisfying the inequality $\mathrm{frk}_p(X) < \mathrm{hfrk}_p(X)$. It is conceivable that $\mathrm{frk}_p(X) = \mathrm{hfrk}_p(X)$.

THEOREM 2.2. *Let M be a compact solvmanifold and L a finite-dimensional CW-complex with a finite fundamental group $\pi_1(L)$. Then*

$$\mathrm{hfrk}_p(M \times L) \le \dim M + \mathrm{hfrk}_p(L).$$

PROOF. Suppose that $G = (\mathbb{Z}_p)^r$ acts freely on a finite-dimensional CW-complex X which is homotopy equivalent to the product $M \times L$. Let $F = \pi_1(L)$, $\Gamma = \pi_1(M)$ and $\pi = \pi_1(G \setminus X)$. Then the free G-action on X induces a short exact sequence

$$1 \longrightarrow \Gamma \times F \longrightarrow \pi \longrightarrow G \longrightarrow 1.$$

Since F is a characteristic subgroup of $\Gamma \times F$, the above short exact sequence yields the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
1 & \longrightarrow & \Gamma & \longrightarrow & \pi/F & \longrightarrow & G & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow{\scriptstyle =} & & \\
1 & \longrightarrow & \Gamma \times F & \longrightarrow & \pi & \longrightarrow & G & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & & & \\
 & & F & \xrightarrow{=} & F & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

Note that M is a $K(\Gamma, 1)$-space whose dimension is the Hirsch length $h(\Gamma)$ of Γ and Γ is a strongly polycyclic group [6, 9]. Note also that the top horizontal exact sequence

$$1 \longrightarrow \Gamma \longrightarrow \pi/F \longrightarrow G \longrightarrow 1$$

is associated to the following maps:

$$\begin{array}{c} \overline{X} \simeq \widetilde{M} \times L \simeq L \\ \Gamma\backslash \big\downarrow \\ X \simeq M \times L \\ G\backslash \big\downarrow \\ G \setminus X \end{array}$$

By Theorem 1.3, we have $\mathrm{rk}_p(G) \leq \mathrm{rk}_p(\pi/F) + h(\Gamma) = \mathrm{rk}_p(\pi/F) + \dim M$. Note that every finite subgroup H of π/F is isomorphic to a subgroup of G. Since π/F acts freely on $\overline{X}$, the subgroup H acts freely on $\overline{X}$, which is homotopy equivalent to L. Hence $\mathrm{rk}_p(\pi/F) \leq \mathrm{hfrk}_p(L)$ and therefore $\mathrm{rk}_p(G) \leq \dim(M) + \mathrm{hfrk}_p(L)$. □

Corollary 2.3. *Conjecture 1.5 implies Conjecture 1.4.*

Proof. Let $G = (\mathbb{Z}_p)^r$. Suppose that G acts freely on a finite dimensional CW-complex X which is homotopy equivalent to $S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k}$. Without loss of generality, we may assume that $n_1 = 1$ and $n_2, \ldots, n_k \geq 2$. Write $M = S^1$ and $L = S^{n_2} \times \cdots \times S^{n_k}$. Conjecture 1.5 implies that $\mathrm{hfrk}_p(L) \leq k - 1$. By Theorem 2.2, $\mathrm{rk}_p(G) \leq 1 + k - 1 = k$. □

Corollary 2.4. *If $G = (\mathbb{Z}_p)^r$ acts freely on a finite-dimensional CW-complex X which is homotopy equivalent to a product $(S^1)^\ell \times Y$, where Y is one of the following:*

(1) $S^n \times S^m$ *where* $n, m \geq 2$*;*
(2) $\prod^k L_p^{2n-1}(a_1, \ldots, a_n)$*, where* $L_p^{2n-1}(a_1, \ldots, a_n)$ *is the standard lens space with* $(a_i, p) = 1$ *for all* i*;*

then $r \leq \ell + 2$ *for case* (1) *and* $r \leq \ell + k$ *for case* (2).

Proof. This follows from combining the results of [8, 16] with Theorem 2.2. □

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JANG HYUN JO, Department of Mathematics, Sogang University,
Seoul 121-742, Korea
e-mail: jhjo@sogang.ac.kr

JONG BUM LEE, Department of Mathematics, Sogang University,
Seoul 121-742, Korea
e-mail: jlee@sogang.ac.kr