Part III.

Spherically-Symmetric Motions in Stellar Atmospheres.

A. - Pulsating Variable Stars.

Summary-Introduction:

Velocity Fields and Associated Thermodynamic Variations in the External Layers of Intrinsic Variable Stars.

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1. – General introduction.

By intrinsic variable stars, we mean those stars which present variations in light, spectrum and radial velocity which cannot be accounted for in terms of purely geometrical or orbital factors; so that we have to appeal to some kind of periodic physical modification of the star.

This class is very large, and it would be quite impossible here to review even briefly the properties of all the different sub-classes (Ledoux and Walraven, 1958; hereafter called reference A); so we shall limit ourselves to one of the best known groups, which comprises the cepheids and the *RR Lyrae* stars. The essential factors which, up to now, have been called upon to explain the properties of the various kinds of intrinsic variable stars, excluding the most irregular or the most violent types (such as novae and super-novae), will be brought up in the discussion of this group.

The cepheids are supergiants of mean absolute bolometric magnitude, M_{bol}, falling in the range -3 to -6 and of spectral type F to G. In a Hertzsprung-Russell diagram, using M_{bol} and $\log T_e$ as co-ordinates, they occur about $\frac{2}{3}$ of the way from the main sequence to the giant branch, where they form a sequence roughly parallel to the latter (cf. ref. A, p. 572). They comprise stars typical of Populations I (disk and spiral arms), often referred to as classical cepheids, whose periods vary from about 2 to 40 days from the less to the most luminous objects of the class; and stars typical of Population II (globular clusters and spherically distributed stars in the galaxy), which show a concentration in two ranges of periods: 1 to 2 days and 13 to 20 days. Stars in the second group are often designated, after the prototype, as *W Virginis* stars. Although the basic phenomena are probably the same, there are many interesting physical differences between the classical cepheids and the *W Virginis* stars, at least in the behaviour of the external layers.

The *RR Lyrae* stars, which belong to Population II, have periods between $\frac{1}{4}$ of a day and 1 day. The exact value of their absolute magnitude, which is roughly of the order of $0.0\, M_{\text{bol}}$, is at present a matter of great concern as it plays an important role in fixing the distance scale of the Universe. Their spectral type falls in a small range around A5. In the Hertzsprung-Russell diagram of a globular cluster, they fill a small definite gap in the horizontal branch. This suggests that any star which, in the course of its evolution, goes through this gap, becomes the seat of this type of variability.

In some of the *RR Lyrae* stars there appears, superposed on the periodic variation referred to above, a very regular modulation of very long period (60 to 1400 times, the main period), which affects both the amplitude and the phase. The resulting variation can be interpreted as a beat phenomenon between two variations of very close periods.

Some cepheids in the range of periods 2 to 3 days present a similar phenomenon, but with a modulation period that is a much smaller multiple (2 to 3 times) of the fundamental period.

The phenomenon of multiple periodicity is perhaps best represented by a group of variables with very short periods, of the order of 0.05 to 0.2 days. Despite the fact that, as far as the periods are concerned, they fall close to the lower end of the *RR Lyrae* variables, they do not probably belong to that class and are sometimes called dwarf cepheids. One of them which has been studied extensively, *AI Velorum*, presents no less than four well marked periods.

A beat phenomenon is also encountered in some of the *β Cephei* stars, a group containing about 12 known members with main periods in the range $3\frac{1}{2}$ to 6 hours and composed of bright blue stars, B1 to B2, somewhat above the main sequence. However, in this case, the two interfering variations must present some significant physical difference, since one of them is associated with a variable broadening or doubling of the lines while the other does not seem to affect the line shapes. This suggests that the type of motion may be more complicated here than in the case of the other variable stars mentioned above. Note also that the light amplitude is very small in this case.

For well observed classical cepheids and *RR Lyrae* stars, the periods can be defined with a very high precision and are very stable. In some cases, slight secular changes have been found, but no systematic trend has been discovered up to now. In any fairly homogeneous group, there are about equal numbers of stars with lengthening or shortening periods.

With their very short periods, the *β Cephei* stars provide a very favorable case for the study of such secular effects and, indeed, there seems to be a tendency, in this group, for increasing periods.

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For the aerodynamicists, let us recall the fundamental characteristics of the variations of a typical classical cepheid, *δ Cephei* (cf. Fig. 1). The light emitted by the star varies periodically, as does its radial velocity, and the two corresponding curves are practically mirror images of each other. This correlation between the two curves extends even to many details particular to individual stars (humps on the ascending or descending branch, etc.) and suggests that there must really exist a close physical relationship between the two.

The amplitude of the light curve varies from about 0.5 m to about 2 m as the period varies from about 3 to 30 days. In the same way, the amplitude of the velocity curve varies from about 10 to 35 km/s. The spectral type, which is related to the temperature and the density, varies in phase with the light. The corresponding variation of the effective temperature, T_{eff}, for *δ Cephei*, is represented in Fig. 1*e*. This is the most direct argument for the existence of physical changes in the star. But one may also recall that the shape of the light and velocity curves, and the phase relationship between the two, exclude any interpretation of the light variation in terms of eclipses. On the other hand, the discussion of the velocity curve as being due to orbital motion leads also to highly improbable results.

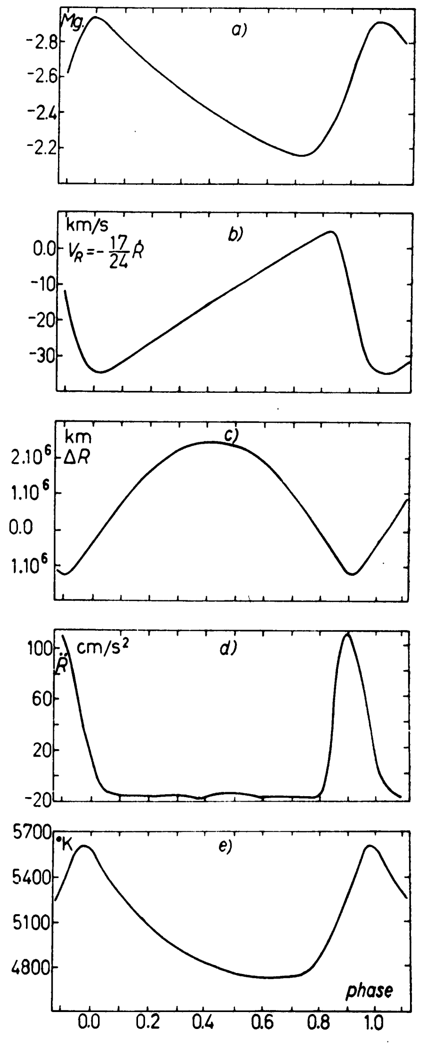

Fig. 1. – *a*) Light curve, *b*) velocity curve, *c*) radial displacement, *d*) acceleration, *e*) effective temperature, for *δ Cephei*.

An explanation of the cepheids in terms of oscillations of a single star was advocated for the first time in a paper by SHAPLEY in 1914. At that time a non-radial oscillation such as that corresponding to a spherical harmonic of degree two was mainly favored because it was thought easier to excite; for instance, by close passage of another star. However, EDDINGTON noted, that this last argument is not very significant and, on general grounds, that one would expect that a purely radial pulsation would be the easiest to maintain. He also pointed to the general spatial symmetry of the phenomenon, an argument which has been confirmed recently by the correlation, found in well observed cepheids, between amplitudes and periods. This result could hardly be expected for any other type of symmetry of the oscillation than a purely spherical one. Thus at present, it is generally believed that, at least in cepheids and probably in the *RR Lyrae* stars, the variations are due to radial pulsations: the star expands and contracts periodically.

In that case, if we change the sign of the radial velocity curve and multiply its ordinates by a factor 24/17 to correct for the averaging over the visible hemisphere and the limb darkening, we get the rate of variation of the radius of the star, R, which, integrated with respect to the time, yields the variation δR of the radius (cf. Fig. 1*c*). In classical cepheids, the relative amplitude $(\delta R/R)_A$ varies from 0.05 to 0.1 at most, while in *RR Lyrae* stars it is a little larger, and lies in the range 0.1 to 0.2. In *W Virginis* stars, it would seem that $(\delta R/R)_A$ may reach appreciably higher values, of the order of 0.3 or larger. However, as will be pointed out later, the evaluation of δR is not quite as straightforward for these stars.

On the other hand, the derivation of $-V_R$ with respect to the time yields the value of the acceleration in the atmosphere (cf. Fig. 1*d*). The results show that these layers are submitted to a strong upward force only for a short time; during most of the period, they fall regularly under a practically constant downward force.

On a simple adiabatic theory, one would expect the star to be hotter at maximum compression. With the usual opacity laws, the variation of the flux computed as a second order non-adiabatic effect would also lead to maximum luminosity at the same phase; and this certainly occurs at sufficiently great depth inside the stars. However, at the star surface, a comparison of Fig. 1*a* and 1*c* shows that maximum luminosity is delayed with respect to the radius variation, and occurs only about half-way on the ascending branch of the radius. In the same way, minimum luminosity does not occur at maximum expansion, but again about half-way on the descending branch. In other words, light maximum and minimum occur for about the same value of the radius, respectively at mid-expansion and mid-contraction. For a sinusoidal variation, this would imply a « quarter phase-lag » of the light-curve with respect to that of the radius, and the effect is usually referred to under this name. This, however, should not distract too much from the actual observations which show that, in some way, the effect is strongly bound to the general symmetry in time of the phenomenon. For instance, in *δ Cephei* the light maximum occurs only about 0.1 of the period late with respect to the maximum contraction; while the light minimum occurs more than 0.3 of the period after the maximum expansion. In other cepheids, like *ζ Geminorum*, for instance, the delays are just about reversed.

The relatively small value of $(\delta R/R)_A$, at least for classical cepheids, might suggest that a linear theory should already provide a fairly good approximation. However one must not forget that the amplitude of other more significant physical variations, such as $(\delta\varrho/\varrho)_A$, may, in the external layers, become fairly large even for small values of $(\delta R/R)_A$. In fact, the observed variations, as illustrated in Fig. 1, exhibit in most cases an appreciable anharmonicity, which may become very strong in some *RR Lyrae* stars. This shows that non-linear effects must be at work, because even if superposition of a few linear modes could, assuming commensurability of their frequencies (which in general is very unlikely), reproduce a *periodic* phenomenon with the observed asymmetry, the maintenance of this shape will depend on the non-linear coupling between these modes. Otherwise they would, in the course of time, grow or fade away individually according to the values and the signs of their damping constants.

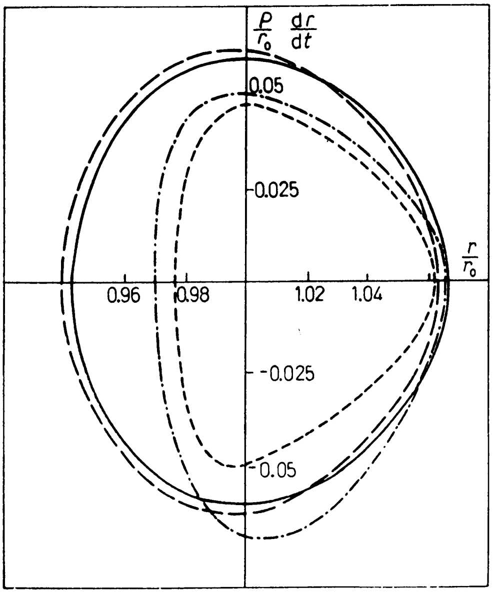

Fig. 2. – Phase diagrams for the observed pulsation of *δ Cephei* (– – –) and *η Aquilae* (–·–·–·–). The thin lines correspond to the theoretical phase diagrams for the homogeneous model (full line) and the standard model (dashed line).

An interesting example, in that respect, is provided by a few stars with multiple periods studied in great details by WALRAVEN (cf. ref. A, p. 22). He has shown that, to recover the observed variation, it is necessary to superpose on the sum of the corresponding harmonic oscillations, a distortion in phase and amplitude which is a function of the instantaneous total amplitude of that sum. Inside homogeneous groups of classical cepheids, it would also seem that the asymmetry increases with the amplitude.

This appears also clearly in a phase-plane $(\delta R, \delta\dot{R})$. There, the asymmetry of the phase-path suggests (cf. Fig. 2) that not only should non-linear terms be taken into account, but also that the non-conservative character of the system plays an important role in the shaping of the motion, at least in the external layers. This is also confirmed by the fact that the conservative non-

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linear cases that have been discussed up to now show that the observed anharmonicity cannot be recovered for finite amplitudes of the order of those observed. Very little work has been devoted to the non-conservative case (for a more complete review cf. ref. A, chap. V) and, up to now, only the linear theory, started by EDDINGTON in 1917, has been developed to some degree of completeness.

Nevertheless, one may expect that such a theory should at least yield periods of the right order of magnitude, give some indication on the run of the amplitude inside the star, and reveal the source of the incipient instability which gives rise to the pulsation.

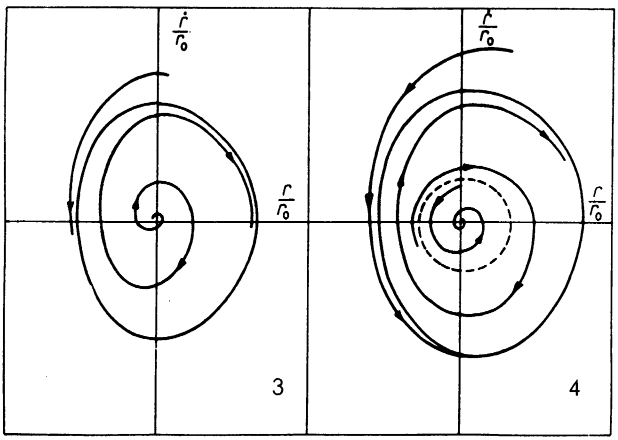

Fig. 3, 4. – Illustrations of soft self-excited oscillation (3) and hard self-excited oscillation (4). Thick full lines: stable limit-cycles; thick dashed line: unstable limit-cycle. Thin lines: phase-paths.

The last point implies that we have to deal with a *soft self-excited oscillation*; *i.e.*, one which starts from an arbitrarily small perturbation of the equilibrium state, and increases, as illustrated on Fig. 3, along a spiralling phase-path. This path tends to a stable limit-cycle along which, on the average, the dissipation factors balance exactly the exciting forces.

This is the point of view which has generally been adopted in this problem, although it would be difficult to advance reliable arguments ruling out definitely the case of *hard self-excited oscillations*. These, as illustrated in Fig. 4, require a finite perturbation capable of pushing the representative point of the system past the first unstable limit cycle. The general disregard of this possibility is probably due to the fact that it seems difficult, in the case of a star, to justify such a finite perturbation on a time-scale comparable to the period of pulsation. For a much slower perturbation such as we encounter normally in stellar evolution, non-adiabatic re-adjustments of the stellar structure would become dominant and could hardly lead to a pulsation such as the one considered here.

However, we have had some indications recently that some phases of stellar evolution could be extremely fast. In this connection, it may be worth-while to keep the possibility of hard self-excited oscillations in mind. Of course, in

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that case the linear theory would be of no avail to elucidate the source of the pulsation.

In the absence of any adequate non-linear theory, it is obvious that we have no information on the possible limit-cycles, not even on their existence; although the latter, as we shall remark briefly later, may appear more likely for some excitation mechanisms than for others.

2. – Linear radial oscillations (*).

2·1. *Development of the pulsation equations.* – In Lagrangian co-ordinates and denoting by δ the variations of the variables following the motion, the small purely radial perturbation of a gaseous spherical star around its equilibrium configuration are governed (ref. A) by the following equations expressing:

1) Conservation of mass

$$\frac{\delta\varrho}{\varrho} = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\,\delta r)\,. \tag{1}$$

2) Conservation of momentum

$$\frac{\partial^2 \partial r}{\partial t^2} = -4\,\frac{\partial r}{r}\,\frac{1}{\varrho}\,\frac{\partial p}{\partial r} - \frac{1}{\varrho}\,\frac{\partial \delta p}{\partial r}\,, \tag{2}$$

where p denotes the pressure.

3) Conservation of thermal energy

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{\mathrm{d}U}{\mathrm{d}t} - \frac{p}{\varrho^2}\,\frac{\mathrm{d}\varrho}{\mathrm{d}t}\,, \tag{3}$$

where U represents the total internal energy

$$U = \frac{3}{2}\,\frac{RT}{\bar{\mu}} + \frac{aT^4}{\varrho} + I\,, \tag{4}$$

(*) In view of the interest shown by aerodynamicists in the question of incipient instability underlying the pulsations of these variable stars, this part, which was intended primarily as an introduction to the problem of the external layers where the observable aerodynamic motions occur, has been amplified considerably. In writing up the final text due consideration has been paid to the numerous questions asked and the interesting comments made by many of them, especially Dr. CLAUSER, Dr. PETSCHEK and Dr. THOMPSON.

with R, the gas constant, $\bar{\mu}$, the mean molecular weight and I, the ionization energy. We may write

$$\frac{dQ}{dt} = \varepsilon - \frac{1}{\varrho r^2}\frac{\partial}{\partial r}(r^2 F(r)) = \varepsilon - \frac{\partial L(r)}{\partial m}, \tag{5}$$

where ε is the rate of nuclear energy generation per unit mass, $F(r)$, the flux per unit surface and $L(r)$, the total flux equal to $4\pi r^2 F(r)$.

Taking the first variation of eq. (3) and noting that, at equilibrium

$$\varepsilon = \frac{\partial L(r)}{\partial m}, \tag{6}$$

we find

$$\frac{\partial \delta p}{\partial t} = \frac{\Gamma_1 p}{\varrho}\frac{\partial \delta\varrho}{\partial t} + (\Gamma_3 - 1)\varrho\left(\delta\varepsilon - \frac{\partial \delta L}{\partial m}\right), \tag{7}$$

where Γ_1 and Γ_3 are the generalized adiabatic coefficients relating the logarithmic variations of the pressure and the temperature to those of the density for a mixture of partly ionized gas and radiation.

The variables p, ϱ and T are related by an equation of state

$$p = \frac{R\varrho T}{\bar{\mu}} + \frac{1}{3}aT^4,$$

and their variations by

$$\frac{\delta p}{p} = \beta\frac{\delta\varrho}{\varrho} - \beta\frac{\delta\bar{\mu}}{\bar{\mu}} + (4 - 3\beta)\frac{\delta T}{T}, \tag{8}$$

where β represents the ratio of the gas pressure to the total pressure. The second term on the right is important only in a region where the ionization of an abundant element is critical and may vary rapidly during the pulsation and, in that case, $\delta\bar{\mu}/\bar{\mu}$ may be expressed fairly simply in terms of $\delta\varrho/\varrho$ and $\delta T/T$.

If instead of p and ϱ, we use T and ϱ as the independent thermodynamical variables we obtain instead of eq. (7)

$$\frac{\partial \delta T}{\partial t} = \frac{(\Gamma_3 - 1)T}{\varrho}\frac{\partial \delta\varrho}{\partial t} + \frac{1}{C_v}\left(\delta\varepsilon - \frac{\partial \delta L}{\partial m}\right), \tag{9}$$

where C_v is a generalized specific heat at constant volume.

The viscous terms which have been neglected here can be shown to have a very small influence on the few first modes of radial oscillation at least as long as only molecular and radiative viscosity come into play. Furthermore, these equations are strictly valid only when radiative equilibrium prevails through the whole star. In presence of convection in some region of the star, they must be generalized and an extra equation expressing the conservation of the kinetic energy of convection must be added. As shown by COWLING, under the hypothesis of isotropic turbulent convection and adopting a mixing length picture, this generalization is not too difficult and the results have been summarized in ref. A.

The main effect is that, in such a region, F must be treated as the sum of two terms; the radiative flux

$$F_R = -\frac{4acT^3}{3\varkappa\varrho}\frac{\mathrm{d}T}{\mathrm{d}r}, \tag{10}$$

where $\varkappa$ is the opacity coefficient, a, the Stefan-Boltzmann constant and c, the velocity of light; and the convective flux which can be defined by

$$F_C = -\varrho l C C_p T\left(\frac{1}{T}\frac{\mathrm{d}T}{\mathrm{d}r} - \frac{\Gamma_2 - 1}{\Gamma_2}\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}r}\right). \tag{11}$$

C is of the order of a root mean square velocity of convection; C_p, a generalized specific heat at constant pressure; and l, a characteristic mixing length. Γ_2 is related to Γ_1 and Γ_3 defined above.

For thermonuclear reactions, we may represent ε by

$$\varepsilon = \varepsilon_0 \varrho T^{\nu}$$

and its variations by

$$\frac{\delta\varepsilon}{\varepsilon} = \frac{\delta\varrho}{\varrho} + \nu_e \frac{\delta T}{T}, \tag{12}$$

where however ν_e may be different from ν due to possible phase-delays in the variations of the abundances of the relevant elements in the course of the pulsation.

In regions in radiative equilibrium we find, according to (10) where we assume that $\varkappa = \varkappa_0 \varrho^{\chi} T^{-n}$

$$\frac{\delta L}{L} = 4\frac{\delta r}{r} + (4+n)\frac{\delta T}{T} - \chi\frac{\delta\varrho}{\varrho} + \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\delta T}{T}\right)\Big/\frac{1}{T}\frac{\mathrm{d}T}{\mathrm{d}r}, \tag{13}$$

and corresponding expressions for regions in convective equilibrium can be written down using (11).

After δT has been expressed in terms of $\delta\varrho$ and δp by means of (8) in (12)

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and (13), we may substitute these expressions in eq. (7) and then proceed to the elimination of δp and $\delta\varrho$ between this equation and eq. (1) and (2). However, this leads to a fifth order differential equation in r, which is very cumbersome. Since, furthermore, the last term in (7) is very small in the greatest part of the star, it has been customary to carry out the elimination of δp and $\delta\varrho$ between (1), (2) and (7) without taking the explicit dependence of that last term on $\delta\varrho$ and δp into account. If we separate out the time-dependence, writing

$$\frac{\delta r}{r} = \xi(r) \exp\,[i\sigma t] \,. \tag{14}$$

this leads to the usual equation

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}r^2} + \frac{\mathrm{d}\xi}{\mathrm{d}r}\left[\frac{4}{r} + \frac{1}{\Gamma_1 p}\frac{\mathrm{d}(\Gamma_1 p)}{\mathrm{d}r}\right] + \xi\left[\frac{\sigma^2\varrho}{\Gamma_1 p} + \frac{1}{r\Gamma_1 p}\frac{\mathrm{d}}{\mathrm{d}r}\{(3\Gamma_1 - 4)p\}\right] = \frac{1}{i\sigma r\Gamma_1 p}\frac{\mathrm{d}}{\mathrm{d}r}\left[(\Gamma_3 - 1)\varrho\left(\delta\varepsilon - \frac{\partial\delta L}{\partial m}\right)\right]. \tag{15}$$

The solution should satisfy the following boundary conditions: at the center $(r=0)$:

$$\delta r = r\xi = 0 \tag{16}$$

and at the surface $(r=R)$, treated as a free surface:

$$\delta p = -\Gamma_1 p\left(3\xi + r\frac{\mathrm{d}\xi}{\mathrm{d}r}\right) - \frac{(\Gamma_3 - 1)}{4\pi r^2 i\sigma}\frac{\mathrm{d}\delta L}{\mathrm{d}r} = 0\,, \tag{17}$$

since ε vanishes well below the surface.

2·2. *Adiabatic pulsation.* – Excluding a thin external layer of negligible mass, the right-hand member of (15), which represents the deviations from adiabaticity due to energy generation and conductivity, is very small. This is simply a consequence of the fact that, in a star, ε is very small compared to the internal energy per unit mass. Thus one may expect to get a good approximation for σ and the run of the relative amplitude $\xi(r)$ in the bulk of the star by dropping this term completely both in (15) and (17). The latter implies that ξ and $\mathrm{d}\xi/\mathrm{d}r$ remain finite everywhere, and we are left with a well defined eigenvalue problem with a discrete spectrum: $\sigma_0, \sigma_1, \sigma_2, \ldots$ associated with the complete set of orthogonal eigensolutions $\xi_0, \xi_1, \xi_2, \ldots$.

Provided Γ_1 does not become smaller than $\frac{4}{3}$ in an appreciable part of the mass, all the eigenvalues are positive (no dynamical instability). The first

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few modes of pulsation (corresponding to the lowest eigenvalues) have been studied numerically for many stellar models (ref. A, Table 12).

For the fundamental mode of oscillation (no node in $0 < r < R$) one can show that σ_0^2 is given by an expression of the form

$$\sigma_0^2 = C(3\bar{\Gamma}_1 - 4)\bar{\varrho}\,, \tag{18}$$

where $\bar{\varrho}$ is the mean density, $\bar{\Gamma}_1$ an appropriate average and C depends on the actual distribution of ϱ inside the star and generally increases with the central condensation $(\varrho_c/\bar{\varrho})$ of the model. Going over to the period P, (18) may be written

$$P_0 = \frac{Q_0}{\sqrt{3\bar{\Gamma}_1 - 4}}\sqrt{\frac{\bar{\varrho}_\odot}{\bar{\varrho}}}\ \text{day}\ . \tag{19}$$

where the subscript « $\odot$ » represents values for the sun, chosen as convenient reference.

On general grounds, the fundamental mode should be the easiest one to maintain. Furthermore, the period decreases rapidly as the order of the mode increases, and this makes the comparison with observations less favorable to the higher modes. In summing up the results of this comparison, we shall thus limit ourselves to the fundamental mode.

In that case, the theoretical value of Q_0 varies between about 0.07 to 0.03 as we go from the less to the most centrally condensed models that have been discussed and have a physical meaning. The most likely models for the cepheids must have a fairly high central condensation to enable the nuclear reactions to proceed at the necessary rate to explain the average luminosity. The corresponding theoretical value of Q_0 may vary between 0.03 and 0.04, depending on the exact constitution of the external layers. For the classical cepheids, assuming masses obeying the usual mass-luminosity relation, the observed value of Q_0 comes out around 0.032 to 0.035 so that the agreement is very reasonable in this case.

However, for cepheids of type II, the same hypothesis for the mass leads to $(Q_0)_{obs} \simeq 0.16$, which is quite incompatible with the theory. But evolutionary considerations suggest that the mass may be much smaller in this case; and if it is of the order of $M_\odot$, $(Q_0)_{obs}$ is reduced to 0.065, which is still very large. Taking into account the finite amplitudes, which are large in this case, may reduce $(Q_0)_{obs}$ further to perhaps 0.052, which at least falls in the range of the possible theoretical values.

In the case of the *RR Lyrae* stars (at least for groups *a* and *b*) the mass has also to be taken appreciably smaller than that derived from the usual mass-luminosity relation to bring $(Q_0)_{obs}$ in the allowed range.

For the long-period variables, $(Q_0)_{obs}$ is also somewhat large, of the order of 0.07 to 0.06, which would not be compatible with the high degree of central condensation required by the energy generation, but here also the uncertainties on the masses are considerable.

For the *β Cephei* star, on the contrary, $(Q_0)_{obs}$ comes out rather too low, of the order of 0.022. Although masses and radii are better defined here, the readjustments necessary to bring $(Q_0)_{obs}$ back in the theoretical range cannot be ruled out.

If one keeps in mind that for light variations associated with orbital motion or the rotation of a single convex body (assuming that the rotation axis is a symmetry axis of order 2) the smallest possible value of Q_0 (contact binaries or limit of rotational stability) is of the order of 0.12, one sees that the previous discussion definitely favors pulsation.

As far as the run of the relative amplitude, $\xi(r)$, inside the star is concerned, it is characterized in all reasonable models and for masses of the order of those that are significant for our problem ($M < 15\,M_\odot$), by a large increase from the center to the surface, which takes place mainly in the external half of the model. In the « standard model », which has a very moderate central condensation ($\varrho_C/\bar{\varrho} = 54$), the amplitude of the fundamental mode ξ_0 increases by about a factor 2 from $r=0$ to $r=R/2$, and increases by another factor 10 from there to the surface: $\xi_R/\xi_C \simeq 20$. But in highly centrally condensed models ($\varrho_C/\bar{\varrho} \simeq 10^6$), which are probably more significant for many of the variables considered, ξ_R/ξ_C may reach values as high as 10^4 to 10^6 depending on whether the external half is predominantly in convective or in radiative equilibrium.

This behaviour becomes more and more pronounced as one goes to higher and higher modes, the amplitude remaining fairly small up to the most external node, and increasing then very abruptly to a large value at the surface.

The behaviour of $\delta\varrho/\varrho$ and $\delta T/T$ is qualitatively similar except that they increase even more rapidly from the center to the surface.

2·3. *Non-adiabatic pulsation.* – We must now take into account the second member of eq. (15) and the complete boundary condition (17). Since $p=0$ at $r=R$ the latter implies that

$$\frac{\mathrm{d}\delta L}{\mathrm{d}r} \to 0 \qquad \text{at} \qquad r = R\,. \tag{20}$$

Furthermore, as we approach the surface, keeping only the terms which increase as p^{-1} and noting that the Γ's may be treated as constants there, eq. (15) becomes

$$\varrho\left[-\Gamma_1 g\frac{\mathrm{d}\xi}{\mathrm{d}r} + \xi\left(\sigma^2 - \frac{3\Gamma_1 - 4}{r}\,g\right)\right] = -\frac{\Gamma_3 - 1}{i\sigma r}\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{4\pi r^2}\frac{\mathrm{d}\delta L}{\mathrm{d}r}\right], \tag{21}$$

where g denotes the gravity $Gm(r)/r^2$. As ϱ tends toward zero at $r = R$, we also have

$$\frac{d^2\delta L}{dr^2} \to 0 \qquad \text{at} \qquad r = R \, . \tag{22}$$

Conditions (20) and (22) show that, in these very external layers, say above $r = r^*$, δL may, to a high degree of approximation, be treated as a constant. Physically, this means that the pressure, the density and the heat capacity of these layers are so small that their differential motion can no longer affect the flux.

However, as we go deeper inside the star, we reach a level—say $r = r_a$—below which the left-hand member of (15) becomes dominant. This level can be taken as fixing the upper limit of the « adiabatic interior ». It is also the level below which the last term on the right of eq. (9) becomes negligible. It occurs usually in a region where T is of the order of a few times 10^4 °K. We shall call the non-adiabatic region between r_a and r^* the « critical layer ».

Below that layer, the right-hand member of (15) may be evaluated by means of the adiabatic solution found previously and, in that part of the star which contains practically the whole mass, it will thus be purely imaginary *i.e.* in phase with $\pm v = d\delta r/dt$. This means that the main correction to the adiabatic solution will be of the nature of a damping corresponding to the addition of an imaginary part to σ, which for the k-mode may be written

$$\sigma_k = \sigma_{k,a} + i\sigma_k' \, .$$

We shall neglect here any possible modification of the real part, which will be taken equal to the adiabatic frequency.

The eigensolutions will also acquire an imaginary part corresponding to a variable phase, so that the general solution will be of the form

$$\left(\frac{\delta r}{r}\right)_k = \exp\left[-\sigma_k' t\right] \xi_{k,a}(r)\sqrt{1 + \mathrm{tg}^2\, \theta_k(r)}\, \cos\left[\sigma_{k,a} t + \theta_k(r)\right] . \tag{23}$$

Very generally, σ_k' will be small compared to $\sigma_{k,a}$; and one may evaluate it by a perturbation method, as was done first by ROSSELAND. Although due to the non-adiabatic external layers, application of the perturbation method is not quite straightforward, it yields values of σ' and $\theta(r)$ which, in practice, are significant, provided that no special circumstances, such as the ionization of an abundant element, occur in the critical layer.

However, as that case is of special interest here, we shall require a more general expression for σ'. This can be obtained from eq. (3), which shows that to maintain one unit mass in steady pulsation, we must provide, per

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period P, an amount of mechanical work

$$W_e = \int_0^P \frac{p}{\varrho_2}\frac{d\varrho}{dt}\,dt = -\int_0^P \frac{dQ}{dt}\,dt\,.$$

The integrands have to be evaluated up to the second order of small quantities, since the integrals of the first order terms vanish. This can be done by integrating once by parts the first integral, and then using for $dp/dt = d\delta p/dt$ its value from eq. (7). But, if, following EDDINGTON, we remark that the change of entropy over a complete cycle must be zero, we have directly

$$\int_0^P \frac{dS}{dt}\,dt = \int_0^P \frac{1}{T}\frac{dQ}{dt}\,dt = \int_0^P \frac{1}{T}\frac{dQ}{dt}\,dt - \int_0^P \frac{\delta T}{T}\frac{dQ}{dt}\,dt = 0\,,$$

or

$$\int_0^P \frac{dQ}{dt}\,dt = \int_0^P \frac{\delta T}{T}\frac{dQ}{dt}\,dt = \int_0^P \frac{\delta T}{T}\left(\delta\varepsilon - \frac{d\delta L}{dm}\right)dt\,,$$

using eq. (5).

From this expression of the dissipation integrated over the whole mass, it is easy to compute the damping constant for the k-mode

$$\sigma'_k = -\frac{1}{2\pi\sigma_{k,a}J_{k,a}}\int_0^M dm \int_0^{P_k}\left[\frac{\delta T}{T}\left(\delta\varepsilon - \frac{d\delta L}{dm}\right)\right]_k dt\,, \tag{24}$$

with

$$J_{k,a} = \int_0^M \xi^2_{k,a}\, r^2\, dm\,.$$

In the adiabatic interior ($m < Ma$), we may substitute the adiabatic solution in the integrand of the numerator and the time factor $\cos^2(\sigma_a t)$ may be integrated out. On the other hand, since $d\delta L/dr$ is negligible for $m > M^*$, and since ε certainly vanishes there, the expression (24) may be written

$$\sigma' = -\frac{1}{2\sigma_a^2 J_a}\int_0^{M_a}\left(\frac{\delta T}{T}\right)_a\left(\delta\varepsilon - \frac{d\delta L}{dm}\right)_a dm \; + \tag{25}$$

$$+\frac{1}{2\pi\sigma_a J}\int_{M_a}^{M^*} dm \int_0^{2\pi/\sigma_a}\left[\left(\frac{\delta T}{T}\right)_a + \left(\frac{\delta T}{T}\right)_{na}\right]\frac{d\delta L}{dm}\,dt\,,$$

where we have dropped the index k as, from now on, we shall be mainly interested in the fundamental mode. This expression is also known as the coefficient of vibrational stability, and the star is said to be vibrationally stable or unstable (overstable) depending on whether it is positive or negative.

Let us first discuss the first term on the right-hand side of (25) which, in some cases, is the only one that matters. Expression (12) shows that $\delta\varepsilon$ is always of the same sign as δT, and the energy generation contributes negatively to σ'; *i.e.* it always tends to increase the amplitude or, in otherw ords, to render the star vibrationally unstable.

As to the term in δL, an integration by parts using eq. (13) and remembering that δL vanishes at the center, gives

$$(26)\quad \int_0^{M_a}\left(\frac{\delta T}{T}\right)_a\left(\frac{\mathrm{d}\delta L}{\mathrm{d}m}\right)_a\mathrm{d}m=\left\{L\left(\frac{\delta T}{T}\right)_a\left[4\xi+(4+n)\frac{\delta T}{T}-\chi\frac{\delta\varrho}{\varrho}+\frac{(\mathrm{d}/\mathrm{d}r)(\delta T/T)}{(1/T)(\mathrm{d}T/\mathrm{d}r)}\right]_a\right\}_{M_a}-$$
$$-\int_0^{M_a}L\left[4\xi+(4+n)\frac{\delta T}{T}-\chi\frac{\delta\varrho}{\varrho}+\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\delta T}{T}\right)\Big/\frac{1}{T}\frac{\mathrm{d}T}{\mathrm{d}r}\right]_a\frac{\mathrm{d}}{\mathrm{d}m}\left(\frac{\delta T}{T}\right)_a\mathrm{d}m\,.$$

In the bracket of the integrated part, the first, third ($\chi>0$) and fourth terms are always of the opposite sign to that of δT; thus they contribute negatively to σ' and reinforce the instability. Physically, they correspond to the different factors that tend to decrease the flux at contraction: the decrease of the radiating surface, the increase of the opacity associated with its proportionality to ϱ, and the decrease of the temperature gradient; and vice-versa, at expansion. As the energy generation, these factors tend to heat up the gas at compression and to cool it at expansion.

On the other hand, the second term in the bracket with ($n>0$) has the same sign as δT, and gives a positive damping which contributes to the stability of the star. Physically it corresponds to the increased radiation power per unit surface at compression, and the decrease of the opacity associated with its dependence on a negative power of T.

As an illustration, let us assume that the first and fourth terms in the bracket amount together to about $-\delta\varrho/\varrho$, then eliminating $\delta T/T$ by means of the adiabatic part of the relation (9), the whole bracket may be written

$$(27)\qquad [(4+n)(\Gamma_3-1)-(\chi+1)]\,\frac{\delta\varrho}{\varrho}\,.$$

For the usual Kramers opacity law ($\chi=1$, $n=3.5$) and a star of fairly small mass ($\Gamma_3\to\frac{5}{3}$), the first term predominates and all together, the integrated term in (26) will have a strongly stabilizing influence. In practice, numerical computations confirm that the same situation prevails for all reasonable opacity laws provided the mass does not become very large.

The same type of reasoning may be applied to the integral in the second member of (26), the different terms having now the opposite effect because of the negative sign before the integral. Thus, on the whole, in the same conditions as above, this term has a destabilizing influence. However, due to the rapid increase of ξ_a, $(\delta T/T)_a$ and $(\delta \varrho/\varrho)_a$ from the center outwards, the integrated term will be largely preponderant; and more so for higher modes since, in that case, parts of the integral on successive regions will tend to cancel out. Thus, on the whole the « conduction » will tend to damp out the oscillation.

We must now compare this stabilizing effect to the destabilizing influence of the energy generation. The latter is limited to a very small central core where all the amplitudes are small while, as we have just seen, the external layers where the amplitudes are large are determinant for the « conduction » effect. Since these opposing influences are proportional to the squares of the amplitudes, it is understandable that, in all cases satisfying our general hypothesis (fairly small masses, small heat capacity of the non adiabatic layer) the coefficient of vibrational stability comes out positive and large. For a not unreasonable model of a cepheid, Cox (1955) found that the destabilizing influence of ε would lead to an increase of the amplitude by a factor e in an extremely long time, of the order of 10^9 years, while the damping time due to « conduction » is only of the order of 10 days. It is obvious that, in such a case, only a strong reversal of the stabilizing trend in the external layers could bring about vibrational instability.

As we go to larger and larger masses, the pressure of radiation becomes an increasingly large fraction of the total pressure and Γ_1, Γ_2 and Γ_3 decrease and tend towards $\frac{4}{3}$. This has two effects: it reduces considerably the rise of the amplitude with r (for $\Gamma \to \frac{4}{3}$, $\sigma_0 \to 0$, $\xi \to C^t$) so that the energy generation is no longer in such an unfavorable position with respect to the « conduction ». Furthermore, the stabilizing effect of the latter decreases, as can be seen from (26) when smaller and smaller values of Γ_3 are used. The result is that, for any law of thermonuclear energy generation, there is always a critical mass above which the stars become strongly vibrationally unstable (LEDOUX, 1941; SCHWARZSCHILD and HÄRM, 1959). Unfortunately, this occurs at much too high masses to be of interest for our problem; furthermore, the instability becomes so strong for small excesses of the mass above the critical value, that it seems likely to lead to strong ejection of material rather than to a regular pulsation.

In all this, we have assumed radiative equilibrium. But the presence of *limited* zones in convective equilibrium does not alter the main conclusion since the convective transfer of energy either increases at contraction and de-

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creases at expansion, thus having a stabilizing influence just as radiative conductivity (time of relaxation of convection short as compared to the period) or does not vary appreciably its effects on stability being very small (time of relaxation long).

Thus, up to now, we haven't discovered any source of vibrational instability of significance for the regular variable stars. However, as illustrated above for massive stars, a lowering of the Γ's—whatever its origin—would favor instability. Long ago, EDDINGTON noted that this could also be brought about by the ionization of an abundant element, at least in limited regions of the star. However, these regions will be so small that they will practically not affect σ or the run of ξ; so that the main effect described above for massive stars will be absent in this case. For this reason, ionization of heavy elements occurring deep in the star will not be of much help and, anyway, with the large preponderance of H and He (98 to 99 percent of the mass) now accepted, such ionization would not even modify the Γ's appreciably. This led EDDINGTON (1942) to propose a qualitative theory in which the instability responsible for the development of pulsation in cepheids was attributed to the ionization of hydrogen. However, this occurs in the critical layer r_a to r^*, where the non-adiabatic terms are very important, and it is essential to take them into account. Later quantitative discussions have failed to provide definite support for Eddington's suggestion, but they have led to some clarification of the important physical factors and to the development by ZHEVAKIN (1960) of a modified version in which the second ionization of helium now plays the fundamental role. As this occurs already deeper in the star, the non-adiabatic aspects of the phenomenon, although still significant, are perhaps not quite as essential as in Eddington's original theory.

Let us note that, due to the large separation of the ionization potentials of H and He, the corresponding ionizations occur in fairly distinct regions. In the middle of such a region (50 percent ionization), the Γ's reach fairly low values depending on the abundance of the element and its ionization potential. But even with a He-abundance by number of about 15 percent, the values of the Γ's associated with its second ionization may be as low as 1.25, according to ZHEVAKIN. In all cases, the mass of the corresponding region is too small for this local change in compressibility to affect appreciably the period of the pulsation or even the local amplitude of the displacement. However the adiabatic variation of the temperature (first term on the right-hand side of (9)) will be much reduced there, and this alone can change the sign of the quantity in brackets in (26) or (27). But it is also found (SCHATZMAN (1956), ZHEVAKIN (1960)) that n decreases considerably below the usual values ($\simeq 3$), and may even become negative in such a region—especially on the external side of it—and this enhances very much the reversal in the effect of the radiative conductivity. Furthermore, taking for instance the situation at compression, the gradient of the modified temperature will be steeper on the internal side of the ionization zone, and less steep on the external side, favoring the inflow of energy in it and hindering its escape (influence of the last term in the bracket), thus also reinforcing the instability.

Of course, we shall have, superposed on those effects, those of non-adiabaticity, a measure of which is provided in a first approximation by

$$q = \frac{[\delta L(r_a) - \delta L(r^*)]_a P}{\pi(M^* - M_a)\, C_v(\delta T)_a}, \tag{28}$$

which is the ratio of the heat accumulating in this layer due to the quasi-adiabatic effects discussed above during a quarter-cycle, to the variation of its heat content due to compressional work during the same interval.

To discuss rigorously these non-adiabatic effects, we would have to continue, for $r > r_a$, the adiabatic solution by the complete complex solution of eq. (25) after $\delta\varrho$ and δT have been expressed in terms of ξ in δL ($\delta\varepsilon = 0$, here). As mentioned under eq. (13), this leads to a complicated high order differential problem. Furthermore, the variable ξ is probably not particularly appropriate because as shown by EDDINGTON and confirmed by ZHEVAKIN (1960), it is affected very little by the non-adiabatic terms, its imaginary part (or its phase-shift across the region, cf. (23)) remaining very small. As a consequence, δp will also remain very close to the adiabatic solution so that, from (7) and (9), the non-adiabatic component of δT will satisfy the following approximate equation

$$\frac{\partial}{\partial t}\left(\frac{\delta T}{T}\right)_{\text{n.a.}} = -\frac{1}{\Gamma_1 C_v T}\left[\Gamma_1 - \frac{(\Gamma_3 - 1)^2 \beta C_v}{(\gamma - 1)c_v}\right]\frac{\partial \delta L}{\partial m} \simeq -\frac{1}{\Gamma_1 C_v T}\,\frac{\partial \delta L}{\partial m}. \tag{29}$$

In this respect, one may verify from the continuity equation that a very small non-adiabatic component $\xi_{\text{n.a.}}$ is sufficient to cause a fairly large readjustment of $(\delta\varrho/\varrho)$, capable of compensating $(\delta T/T)_{\text{n.a.}}$ without any appreciable change in $\delta p/p$.

If we assume with EDDINGTON that the dominating term on the right of (13) is the one in $\delta T/T$, and that, in the critical layer $(r_a - r^*)$, the non-adiabatic part of $\delta T/T$ is largest, eq. (29) becomes

$$\frac{\partial}{\partial t}\left(\frac{\delta L}{L}\right) \simeq -\frac{L(4+n)}{\Gamma_1 C_v T}\,\frac{\partial}{\partial m}\left(\frac{\delta L}{L}\right), \tag{30}$$

which admits a solution

$$\frac{\delta L}{L} = \left(\frac{\delta L}{L}\right)_{M_a} \cos\left[\sigma_a t + \varphi(m)\right], \tag{31}$$

with a phase-lag

$$\varphi(m) = -\frac{2\pi \int\limits_{M_a}^{m} \Gamma_1 C_v T \,\mathrm{d}m}{(4+n)LP}\,, \tag{32}$$

defined essentially as the ratio of the heat content of the layer to the total energy radiated across it in one period. This confirms our previous statement that the layers with very small heat capacity cannot affect the flux.

Let us go back now to the discussion of the vibrational stability. Taking (29) into account, the time integral involving $(\delta T/T)_{\text{n.a.}}$ in the last term of (25) cancels out; and using (31), the expression (25) of σ' may be written

$$(2\sigma_a^2 J_a)\sigma' = -\int\limits_0^{M_a} \left(\frac{\delta T}{T}\right)_a \delta\varepsilon_a \,\mathrm{d}m - \int\limits_0^{M_a} (\delta L)_a \frac{\mathrm{d}}{\mathrm{d}m}\left(\frac{\delta T}{T}\right)_a \mathrm{d}m + \tag{33}$$

$$+ \left[\left(\frac{\delta T}{T}\right)_a\right]_{M^*} (\delta L)_{M_a} \cos\,[\varphi(M^*)] - \int\limits_{M_a}^{M^*} (\delta L)_{M_a} \cos\,[\varphi(m)] \frac{\mathrm{d}}{\mathrm{d}m}\left(\frac{\delta T}{T}\right)_a \mathrm{d}m\,.$$

As in our discussion of the quasi-adiabatic part of (25) or of (26), the first two terms in (30) contribute to the instability while the integrated stabilizing term in (26) is replaced here by the third term proportional to $\cos\,[\varphi(M^*)]$. As long as $|\varphi(M^*)|$ is $<\pi/2$, it is still positive and reinforces the stability.

But as we increase the heat capacity (or C_v) of the layer $(M^* - M_a)$ (and the ionization of an abundant element is the only way to achieve this), $|\varphi(M^*)|$ also increases; and the contribution of the third term to the stability decreases until for $|\varphi(M^*)| = \pi/2$, it vanishes altogether. In that case, the last term has a small stabilizing influence $[\mathrm{d}(\delta T/T)/\mathrm{d}m < 0$ in the region where $\varphi(m)$ is small$]$ which may at most balance the second term in (33). If $|\varphi(M^*)|$ increases above $\pi/2$, the third term now reinforces the instability, but the stabilizing contribution of the last term increases too $[\mathrm{d}(\delta T/T)/\mathrm{d}m > 0$ towards the top of the critical region where $\cos\varphi(m) < 0]$ and EDDINGTON thought that this indicated that the minimum dissipation (or maximum instability) would occur for $\varphi(M^*)$ in the vicinity of $\pi/2$. As one may well admit that pulsation occurs only when this condition of maximum instability is realized, this would, at the same time, explain the observed phase-lag between displacement and the flux in cepheids. Furthermore it would open the way to an explanation of their period-luminosity relation (or of their distribution in a narrow band in the H-R diagram) since, apart from the mass-luminosity relation and the period-mass-radius relation provided by (19), this condition of maximum instability (or $\varphi(M^*) = \pi/2$) would furnish a third relation between L, M and R and thus permit the elimination between the three equa-

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tions of two of the parameters say M and R (or, in the H-R diagram, of M and P).

However, we may note immediately that in Eddington's scheme, the final source of instability is still the first term in (33) due to the energy generation; and we have seen already that, for likely models of supergiants like the cepheids, this yields a much too small rate of increase of the amplitude. Thus what we really want to find is not just the cancellation of the dissipation but a negative value for it capable of bringing any small pulsation up to an appreciable amplitude in a reasonable time.

First, we could try to verify whether Eddington's views on maximum instability occurring for $\varphi(M^*) \simeq \pi/2$ are correct. It seems that when $\varphi(M^*)$ becomes greater than $\pi/2$, the destabilizing influence of the integrated term in (33) may very well dominate any possible stabilizing effects of the last term; and it may be worth-while to investigate this point, even if it means losing the right phase-lag in this approximation.

However, Eddington's approach suffers from other difficulties. Even in the ionization zone of hydrogen (nearest to the surface and strongest non-adiabatic effects), it is not at all certain that the non-adiabatic terms are so dominant as to justify the corresponding hypothesis in the establishment of (30). Furthermore, this equation rests also on the assumption that the term in $\delta T/T$ is largest in eq. (13). But in the ionization zone, Γ_3 and n tend to become so small that, at least in the quasi-adiabatic approximation, this term is far from dominant. In particular, the term proportional to the gradient of δT may become large close to the extremities of the critical layer.

We then suspect that the variation of $(\delta L/L)$ in the critical layer may indeed be very different from the simple phase-shift given by (31), and that it is going to be necessary to solve the non-adiabatic equation much more carefully. Furthermore, convection tends to get established in those layers of low Γ's, and if it takes an appreciable part in the energy transport, it will certainly decrease the instability.

Although no detailed discussions of the non-adiabatic effects of the ionization zone of hydrogen exist, one is tempted to conclude from Schatzman's quasi-adiabatic treatment (1956), when corrected by the introduction of an appropriate limit M^* to the integrals in σ (cf. ref. A, Section 69), that it occurs too far out to provide the necessary instability.

This favors Zhevakin's point of view that the source of instability should be looked for deeper in the star, in the region of the second helium ionization. According to him, although convection arises in this layer, the superadiabatic excess of the radiative gradient remains so small that the flux is still mainly transported by radiation, so that δL is still given by (13). In that case, ZHEVAKIN finds that vibrational instability prevails for a whole range of values of q, as given by (28), corresponding to all possible phase-lags from practically

0 to 180° depending on the ratio of the mass of the critical layer to the mass above it, a circumstance in which he sees the possibility of explaining practically every type of observed stellar variability. However, no account has been taken of the first ionization zone of helium, or of that of hydrogen which, despite the fact that they may not suffice to cause instability, may still affect the phase-lag appreciably. One should note also that, from his latest computations (ZHEVAKIN, 1960), it appears that already a large fraction of the total negative dissipation arises below what he calls the critical layer of the second helium ionization.

In general, although ZHEVAKIN has devised an ingenious algebraic method based on the subdivision of the star into discrete layers to solve his non-adiabatic equations in the external zone, his models and the application of his method itself (for instance the whole critical layer is one of the discrete shells) are still very rough, so that the results fail to be completely convincing.

COX (1958) has also discussed the problen of the non-adiabatic layers, trying to formalize some of its aspects; for instance, by assimilating the effect of the critical ionization to a sudden and large drop in δL, so as to permit a simple mathematical treatment in terms of the parameter $\varphi(M^*)$ as defined by (32). Later COX and WHITNEY (1958) and COX (1959), using this formalism, found that, if the region of the second helium ionization is responsible for the sudden change in δL, then the condition $\varphi(M^*) = \pi/2$, used as a criterion for maximum instability, is compatible with the observed period-luminosity relations for cepheids of types I and II.

Since then, new investigations by COX (1960), which provide certainly the best treatment of the non-adiabatic layers ever attempted, have confirmed the destabilizing effect of the second helium ionization zone for reasonable abundances of helium (15 percent, by number) assuming that radiative transfer is dominant there. However, the negative dissipation in that region is of the same order as the estimated positive dissipation in the interior so that his results are somewhat unconclusive as far as the overall vibrational instability of the star is concerned. Nevertheless, it may be noted that the condition of minimum total dissipation in the star (this time without any assumption as to the corresponding phase-lag $\varphi(M^*)$) does again lead to a fairly satisfactory period-luminosity relation for the Population I. cepheids. Due to the low surface gravities of the type II cepheids, this mechanism does not seem to work in their case.

Let us add that all these computations have been made for very idealized models of the external envelopes and neglect the effects of the first helium and hydrogen ionization. While the consequences of the first point are difficult to foresee, it may be reasonable to expect that, if taken properly into account, the second may improve some of the results and increase somewhat the negative dissipation.

Apart from the complexity of the rigorous non-adiabatic equations, one major difficulty resides in the fact that we do not have, at present, any reliable model for the interior of these variables and this may affect considerably the gradient of the different perturbations δr, $\delta \varrho$, δT in the external layers.

Nevertheless, the results reported above show definitely that the ionization of an abundant element in the external non-adiabatic layers, with the corresponding changes in the values of the Γ's, of the opacity and of the temperature gradients, may lead to the accumulation of heat at the expense of the radiative flux in that layer during the phases of compression, and thus produce instability if the heat capacity of the layer including the displacement of ionization equilibrium is large enough.

A general advantage of negative dissipation as a source of the vibrational instability responsible for the pulsation is that its effects have a natural limit. In our case, once the amplitude is large enough so that, at compression, practically all the atoms primitively in a critical stage of ionization are completely ionized, the source of the instability vanishes and positive dissipation takes over, limiting the amplitude to some finite value.

However, a word of caution may be in order. All the previous discussions were supposed to refer to the fundamental mode of oscillation. But the behavior of the first mode or even the second is not so different in the external layers from that of the fundamental mode; and the arguments, as far as they have been developed, would apply just as well to any of these modes. The viscous dissipation is not very much higher either for these first few modes (COUNSON, LEDOUX, SIMON, 1956) and, once excited, it will be difficult to get rid of them, although no traces of such modes are found in most of the observed light or velocity curves. This seems to be one of the fundamental difficulties associated with pushing the source of the instability far out into the external layers.

3. – The atmospheric problem.

All the observations refer directly to the atmospheric layers, and it is obvious that our views on the general phenomenon depend strictly on a correct interpretation of these observations. However, if one admits that the observed radial velocity fields and light-variations are conclusive evidence of pulsations, the preceding sections may lead him to think that the atmosphere plays a negligible role in the problem since its mass and its heat capacity are so small that it can neither affect the period $P = 2\pi/\sigma$ (cf. eq. 18) nor the coefficient of vibrational stability σ' (cf. eq. 25).

But this is not quite true, because the boundary conditions depend very much on what we believe is the structure of the most external layers. In the preceding discussion, we have always assumed that the pressure vanishes at some point, $r = R$, insuring perfect reflection there. If this is correct, the oscillations, at least in the linear approximation, should have essentially the character of a standing wave right through to the surface except for the very small phase-shift $\theta(r)$ due to non-adiabaticity (cf. eq. 23).

If on the other hand the star had no such sharp boundary but would fade out gradually into the surrounding interstellar medium, the wave could take a marked progressive character in the external layers and an appreciable part of the wave energy would be lost to that medium, increasing appreciably the damping (cf. ref. A, section 68; SCHWARZSCHILD and HÄRM, 1959). Not much is known of the exact circumstances under which this would occur. If we assume that the star is surrounded by an extensive hot corona with a much lower density than the atmosphere proper, an appreciable fraction of the energy could still, even at the wavelengths considered here, filter through the density discontinuity separating the atmosphere from the corona.

Furthermore, in the absence of a proper boundary, the amplitude would in the linear approximation increase indefinitely, leading unavoidably to a breakdown of this approximation, to the formation of shock-waves and to non-linear damping effects. Of course, for finite amplitudes, this type of effect may already occur in the atmosphere proper, even in the presence of a sharp boundary, and may indeed be one of the mechanisms which, in the presence of vibrational instability, stops the increase of the amplitude and stabilizes it at some finite value.

There are two approaches to this aspect of the problem. One may, for instance, build purely theoretical models, try to solve the corresponding aerodynamical problems completely and compare the solutions with the observations until some kind of fit is found. However, at the present time, this is likely to lead to a great many useless trials in all kinds of directions. Instead we may start directly from the observations, which we analyse in as much detail as possible and, if necessary, we make new observations to find out what is the actual behaviour of the external layers in the course of the pulsation. In particular, we try first to answer the two following questions, which are not quite independent:

1) Does the atmosphere, at any instant, deviate appreciably from the normal atmosphere of a non pulsating star, and what are these deviations? Or can we, at each phase, explain its properties by the usual theory of static atmospheres?

2) What is the character of the wave in the external layers (standing, progressive, shock)?

Our two sources of information are the distribution of the radiation in the continuum and the line-spectrum.

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3·1. *The continuum.* – The study of the continuum has improved continuously with the development of better and better techniques. At first it reduced essentially to the determination of a color index or at most a few color indices. This type of study culminated in the six color photoelectric photometry of STEBBINS and collaborators (STEBBINS, 1953).

Spectrophotometry, which consists in measuring the intensity of the continuous spectrum between the absorption lines at as many wave-lengths as possible, provides, in principle, an ideal method. However, as long as its application had to be made photographically, it presented in practice many limitations, and required an enormous labor in transforming from photographic density into true intensities. But direct photoelectric scanning with high resolution of the spectrum has now become possible, achieving a considerable gain in precision and speed, and we shall describe later some of the results obtained by J. B. OKE using this method.

If we denote by an asterisk, values relative to some standard star, the « relative gradient »

$$\Delta\Phi = -\,2.3\,\frac{\Delta \log\,(I_\lambda/I_\lambda^*)}{\Delta(1/\lambda)}\,,$$

is fairly constant in rather large intervals of λ and can be used to characterize the distribution of energy in at least a part of the spectrum. If the stars were radiating like black-bodies, we would have

$$\Delta\Phi = \frac{C_2}{T}\,(1-\exp\,[-\,C_2/\lambda T])^{-1} - \frac{C_2}{T^*}\,(1-\exp\,[-\,C_2/\lambda T^*])^{-1}\,,$$

with $C_2 = 14\,300$ if λ is expressed in Å and T in °K. In the range of temperatures, considered, we may write

$$\Delta\Phi \simeq \frac{C_2}{T} - \frac{C_2}{T^*}\,,$$

which defines the color temperature of the star and is very closely related to the color index C.I. $= m_{\lambda_2} - m_{\lambda_1}$, if λ_2 and λ_1 lie in the same range of wavelengths.

Around 1940, BECKER and STROHMEIER found that they could represent relatively well the distribution of intensity in cepheids at any phase by means of two gradients, corresponding respectively to the following ranges of λ: $(6\,500 \div 4\,800)$ Å and $(4\,800 \div 3\,900)$ Å. They derived corresponding color temperatures and their variations in the course of the pulsation.

These temperatures however presented considerable differences with the « radiation temperatures » defining the surface brightness of the star. As a rule, the color temperatures are larger at maximum light and smaller at minimum light than the corresponding radiation temperatures. Thus, the amplitude of the color temperature is much larger than that of the radiation temperature. This departure from black-body radiation points to a real difficulty in the straightforward interpretation of the variations in light and color in terms of the variation of the radius and temperature.

CANAVAGGIA (1949), also determined two gradients in the continuous spectrum; before (4 800 to 4 000 Å) and after (3 600 to 3 100 Å) the Balmer discontinuity for *δ Cephei*, *η Aquilae* and *ξ Geminorum*. She also evaluated the Balmer discontinuity $D = \log (I_{3700+}/I_{3700-})$ which varies more or less in parallel to the brightness of the star. These variations affect strongly the gradient in the violet, so that the color temperature derived from it has no physical meaning. On the other hand, D is very sensitive to the value of the gravity and can be used to determine its effective value, g_{eff}, at different phases.

In principle, six-color photometry (STEBBINS used λλ 10 300(I), 7 190(R), 5 700(G), 4 880(B), 4220(V) and 3 500(U)) yields a much larger amount of information on the continuous spectrum than the determination of one or two gradients since the differences between any two of the six corresponding curves may be used as so many measures of the variations in color of the star during the pulsation. However, one must keep in mind that the band-widths of the filters are quite considerable (a few hundreds Å) so that the influence of the absorption lines is not eliminated. In particular, the amplitude of the U-curves is still strongly affected by the Balmer discontinuity and, in many investigations, it cannot be used.

Furthermore, these measures are affected by the general interstellar reddening, an effect which was neglected in the earlier attempts at interpretation (HARRIS, 1954).

One of the first effects established by the six-color observations is the phase-shift between the light-curves in different colors; the maximum and the minimum, which remain equally distant, occurring progressively later as one goes from the U-curve to the I-curve, the phase retardation for this last curve amounting to about $0.05\,P$. As predicted by VAN HOOF (1943), this is really what one should expect on the simple picture of a star radiating like a black-body due to the fact that in a cepheid, according to the usual interpretation of the velocity curve, R increases while T_e goes through its maximum. This simple approach should be corrected for the black-body deviations; but WESSELINCK (1947) has shown that, qualitatively, the effect remains the same.

Even if exact quantitative agreement has not been reached, this effect and its qualitative interpretation certainly support the pulsation theory and the ordinary interpretation of the radial velocity curve as being practically identical to that of the photosphere.

But the six-color observations can yield other tests of the pulsation theory, which are all related to a very simple criterion first proposed by BAADE (1926): if a cepheid radiates like a black-body, at all phases,

$$L(t) = 4\pi R^2(t)\sigma T_e^4(t) \tag{34}$$

and if $T_e(t)$ is known throughout the cycle as a function of the phase (from spectral type or color-index), the relative variation of the radius $R(t)/R_0$ may be computed from (34) and should agree in phase and amplitude with the displacement $(R(t) - R_0)$ derived from the radial velocity curve for a value of R_0 compatible with the average luminosity L and the average effective temperature $(T_e)_0$ of the star. Of course, Baade's criterion may be formulated in terms of monochromatic luminosities as well.

The difficulties encountered in the first application (BOTTLINGER, 1928) were attributed to deviations from the black-body laws, which were confirmed by BECKER's investigations (1940). Then, the main step in the application of Baade's test is to find a correlation between the color-index C.I. $= m_{\lambda_2} - m_{\lambda_1}$ and the surface brightness b_{λ_1} expressed in magnitudes in a small interval around λ_1, say

$$b_{\lambda_1} = f(\text{C.I.})\,. \tag{35}$$

In the absence of a detailed theory of stellar atmospheres, such a relation could only have an empirical basis. BECKER himself had already obtained one by comparing ΔC.I.'s and the Δm_{λ_1}'s between the phases of maximum and minimum in a series of cepheids *admitting that, at these phases, the radii are equal* so that the Δm_{λ_1}'s reduce to the Δb_{λ_1}'s. Of course, this assumption is a consequence of the usual interpretation of the radial velocity curve. It implies already to a certain extent that the pulsation theory is correct; and that the variation of the photospheric radius, R_p, is parallel to that of the radius, R_v, characteristic of the level where the absorption lines are formed. With his relation (35) BECKER found that Baade's test was, on the average, well satisfied but the overall precision was not very high.

VAN HOOF (1943) later extended Becker's method to all pairs of phases in a given cepheid which, according to the velocity curve, have the same radius. This is sufficient to establish a relation of type (35), which can usually be written

$$\Delta b_{\lambda_1} = a_{\lambda_1}\Delta(\text{C.I.})\,. \tag{36}$$

If the variation of C.I. is known through the cycle, the variation of $(R_p)_0$

between any two phases 1 and 2 can be obtained from

$$(\Delta m_{\lambda_1})_{1,2} = -\,2.151\,\frac{(\Delta R_p)_{1,2}}{(R_p)_0} + a_{\lambda_1}(\Delta \mathrm{C.I.})_{1,2}\,, \tag{37}$$

where $(R_p)_0$ is the photospheric radius at some phase, say the mean radius. Identifying $(\Delta R_p)_{1,2}$ given by (37) to the corresponding displacement $(\Delta R_v)_{1,2}$, obtained by integration of the velocity curve, one can determine a value of $(R_p)_0$ for each pair of phases (1, 2) considered. Since the method is rather sensitive to small errors in the empirically determined values of $(\Delta m_{\lambda_1})_{1,2}$ and $(\Delta \mathrm{C.I.})_{1,2}$, an agreement in order of magnitude between the different values of $(R_p)_0$ is generally considered a satisfactory test.

WESSELINCK (1946) took a new step in considering phases of equal C.I., freeing himself of relations of the type (36), although he still assumes that Δb_{λ_1} vanishes with Δ(C.I.). Now, if 1 and 2 denote phases corresponding to the same C.I., eq. (37) yields

$$(\Delta m_{\lambda_1})_{1,2} = -\,2.151\,\frac{(\Delta R_p)_{1,2}}{(R_p)_0}\,.$$

As before a set of $(R_p)_0$ can be derived, which should be consistent.

The six-color photometry gave a new impetus to these methods; in particular the corresponding phases in Wesselinck's method could be determined by a much better match in colors (WESSELINCK 1947, STEBBINS, KRON and SMITH 1952). On the whole, in the case of the cepheids, the values of the mean radius thus obtained show a reasonable agreement with other determinations.

A combination of Van Hoof's and Wesselinck's methods, with an empirical determination of the a_λ's in the different relations (36) corresponding to Stebbins' six color observations, can lead also to the absolute values of the radius as a function of phase (OPOLSKI and KRANÈCKA 1956).

However only a theoretical discussion can reveal the complete meaning of these tests. The simplest approach is to assume that, at each phase, the atmosphere adjusts itself practically instantaneously to the radiative flux coming from the interior and to the effective gravity g_{eff}

$$g_{\mathrm{eff}} = \frac{GM}{R^2} + \ddot{R}\,, \tag{38}$$

where R and $\ddot{R}$ are the instantaneous values of the radius and the acceleration, which is supposed uniform throughout the atmosphere.

One may then build a series of static model atmospheres for an appropriate range of values of T_e and g_{eff}. Each of these models gives the flux, F_λ, as a function of frequency and a value for the Balmer discontinuity D, which is rather sensitive to g_{eff}. It remains then to pick out the models that reproduce at each phase the observed six-color magnitudes and D. Then the relative variation of the photospheric radius may be computed from

$$(m_\lambda)_i - (m_\lambda)_0 = 2.5 \log \left(\frac{F_{\lambda,0}}{F_{\lambda,i}}\right) - 5 \log \left(\frac{R_{p,i}}{R_{p,0}}\right), \tag{39}$$

where the left-hand member represents the observed magnitude difference at the wavelength λ between any pair of phases i and 0 and the F_λ's, the theoretical flux given by the appropriate models.

This can be repeated for a series of colors λ, providing an internal test on the values of $R_p/R_{p,0}$. As a rule, the observations in the ultraviolet are not used because they are strongly affected by the Balmer discontinuity. Moreover, the theoretical F_λ's should really refer, not to a given wavelength but to the total flux passing through each of Stebbins' filters.

The relative variations of the radius $R_p/R_{p,0}$ thus determined should agree in phase with those of $[R_v - (R_v)_0]$. The agreement in amplitude permits one to determine the mean radius $R_{p,0}$. Furthermore, the values of the g_{eff}'s fixed by the fitting of the models to the observations (especially D) should agree with those which can be computed from (38), provided a likely value of the mass is known.

The first attempts (HITOTUYANAGI, 1952; CANAVAGGIA and PECKER, 1952*a*, 1952*b*, 1953; LEDOUX and GRANDJEAN, 1954) to carry this test through encountered considerable difficulties. The method is essentially equivalent to a theoretical determination of the a_λ's in (36) for the different wavelengths used; and the models, apparently like the black-body, yielded too large values for these.

However, further corrections to the observations were necessary due to the effect on the continuum of the variations in intensity of the absorption lines with phase and to the interstellar reddening (HARRIS, 1954; CANAVAGGIA, 1954, 1955). As shown by WHITNEY (1955), the first factor acts in the right direction to improve the agreement between R_p and R_v, and this was partially confirmed by HITOTUYANAGI and VIJ-IYE (1956) in an investigation based on photographic spectrophotometric data.

Recently J. B. OKE (1960) has considered the problem again for *RR Lyrae* and *η Aquilae*, using photoelectric scanning of the spectrum, and he has kindly sent a detailed summary of his results in advance of publication. In the case of *RR Lyrae*, the observations were made at the Cassegrain focus of the 100 inch telescope and the resolution of the scan extending from 6000 to 3300 Å is approximately 5 Å in wavelength and 15 min in time.

HD 182487, whose energy distribution had been calibrated on an absolute scale by comparison with Vega using Code's results, was used as a standard; and monochromatic light curves of the observed differences between *RR Lyrae* and HD 182 487, outside the atmosphere, were plotted for 24 wavelengths. Corrections for absorption lines were carefully computed from intensity tracings of Sanford's 10 Å/mm spectrograms. No corrections for interstellar reddening were applied since, according to STRÖMGREN, it is very small in the *RR Lyrae* region. By adding to the corrected measurements the absolute energy distribution of HD 182 487, the absolute energy distribution in the spectrum of *RR Lyrae* was obtained at each phase.

By comparison with the absolute flux of model atmospheres computed by DE JAGER and NEVEN, effective temperatures, $T_{\text{eff}\,i}$, were determined primarily by the slope of the energy distribution to the red of $\lambda\, 4\,000$; effective gravities g_{eff} were found primarily from the Balmer discontinuity. It is believed that apart from uncertainties in the theoretical models, $\theta_{\text{eff}} = 5\,040/T_{\text{eff}}$ can be determined to within 0.005 and $\log g_{\text{eff}}$ within 0.1 to 0.2. The final effective range of T_{eff} is $(5\,900 \div 7\,200)$ °K which, according to OKE, compares favorably with other lines of evidence.

Formula (39) was then used to determine $(R_{p,i}/R_{p,0})$; and assuming $R_p \equiv R_V$, comparison with the displacements $(R_{V,i} - R_{V,0})$ derived from Sanford's radial velocity curve yielded for R_0 (identified here with $R_{\max}$)

$$R_0(\equiv R_{\max}) = (8.3 \pm 0.7) R_\odot \,.$$

The general agreement in phase and amplitude is quite good.

Finally with $(GM/R_0^2) = 475$, which corresponds to $M = 1.2\, M_\odot$, formula (38), in conjunction with the radial velocity curve differentiated to yield $\ddot{R}$, was used to compute theoretical g_{eff}'s. The comparison between these and the observed values is also very satisfactory. The value of Q_0 in the relation (19) comes out of the order of 0.03, using a mean radius $R = 7.8\, R_\odot$.

For *η Aquilae*, the method followed was essentially the same, using a correction for interstellar reddening of 0.14 in the B-V scale as determined by KRAFT. A comparison between the observed absolute distribution of the flux in the true continuum (corrected for absorption lines), and that in model atmospheres computed by CANAVAGGIA and PECKER, yielded the values of T_{eff} and g_{eff}. The temperature range found in this way is from 5 320 °K to 6 140 °K.

The comparison between the radii derived from the radial velocity curve and those derived as shown above from relation (39) with a value of the minimum radius (taken here as the standard R_0)

$$R_{\min} = 52.3 \pm 2.1\, R_\odot$$

shows a good overall agreement in phase and amplitude. The value of M_V turns out to be equal to -3.85 ± 0.3 at mean light which is in good agreement with Kraft's value of ~ -3.6.

The essential difference with most of the previous treatments is the smaller range in T_{eff} and it is this, as already noted by GRANDJEAN and LEDOUX, which brings the agreement between the two radii.

The most obvious conclusion is that ordinary stellar model atmospheres computed at each phase, assuming radiative and hydrostatic equilibrium, seem to reproduce the observations and that there is no need to distinguish very carefully between the photospheric radius R_p and the radius R_V of the level where the weak absorption lines are formed.

Since the models used are computed for an assumed constant total flux through the atmosphere, it would seem difficult to reconcile these conclusions with the passing through the atmosphere of a strong isothermal shock-wave, which would act as a local moving heat source, creating a difference between the flux F_1 on the external side and F_2 on the internal side of the order of (cf. ref. A, section 92)

$$F_1 - F_2 = \frac{p_1}{2}\left(1 + \frac{p_2}{p_1}\right)(v_2 - v_1) , \tag{40}$$

which can become quite appreciable. Let us note too, in that respect, that in the case of *RR Lyrae*, OKE has observed no extra continuous emission in the U-V during the abrupt ascending branch phase but, in this respect, his observations may not refer to the most favorable moment in the 41-day cycle. The non-linear transfer of momentum associated with such a wave would also affect the effective gravity g_{eff}. However these investigations relate to stars with continuous velocity curves and no or very weak emission lines (except *RR Lyrae*, at some phases) and the conclusions may not necessarily apply to stars like *W Virginis*.

A few other hydrodynamical inferences can also be drawn from this type of investigation. Once model atmospheres have been fitted at a series of phases as described above, they can be used to follow, throughout a cycle, the Lagrangian variations in density of a given element defined by the constant mass above it.

In the case of η *Aquilae*, results derived by this method (LEDOUX and GRANDJEAN, 1955) were in good qualitative agreement with those obtained previously by M. and B. SCHWARZSCHILD and W. S. ADAMS (1948) from a quantitative discussion of the intensities of the lines of neutral and ionized iron at 20 selected phases. In both cases, the variation of the density showed a much better agreement in shape and phase with the velocity curve than with the displacement curve, suggesting a predominant progressive character for the wave. However, as shown by these authors, a straightforward inter-

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pretation on this basis leads to an impossibly high speed of propagation in the atmosphere.

This led them to consider a composite atmosphere with a hot corona, in which the wave can travel outward at high speed. Although, on this picture, the wave becomes again stationary at some depth in the atmosphere, there is an intermediate layer where it resembles that in the corona and, assuming that the observations refer to that level, the above explanation may then become acceptable.

However, apart from the fact that this is based on a linear theory, it is likely that a more realistic treatment would show that this layer is extremely shallow and situated at such a small optical depth that it cannot affect the lines. Furthermore, this explanation could certainly not be extended to the results derived from model atmospheres, since some of them refer to particles at fairly great optical depths. In this case, a comparison between different particles shows that the phase of the density variation could be explained by very small variations with height of the phase and (or) the amplitude of the velocity curves, which may correspond mainly to an increase of the anharmonicity with height.

3·2. *Evidence from the line spectrum.* – The bifurcation, through Doppler shifts, of the spectral lines of *W Virginis*, *RR Lyrae*, and *β Cephei* stars is convincing evidence for the existence of very steep gradients in their atmospheres. In fact the presence of emission lines associated with upward-moving gas indicates the existence of a high temperature region such as might be associated with a shock discontinuity.

Unfortunately the data available are grossly inadequate for the determination of the actual velocity fields. The principal limitation is the fact that an emergent pencil of radiation integrates information from a wide range of atmospheric levels and conceals the details of its source. Furthermore, the purity of spectra presently available is insufficient to allow a precise determination of the profiles of spectrum lines.

In sum, the *a priori* expectation that valuable data on the velocity field might be obtained from profiles and velocity-shifts of spectral lines has not been fulfilled.

In view of the paucity of reliable data, there seems little point in interrupting this text with a discussion of discordances between the opinions of various astronomers. We shall however abstract some relevant papers in the Appendix.

R. G. TESKE (1960) has constructed *theoretical* line profiles for pulsating atmospheres with and without velocity gradients. His aim was to provide a theoretical framework within which to examine available data and to suggest what is desired from future observations.

Two of his results are of particular significance for the interpretation of spectroscopic data on pulsating atmospheres.

He has shown that the curve of growth for lines formed by pure absorption is affected by a velocity gradient in a manner which roughly mimics the effect of microturbulence.

Line profiles produced by a pulsating atmosphere are asymmetric, since the line-wings are formed at greater depths than the line-centers. An asymmetry will exist whether or not there is a vertical gradient of velocity. In the absence of such a gradient, this asymmetry is produced by the fact that observed radiation represents an integration over the spherical stellar disc. We shall designate the asymmetry arising in this case as the « zero-gradient asymmetry. »

TESKE has re-examined earlier work on the observed line asymmetries and the velocity differences between lines of different strengths, and he concludes that these two problems are quite closely related.

There are several cases (*e.g.* the *RR Lyrae* variables) in which large velocity differences are definitely observed. These can amount to several tens of km/s, and are most pronounced between the hydrogen lines and the weak iron lines. The interpretation of these differences in terms of velocity gradients has not yet been put on a quantitative basis.

In the case of small velocity-differences, the problem is complicated by the subjective nature of the data reduction. The published line-shifts are based on micrometer measures of the « position » of absorption lines. For asymmetric lines, the results will be different for the center of gravity, the minimum of intensity, or the midpoint of the wings of the absorption lines. Unfortunately, it is not clear which of these positions is actually measured by an individual measurer.

On the assumption that the micrometer measures refer to the deepest point of the line profile, TESKE shows that apparent differential velocities will be recorded even in the zero-gradient case. Further, these differences will be a function of the depth of line formation and will therefore mimic the effects of a velocity gradient. TESKE concludes that many of the « velocity differences » discussed in the literature may be produced by zero-gradient asymmetries.

Evidently more precise and objective data are needed before we can specify the origin of these small velocity differences.

4. – Some theoretical aspects of the dynamical behavior of stellar atmospheres.

In view of the virtual impossibility of a direct empirical approach to the determination of atmospheric velocity structure, we shall now consider the theoretical approach through synthesis.

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To provide a foundation for this discussion, we first describe some properties of stellar atmospheres relevant to their dynamical behavior.

4·1. *Properties of the undisturbed stellar atmosphere.* – The differential equations governing the dynamics of stellar atmospheres contain a set of dimensionless parameters whose enumeration is useful.

Let

$H = \mathscr{R}T/\mu g$ = atmospheric scale-height, μ = mean molecular weight,

$c = \sqrt{\gamma \mathscr{R}T/\mu}$ = sonic velocity in the atmosphere,

g = stellar surface gravity,

P = period of pulsation,

p = gas pressure.

Then the dimensionless ratio of scale-height to photon free path is $H\varkappa\varrho$ where $\varkappa$ is the absorption coefficient and ϱ the gas density. Although the scale-height does not vary rapidly through the atmosphere, the density does; so, consequently, does $H\varkappa\varrho$. Defining the optical depth of a layer through

$$\tau = -\int_x^\infty \varkappa\varrho \,\mathrm{d}x ,$$

and utilizing the approximate constancy of H, we see that $H\varkappa\varrho$ at any level is roughly equal to the optical depth of the level. For the observable atmosphere, therefore, $H\varkappa\varrho \simeq 1$.

A thermal parameter may be constructed in the following way. The quantity $\varkappa\varrho\sigma T^4$ is, for a «grey» gas, the rate of emission from an optically thin volume element. The ratio $p/\varkappa\varrho\sigma T^4$ is the time to radiate the thermal energy of this element at constant temperature. The ratio H/c is the time for a small perturbation to travel one scale height. Therefore the order of magnitude of the following parameter, α_T, indicates the extent to which an optically thin perturbation will be affected by radiation while travelling through the atmosphere

$$\alpha_T = \frac{\varkappa\varrho\sigma T^4}{p}\,\frac{H}{c} .$$

Using $H\varkappa\varrho \simeq 1$, this simplifies to

$$\alpha_T = \frac{\sigma T^4}{pc} .$$

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This parameter may be regarded as an index of departure from adiabaticity. Adopting a hydrogen atmosphere at $T = 5\,000°$,

$$\alpha_T \simeq \frac{5 \cdot 10^3}{p},$$

and since $10^3 < p < 10^4$ for typical pulsating atmospheres at $\tau = 1$, we see that α_T is of the order of unity. Thus, deviations from a radiative equilibrium distribution of temperature tend to smooth out in a time comparable to the acoustic delay for one scale height.

TABLE I. – *Physical properties of three pulsating stars.*

	δ Cephei	RR Lyrae	W Virginis
P (days)	5.4	0.57	17.3
R (cm)	$3.7 \cdot 10^{12}$	$4.6 \cdot 10^{11}$	$4.4 \cdot 10^{12}$
g (cm/s^2)	100	800	10
T (°K)	6000	6800	6000
H (cm)	$8 \cdot 10^9$	$1.2 \cdot 10^9$	$1 \cdot 10^{11}$
Δv (km/s)	40	100	55
c (km/s)	9.1	9.6	9.1
α_λ	50	40	14
α_D	.086	.25	.37
$(\sigma/\pi)T^4$	$2.4 \cdot 10^{10}$	$3.9 \cdot 10^{10}$	$2.4 \cdot 10^{10}$

A geometrical parameter of interest is the ratio of pulsation period to H/c, the acoustic delay time of one scale height, *i.e.*

$$\alpha_\lambda = \frac{Pc}{H}.$$

This quantity may also be considered as the ratio of the atmospheric pulsation wavelength to the atmospheric scale-height.

This quantity is also of interest in conjunction with α_T in indicating the extent to which pulsational disturbances of the temperature from the radiative distribution are suppressed by radiative transfer. Values of $\alpha_\lambda \gg 1$, such as are tabulated above, indicate that, taken grossly, the temperature distribution in a pulsating atmosphere is not very different from the equilibrium distribution. It must, of course, differ quite significantly during the brief intervals when steep velocity gradients lie near $\tau = 1$.

As an index of the degree of departure from hydrostatic equilibrium we may construct the dynamical parameter α_D through

$$\alpha_D = \frac{v}{Pg},$$

where v is the amplitude of the observed pulsation velocity-curve. Writing $v = Mc$, where M is the Mach number, and using

$$H = \frac{c^2}{\gamma g},$$

we find

$$\alpha_D = \frac{M\gamma}{\alpha_\lambda}.$$

In Table I we list some of the stellar dimensions relevant to the present discussion for three prototype stars. The stars are listed in order of increasing α_D and increasing α_λ. It is interesting to note that, observationally, this sequence is also one of increasing evidence for discontinuity in the atmospheric velocity distribution. That is, *δ Cephei* shows no discontinuity, ***RR Lyrae*** shows it only in the strongest, high-level, lines and *W Virginis* shows it for all lines. This coincidence may be interpreted in the following way (WHITNEY, 1955).

A large value of α_D indicates that the atmosphere is driven violently by the interior pulsations and deviates significantly from hydrostatic equilibrium. Thus a large α_D should be propitious for the production of velocity discontinuity.

In a crude way, the smaller the value of α_λ, the greater is the distance in wavelength that the pulsation wave must travel in traversing the atmosphere. This is also propitious for the development of shocks.

Thus, the sequence of increasing discontinuities is consistent with the related observational parameters.

4·2. *Outline of the dynamical problems.* – These problems may be listed as follows:

a) The development of a shock discontinuity from a continuous wave of moderate amplitude.

We face here the problem of relating the atmospheric wave to the conditions at the exterior limit of the nearly-adiabatic, small-amplitude pulsation of the interior. This conceptual dichotomy between the interior and the atmosphere is quite artificial, but the fact remains that it can be useful since the differential equations governing most of the interior are considerably simpler than those for the atmosphere. One specific example of the « useful misuse » of this dichotomy has been the replacing of the interior-atmosphere transition region by a solid oscillating piston. This model provides a handy boundary condition for the treatment of atmospheric waves (WHITNEY, 1956) but must give an inaccurate picture of the development of atmospheric shocks.

b) The derivation of conditions immediately behind the shock and the behavior of the ionizing hydrogen within the shock transition.

c) The description of the physical nature and spectroscopic properties of the regions of recombination and radiation behind the shock.

d) A determination of the overall temperature distribution and radiation field in an atmosphere containing a radiating shock.

Of particular interest in this regard is the effect of radiation leakage from the shock into the atmosphere lying ahead of the shock.

e) The growth or decay of a strong shock moving up along a density gradient.

In the next section we write the non-linear equations applicable to atmospheric waves and in succeeding sections we attempt preliminary discussion of some of the questions outlined above.

4·3. *The relevant equations.* – We shall restrict ourselves to the one-dimensional case, since the depth of the stellar region of interest here is considerably less than its radius of curvature. We also neglect viscosity.

In their Eulerian form, the conservation equations may be written as follows for the one-dimensional case.

Conservation of mass:

$$\frac{\partial \varrho}{\partial t} = -\frac{\partial}{\partial x}\varrho v\,. \tag{41}$$

Conservation of momentum:

$$\frac{\partial}{\partial t}\varrho v + \frac{\partial}{\partial x}\varrho v^2 = -\frac{\partial p}{\partial x} - g\varrho\,. \tag{42}$$

Conservation of energy:

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{3}{2}p + E_i + \frac{1}{2}\varrho v^2\right) &= \\ &= -\frac{\partial}{\partial x}pv - gv\varrho - && \text{(work)} \\ &-\frac{\partial}{\partial x}v\left[\left(\frac{3}{2}p + E_i + \frac{1}{2}\varrho v^2\right)\right] + && \text{(convection)} \\ &+ Q_r + Q_c\,, && \text{(radiation and conduction).} \end{aligned} \tag{43}$$

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The conservation of energy requires that the rate of change of energy per unit volume equal the sum of the rates at which work is done on the element and energy is carried into the element by mass motion, radiation and conduction. E_i is the density of excitation and ionization energy and g is the gravitational acceleration, which is directed toward negative x.

Radiative transfer:

$$\cos\theta \frac{\partial I_\nu(x, \theta)}{\partial x} = -\varkappa_\nu \varrho I_\nu + j_\nu \varrho \,. \tag{44}$$

$I_\nu(x, \theta)$ is the specific intensity of radiation at x in a direction making an angle θ with the x axis, $\varkappa_\nu$ and j_ν are the mass absorption and emission coefficients.

Integration of the transfer equation over all solid angles and all frequencies leads to the following expression for Q_r, the radiative input term

$$Q_r = 2\pi \frac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty \mathrm{d}\nu \int_{-1}^{1} I_\nu \cos\theta (\mathrm{d}\cos\theta) = 4\pi\varrho \int_0^\infty (\varkappa_\nu \bar{I}_\nu - j_\nu)\, \mathrm{d}\nu \,, \tag{45}$$

where the bars denote mean values with respect to direction.

It may be convenient to cast Q_r into another form which is obtained by formal integration of the transfer equation. Define the differential of optical depth by

$$\mathrm{d}\tau_\nu = -\varkappa_\nu \varrho \, \mathrm{d}x \,, \tag{46}$$

divide the transfer equation by $\varkappa_\nu \varrho$ and integrate. The following expression then results,

$$Q_r = \frac{\mathrm{d}}{\mathrm{d}x} \int_0^\infty F_{\nu r} \, \mathrm{d}\nu \,, \tag{47}$$

where

$$F_{\nu r} = 2\pi \int_\tau^\infty S_\nu(t) E_2(t - \tau)\, \mathrm{d}t - 2\pi \int_0^\tau S_\nu(t) E_2(\tau - t)\, \mathrm{d}t \,, \tag{48}$$

$$S_\nu = j_\nu / \varkappa_\nu \,. \tag{49}$$

4·4. *The isothermal equations and a kinematic model for W Virginis.* – Temperature fluctuations in a stellar atmosphere are smoothed very rapidly by radiative transfer of energy. This fact, and the great simplification thereby introduced, made it appropriate to adopt the isothermal form of the dynamical equations for preliminary theoretical studies of atmospheric pulsations.

The use of the isothermal approximation clearly eliminates the possibility of directly synthesizing the atmospheric thermal structure and radiation field. However the purely kinematical results have been useful in two ways. On the one hand, they have provided a model with which to interpret several aspects of the velocity curves of *W Virginis* stars. On the other hand they provide an approximate velocity field with which to study the relative importance of the compressional work term and the radiation term of the energy equation.

The differential equations obtained by setting $T = \text{constant}$ may be directly integrated from adopted boundary conditions. The integration is, however, subject to numerical instability associated with the generation of parasitic and exponentially-growing oscillatory solutions of the finite-difference equations (Courant-Friedrichs instability). These oscillations may be suppressed over most of the flow by proper choice of mesh intervals, but they inevitably appear in regions of steep gradients.

A more powerful method of eliminating this difficulty is through the use of the method of characteristics.

From eq. (41) and (42), *i.e.*,

$$\frac{\partial \varrho}{\partial t} + \frac{\partial}{\partial x}\varrho v = 0\,,$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\varrho}\frac{\partial p}{\partial x} - g\,,$$

we may derive the dimensionless equations

$$\frac{\partial}{\partial \tau}R + (u+1)\frac{\partial}{\partial \eta}R = 0\,, \qquad R = u + \ln\varrho + \tau\,; \tag{50}$$

$$\frac{\partial}{\partial \tau}S + (u-1)\frac{\partial}{\partial \eta}S = 0\,, \qquad S = u - \ln\varrho + \tau\,. \tag{51}$$

Where we have defined the new variables through

$$\tau = t\,\frac{c}{H}\,,$$

$$\eta = x/H\,,$$

$$u = v/c\,.$$

The density is in units of the density at some convenient point, *e.g.*, $(x = 0,\ t = 0)$.

Evidently R and S are constant along lines of slope $u + 1$ and $u - 1$, respectively. Further, the values of u and ϱ may be derived from R, S and τ

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through inversion of the defining relations. The shock transitions are handled with the Rankine-Hugoniot relations, and the integrations proceed in a straightforward manner.

Results of integrations for three sets of boundary conditions have been published (WHITNEY, 1955). These integrations started from $v = 0$ throughout the gas for $t < 0$. A piston at the bottom of the atmosphere was stationary for $t < 0$ and driven sinusoidally with a velocity amplitude equal to sonic velocity for $t > 0$. The atmosphere was assumed to extend infinitely far upwards.

The initial shock developed quite close to the piston and travelled out into the stationary atmosphere with an increasing velocity. The motion of subsequent shocks was complicated by the preceding disturbances of the atmosphere and the integration were, perforce, stopped after several piston oscillations. By this time, the flow within several scale heights of the piston had apparently relaxed to a nearly cyclic state, although at greater heights the transients associated with the onset of pulsation still dominated the flow.

It is, of course, very dangerous to draw any conclusions about the cyclic flow in pulsating atmospheres from these limited « initial value » solutions. However, two features of the results are worth mentioning.

First, these large amplitude waves transferred momentum to the atmosphere, « levitating » it and reducing the average density gradient by a factor four. We may expect a similar effect in a real pulsating atmosphere. However, the piston frequencies employed in these integrations were higher than are appropriate for real atmospheres. Lautman's results (1956) indicate that at lower frequencies the effect may be less pronounced.

Second, the atmospheric density distribution, which would have resembled a N-wave if the initial medium had been homogeneous, has a stairway profile. That is, the density was nearly constant, at any given time, between succeeding shock discontinuities.

The following kinematic model was synthesized from these partial results and applied to *W Vir* (WHITNEY, 1956):

1) Shock fronts are generated and they travel upward with constant strength.

2) All particles between shocks are subjected to the same downward acceleration, and this acceleration is independent of time. Thus, particle trajectories are parabolic on the space-time plane. In passing through a shock front, each particle has its velocity impulsively reversed, but its speed unchanged.

On this model, the shock velocity is constant with respect to the center of mass of the star as well as with respect to the matter flowing into it from above.

Four quantities specify the model:

a) The velocity of shock propagation. This is chosen so that the velocity discontinuity at the shock is compatible with the amplitude of the observed velocity curve.

b) The oscillation period in dimensionless units. This parameter is equivalent to α_λ, defined in Section **4**·1.

c) The downward acceleration. This quantity is derived from the previous two and the requirement that there be no net flow of matter.

These quantities specify the velocity field. From the continuity equation and the fourth quantity, the density at an arbitrary point, the density distribution may be determined for the entire atmosphere.

Relation of this model to the atmosphere of *W Virginis* entailed a lengthy process of successive approximations to the spatial distribution of temperature, total pressure, electron pressure and optical depth. Empirical data were taken from ABT's (1954) study.

The most serious approximation involved in construction of this model is that the relation between temperature and optical depth is that appropriate to radiative equilibrium. The two sources of departure from radiative equilibrium are 1) the time variations of radiant flux from the stellar interior into the bottom of the atmosphere and 2) the conversion of kinetic energy into thermal energy represented by the work terms in the energy equation.

The second of these effects will be important over a limited interval of depth within the atmosphere. That is, the radiation produced by compressional heating across a shock discontinuity will lead to significant departure from radiative equilibrium in the immediate neighborhood of the shock. We have not yet investigated this situation and cannot apply the present model to *W Virginis* during those phases when the shock is traversing the observable atmosphere.

However, when the shock has passed above the region of spectrum formation or before it has entered this region from below, the assumption of radiative equilibrium will not be grossly in error. The compressional term in the energy equation will then be negligible in the region of spectrum formation.

Also calculation of the relaxation of the temperature distribution in a static atmosphere (WHITNEY, 1955*a*) after an abrupt change of the radiant flux from below indicates relaxation times of the order of an hour or less. This is a small fraction of the pulsation period for *W Virginis* stars (10 to 20 days), so that the variation of the radiant flux into the bottom of the atmosphere will not in itself lead to significant departures from radiative equilibrium.

One specific result of the construction of this model for *W Virginis* of particular value relates to the velocity dispersion within the region of line formation. In sum the situation is the following:

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When the shock rises into the visible portion of the atmosphere, the pressure and opacity in the rising material are very high. Therefore the dispersion of pulsation velocity within the visible material is small. At this phase we observe spectrum lines from matter above and below the shock and doubled lines are produced. As matter streams down through the rising shock the total optical thickness of the overlying atmosphere diminishes, finally becoming so small that this matter becomes invisible. Concurrently, the opacity of the rising gas behind the shock decreases and a wide region behind the shock contributes to the spectrum. Within this region of spectrum formation the dispersion of pulsation velocity becomes as large as 10 to 20 km/s.

This dispersion of pulsation velocity will broaden and strengthen the absorption lines in much the same manner as « micro-turbulence ». Such an effect has indeed been detected by ABT (1954) although he has interpreted it as a real turbulent wake behing the shocks. The present model suggests that the observations can be interpreted quite naturally in terms of the gradient of the pulsational velocity.

All that can conservatively be said about the proposed model for *W Virginis* is that it is not in obvious conflict with available data, it reproduces the gross feature of the observed velocity curve, and it has given some insight into the structure of a pulsating atmosphere.

4·5. *Previous work on the structure of shocks in the presence of radiation.* – In this section we shall comment briefly on published investigations concerning the effects of radiation on flow behind shocks.

This work has been carried out with the assumption of asymptotic approach to a uniform medium at great distances from the shock. The conservation equations may be written in the steady form, dropping the explicit time-dependence,

$$\frac{\mathrm{d}}{\mathrm{d}x}\varrho v = 0\,, \tag{52}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\varrho v^2 + p) = 0\,, \tag{53}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{v\left(\frac{3}{2}p + E_i + \frac{1}{2}\varrho v^2\right)\right\} + \frac{\mathrm{d}}{\mathrm{d}x}pv - Q = 0\,, \tag{54}$$

where E_i designates the energy associated with internal degrees of freedom.

Setting $E_i = 0$ and integrating over a discontinuity of infinitesimal thickness leads to the usual Rankine-Hugoniot (R-H) relations.

SACHS (1946) has introduced radiation pressure and energy density, writing

$$p = p_g + \tfrac{1}{3}aT^4$$

$$E_i = aT^4$$

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and explicitly neglecting transfer of energy by radiation. He has derived the resulting transition relations, but since radiative transfer is important in the stellar atmosphere his results are not applicable to the present problem.

SEN and GUESS (1957) and MARSHAK (1958) have attempted to include radiative transfer and they evaluated Q from the radiative diffusion approximation. They neglected the radiative contributions to pressure and energy density. The former authors were interested in the structure of the shock transition, but the diffusion approximation

$$F_r = -\frac{4acT^3}{3\varkappa\varrho}\frac{\mathrm{d}T}{\mathrm{d}x}$$

is not appropriate for such a study. The condition for applicability of the diffusion approximation, *i.e.* that the temperature does not vary significantly over distances comparable to $1/\varkappa\varrho$, the photon mean-free-path, cannot be fulfilled in the region of the shock transition.

The radiative term must be evaluated properly from the transfer equation. The obvious mathematical difficulty introduced by the radiative term is that conditions at any point are, in principle, influenced by conditions throughout the medium. The governing equations cannot rigorously be reduced to purely differential form but must contain an integral term.

In a recent investigation explicitly recognizing this feature of radiative transfer, KUBIKOWSKI (1959) has evaluated the energy term from eq. (48). However, his reduction of the resulting expression to a tractable form is incorrect and, as noted in the next section, his results are grossly in error.

4·6. *Some features of strong shocks in hydrogen.* – We wish now to investigate some properties of strong shock transitions in an atmosphere composed of hydrogen. Points of principal interest are *a*) the effects of ionization within the shock transition; *b*) the temperature distribution behind the shock in the presence of radiative cooling; and *c*) the effects of radiative transfer to the region ahead of the shock.

Our interest lies in shocks moving parallel to a negative density-gradient, so in principle we should not employ the steady-state equations derived for a medium which is homogeneous at great distances from the shock. However, since the thickness of the region of interest is generally much less than one scale height of atmospheric density, we shall take advantage of the great simplification afforded by assuming the medium to be homogeneous in front of the shock, neglecting the gravitational forces.

Although the phenomena across and behind the shock should be treated with a unified theory, it is convenient to consider separately the following regions: 1) a homogeneous region in front of the shock; 2) the transition region for the external degrees of freedom in which conduction and viscosity

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determine the physical conditions; 3) the relaxation region for internal degrees of freedom in which ionization and excitation take place; 4) the region in which radiative cooling takes place.

Although these distinctions are artificial, they are convenient; and the only one whose validity may be seriously questioned is that between regions 3) and 4).

Region 2), in which the external degrees of freedom relax but the internal degrees remain unexcited, will be a region of high temperature, but it is unobservable astronomically. As the gas moves into region 3), ionization takes place, the temperature will drop, and the density will rise. Since we assume the gas in front of the shock to be neutral, no radiation can occur until excitation and ionization have commenced. We shall therefore assume that *no* radiation occurs until ionization equilibrium is established. This model represents a considerable simplification of the physical situation, but it should be adequate for the present survey. In particular we note that radiation from the shock will dissociate the gas entering the shock and this may have a profound influence.

Let subscript 1 designate the undisturbed gas flowing into the shock and subscript 3 designate the gas at the back of region 3) after ionization has taken place.

In a co-ordinate system moving with the shock front, the R-H transition relations for the steady-state may be written

$$\varrho_3 v_3 = \varrho_1 v_1 \,, \tag{55}$$

$$p_3 + \varrho_3 v_3^2 = p_1 + \varrho_1 v_1^2 \,, \tag{56}$$

$$v_3(\tfrac{5}{2}p_3 + E_{i3} + \tfrac{1}{2}\varrho_3 v_3^2) = v_1(\tfrac{5}{2}p_1 + E_{i1} + \tfrac{1}{2}\varrho_1 v_1^2) \,. \tag{57}$$

E_i is the density of ionization energy, $E_i = N_e \chi$, and we neglect excitation energy.

We shall reduce these equations with the following approximations.

1) The pre-shock gas is not ionized, so $E_{i1} = 0$.

2) The shock is sufficiently strong that we may set $\varrho_1 v_1^2 \gg p_1$. This is equivalent to assuming that the square of the Mach number, M, is much greater than unity. For cases of astronomical interest $10 < M^2 < 100$, so we indeed deal with strong shocks.

3) For the strong shock we also have $\varrho_1 v_1^2 \gg \varrho_3 v_3^2$. From these approximations we may derive

$$p_3 = \varrho_1 v_1^2 \,, \tag{58}$$

$$\begin{aligned} \tfrac{5}{2} p_3 v_3 + N_e \chi v_3 &= \tfrac{5}{2} \varrho_1 v_1^2 v_3 + N_e \chi v_3 \\ &= \tfrac{1}{2} \varrho_1 v_1^3 \,. \end{aligned} \tag{59}$$

If p_1, v_1, and ϱ_1 are chosen we may solve for conditions behind the shock provided we employ the ionization relation $N_e = N_e(p, T)$. For we may write

$$v_3 = \frac{\frac{1}{2}\varrho_1 v_1^3}{\frac{5}{2}\varrho_1 v_1^2 + N_e\chi}, \tag{60}$$

and with the equation of continuity we find

$$\varrho_3 = \varrho_1 \frac{v_1}{v_3} = \frac{\frac{5}{2}\varrho_1 v_1^2 + N_e\chi}{\frac{1}{2}v_1^2}. \tag{61}$$

We find p_3 from eq. (58) and guess a value of T_3. From eq. (61) and the ionization relation we find ϱ_3. The value so derived must satisfy the equation of state for pure hydrogen

$$p = N_e kT + \mathscr{R}\varrho T$$

and we adjust T_3 until a self-consistent set of ϱ_3, v_3, T_3, p_3 is obtained.

Results of a set of calculations for conditions representative of classical cepheids are given in the table below.

TABLE II. – *Shock transition in pure hydrogen*

$T_1 = 5000°$, $\log p_1 = 3.0$, $\log \varrho_1 = -8.62$,

v_1 km/s	M_1^2	Temperature $T_2 \cdot 10^{-3}$	Temperature $T_3 \cdot 10^{-3}$	$\log(p_3/p_1)$	$\log(\varrho_3/\varrho_1)$	α	$\log N_e$
30	13	25	10	1.33	.96	.15	15.32
40	23	41	11.5	1.58	1.08	.36	15.81
50	36	61	13.2	1.78	1.15	.61	16.11
60	52	86	15.5	1.94	1.17	.90	16.29
65	61	100	27	2.01	1.11	.99	16.27
70	71	116	27.5	2.07	2.07	1.0	16.19

Assumed conditions in front of the shock are given at the head of the table. The quantities M_1^2 and α are defined by

$$M_1^2 = \frac{v_1^2}{c_1^2}, \qquad c_1^2 = \frac{5}{3}\mathscr{R}T_1,$$

$$\alpha = \frac{N_e}{N_H + N_e}, \qquad N_H = \text{density of neutral hydrogen}.$$

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Two values are given for the temperature corresponding to each velocity. T_2 is the kinetic temperature immediately behind the shock before the onset of ionization. This temperature is computed from the R-H relations for a non-ionizing gas with $\gamma = \frac{5}{3}$. T_3 is the temperature to which the gas relaxes before radiation occurs but after the ionization takes its equilibrium value.

These tables may be summarized as follows.

1) The density ratio between region 3) and 1) is about ten and is insensitive to the velocity. For velocities higher than those tabulated this ratio will fall to the value four appropriate to a non-ionizing gas.

2) Due to the energy sink provided by ionization, the temperature rise is decreased by a factor of from 2.5 to 5 within the range of this table.

3) Despite the ionization, the pressure ratio increases roughly as M_1^2 and is about 30 per cent greater than the value corresponding to no ionization.

We turn now to the structure of region 4) in which radiative loss of energy proceeds.

We can make a preliminary estimate of the dimensions of region 4) in the following manner. Assume that the loss of energy takes place at constant pressure and temperature and neglect all terms in the energy equation (43) except

$$\frac{\mathrm{d}(vE_i)}{\mathrm{d}x} = Q\,. \tag{62}$$

The isothermal assumption is suggested by the large heat reservoir provided by the ionization energy.

The velocity entering this expression is relative to the shock; we designate it by v_4 and assume it to be constant. Then eq. (62) becomes

$$v_4 \chi \frac{\mathrm{d}N_e}{\mathrm{d}x} = Q\,. \tag{63}$$

Finally, neglect energy *absorbed* in the cooling region, and for the emission term we neglect all but the free-bound emission, setting

$$Q = -\,.535 \cdot 10^{-21} \frac{N_e^2}{T^{\frac{1}{2}}} \text{ erg s}^{-1}\text{ cm}^{-3}\,.$$

Letting N_{e_i} be the electron density at the beginning of the radiating region,

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and N_e the electron density at a distance Δx from this point, we find

$$\frac{\Delta x}{v_4} = 4.0 \cdot 10^{10} \frac{T^{\frac{1}{2}}}{N_{e_i}} \left(\frac{N_{e_i}}{N_e} - 1 \right).$$

We assume $v_4 \simeq v_3 = v_1 \varrho_1 / \varrho_3$.

Arbitrarily setting $N_{e_i}/N_e = 11$, we may define the characteristic time, τ, and the characteristic thickness L as those dimensions corresponding to virtually complete radiation of the ionization energy. Then

$$\tau = \frac{\Delta x}{v_4} = 4 \cdot 10^{11} \frac{T^{\frac{1}{2}}}{N_{e_i}}, \tag{64}$$

$$L = \tau v_4 = \tau v_1 \, \varrho_1 / \varrho_3 . \tag{65}$$

From the results given in Table II we derive the characteristic dimensions given in Table III.

TABLE III.

v	τ (s)	L (m)
30	.02	60
50	.003	9
70	.004	25

For the purpose of comparison with recent work by KUBIKOWSKI (1959), we have applied these equations to the conditions examined by him. The resulting cooling time is two orders of magnitude shorter than Kubikowski's value of 180 seconds.

We now proceed with a more detailed analysis of region 4) through numerical integration of the conservation equations.

We construct a one-dimensional space mesh labelling the points with index i, such that

$$x_i = i \, \Delta x , \qquad (i = 3, 4, \ldots). \tag{66}$$

The initial point x_3 is identified with the rear of region 3), *i.e.*, the point at which ionization equilibrium has been established and radiation commences. The mass and momentum equation may be integrated to give, again for the steady-state,

$$\varrho_{i+1} v_{i+1} = \varrho_i v_i , \tag{67}$$

$$p_{i+1} + \varrho_{i+1} (v_{i+1})^2 = p_i + \varrho_i (v_i)^2 . \tag{68}$$

Define the quantity u through

$$u = \tfrac{5}{2}p + E_i + \tfrac{1}{2}\varrho v^2 . \tag{69}$$

The energy equation then integrates to

$$v_{i+1}u_{i+1} = v_i u_i - \int_{x_i}^{x_{i+1}} Q(x)\,\mathrm{d}x . \tag{70}$$

The net quantity $Q(x)$ is the net rate at which energy is absorbed, per unit volume, at the point x. For the present case, $Q(x)$ may be considered as the difference between the following two rates:

1) The rate, I_{fb}, at which energy is lost from the element through radiative recombinations. We again adopt

$$I_{fb} = .535\cdot 10^{-21}\,\frac{N_e^2}{T^{\frac{1}{2}}} . \tag{71}$$

2) The rate, A_{bf}, at which energy is absorbed from the radiation field by the inverse process of radiative ionizations. The evaluation of the ionization rate requires determination of the radiation field. This, in turn, involves integration over the radiation sources distributed throughout the atmosphere. However, since the shock region of interest is optically thin, we shall neglect the exchange of energy between regions of the shock, in a first approximation. We assume that the remaining atmosphere is isothermal and constitutes a radiation bath equivalent to a black-body at temperature T_1.

With this model we may evaluate the rate of ionization from the rate of recombination through the principle of detailed balancing. Write

$$A_{bf} = C(T_1)N_{\mathrm{H}} , \tag{72}$$

and set

$$\begin{cases} I_{fb}(T_1) = A_{bf}(T_1) , \\ N_e = N_p . \end{cases} \tag{73}$$

Then

$$C(T_1) = \frac{.535\cdot 10^{-21}}{T_1^{\frac{1}{2}}}\,(N_e^2/N_{\mathrm{H}})_* .$$

The quantity $(N_e^2/N_{\mathrm{H}})_*$ is a function of T alone and the asterisk denotes that it is to be evaluated from the Saha equation for the temperature T_1.

Combining these relations we have

$$(75)\qquad Q = A_{bf} - I_{fb} = .535\cdot 10^{-21}\,\frac{N_e^2}{T^{\frac{1}{2}}}\left\{\frac{T^{\frac{1}{2}}}{T_1^{\frac{1}{2}}}\left(\frac{N_e^2}{N_H}\right)_* \frac{N_H}{N_e^2} - 1\right\}.$$

Throughout most of the relaxation process, the gas departs significantly from equilibrium and

$$(76)\qquad \frac{N_e^2}{N_H} \ll \left(\frac{N_e^2}{N_H}\right)_*.$$

PETSCHEK and BYRON (1957) have emphasized that significant differences between the electron and atom kinetic temperature can arise during relaxation to ionization equilibrium through electron-atom collisional ionizations behind a shock. This ionization process cools the electron gas so severely that the rate of approach to equilibrium behind strong shocks in argon is determined not by the collisional ionization cross-section but rather by the rate at which energy can be fed to the electron gas by elastic electron-atom collisions.

This cooling of the electron gas will be important in region 3) in which the hydrogen is being collisionally ionized. We do not explicitly treat region 3) in this discussion, however.

In the radiative cooling region under discussion here, the electron concentration is *decreasing* through recombination, and there is no strong tendency toward a difference between atom and electron temperatures ([1]).

Integrations of the complete set of conservation equations have been carried over the radiative cooling regions behind the shocks of velocity 30 km/s, 50 km/s, and 70 km/s listed in Table IV. The corresponding initial conditions are listed below.

TABLE IV. – *Initial conditions for radiative cooling.*

v_1	ϱ (g/cm³)	p (dyn/cm²)	T (°K)	N_e	x	v (km/s)
30	$2.19\cdot 10^{-8}$	$2.14\cdot 10^{4}$	10000	$2.09\cdot 10^{15}$	.15	3.29
50	$3.39\cdot 10^{-8}$	$6.02\cdot 10^{4}$	13200	$1.29\cdot 10^{16}$	.61	3.54
70	$2.57\cdot 10^{-8}$	$1.18\cdot 10^{5}$	27500	$1.55\cdot 10^{16}$	1.00	6.54

Profiles of the temperature, density and Q are given in Fig. 5-6 for these cases ([2]). The pressure profile is not plotted since in all cases the pressure increased by less than ten percent during the cooling.

([1]) The energy-dependence of the recombination cross-section will produce a very small difference between these temperatures, but we neglect it.

([2]) Examination of these profiles shows that thermal conduction is completely trivial in this cooling region.

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We recall the fact, noted above, that the collisional relaxation to ionization equilibrium also takes place at essentially constant pressure (cf. PETSCHEK and BYRON, 1957). Put in different terms, the pressure distribution, in the

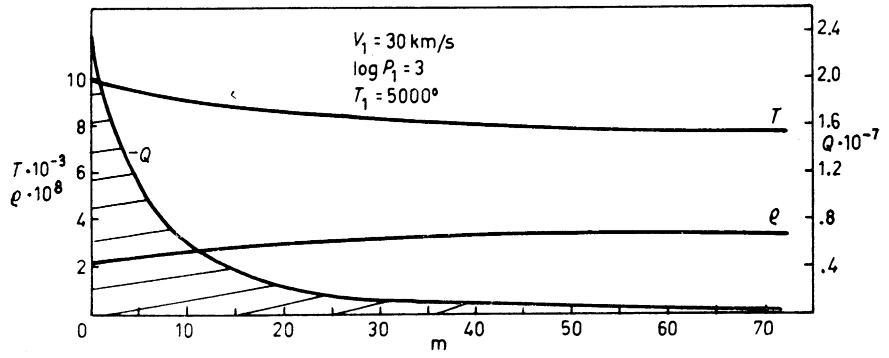

Fig. 5. – Spatial profiles of density, temperature, and emission rate behind a steady shock front in pure hydrogen. Shock velocity, 30 km/s; temperature and pressure in front of the shock, 5000 °K and 10^3 dyn/cm^2, respectively. The coordinate system moves with the shock front, which is at the left border. Matter streams to the right.

presence of ionization and radiative recombination, is very nearly what it would be for a perfect gas in adiabatic flow.

These profiles indicate that for 30 km/s $< v_1 <$ 50 km/s and for the assumed initial pressure, eq. (65) gives a good approximation to the width of the cooling region. For $v = 70$ km/s the cooling proceeds more rapidly than given by this equation. This occurs because the mean temperature is lower and the mean electron density is higher during cooling than assumed in eq. (65).

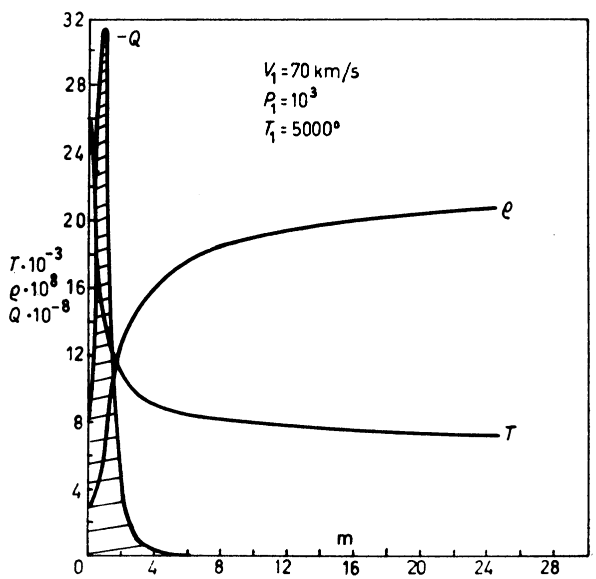

Fig. 6. – Same as Fig. 5 except that the shock velocity has been increased to 70 km/s. Note that the free-bound emission rate initially increases and then decreases with distance from the shock front. Also the thickness of the high-temperature region is reduced at the higher velocity.

Table V summarizes some pertinent properties of the cooling process.

Columns two and three give the enthalpy and the density of ionization energy immediately behind the shock. For the strongest shock these quantities become nearly equal.

TABLE V. – *Some properties of the radiating shocks.*

v_1	$\frac{5}{2}p$	N_e	$\frac{1}{2}\varrho_1 v_1^3$	$\int Q\,\mathrm{d}x$	$\dot{N}_\nu$	$\dot{N}_H$
30	$5.35\cdot10^4$	$4.6\cdot10^5$	$3.2\cdot10^{10}$	$1.9\cdot10^{10}$	$.5\cdot10^{21}$	$4.5\cdot10^{21}$
50	$1.50\cdot10^5$	$2.8\cdot10^5$	$1.5\cdot10^{11}$	$1.1\cdot10^{11}$	$3\ \cdot10^{21}$	$7.5\cdot10^{21}$
70	$2.95\cdot10^5$	$3.4\cdot10^5$	$4.2\cdot10^{11}$	$3.7\cdot10^{11}$	$1\ \cdot10^{22}$	$1.0\cdot10^{22}$

We have evaluated the total emission per $\mathrm{cm}^2\cdot\mathrm{s}$ from the shock and these values are labelled $\int Q\,\mathrm{d}x$. As is expected, the total emission approaches $\frac{1}{2}\varrho_1 v_1^3$, the rate of kinetic energy flow across the shock.

Finally the last two columns compare the flux of ionizing photons, $\dot{N}_\nu$, across the shock from behind, with the flux of hydrogen atoms, $\dot{N}_H$, across the shock from in front. We have assumed that the average photon energy, $h\nu \simeq 14$ eV, and have written

$$\dot{N}_\nu = \frac{\int Q\,\mathrm{d}x}{2h_\nu},$$

where the factor $\frac{1}{2}$ accounts for the fact that this fraction of photons will be emitted toward the shock front.

If we make the limiting assumption that 1) the material in front of the shock is sufficiently opaque that all photons will produce ionization and 2) no recombination takes place ahead of the shock, we see that material streaming into the weaker shocks will receive only a slight additional ionization from the shock radiation. For the strongest shock, however, the material flowing into the shock will be highly pre-ionized.

The pre-ionization will have a profound influence on the structure of the shock. Therefore the present calculations for $v_1 \geqslant 50$ km/s must be grossly incorrect and should be used simply to indicate the importance of pre-ionization.

APPENDIX

We assemble here the best data available concerning the velocity fields in pulsating atmospheres. We shall restrict attention to data obtained with high-dispersion spectra, since results based on moderate dispersion are unreliable.

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The literature being rather limited, we have included brief summaries of the relevant papers rather than summarizing all of the data in a single discussion. Comments by the present writers accompany several of the summaries.

1. – Summaries of papers on classical cepheids.

1. SANFORD, R. F.

Ref.: *Ap. J.*, **123**, 201 (1956).

Subject: Examination of 10 Å/mm 100″ coude plates of two classical cepheids *SV Vul* ($P = 45$d) and *T Mon* ($P = 27$d).

Presents velocity curves for FeI lines and residual for Fe II, Ti II, Sr II, H. The author makes two comments on Fe I curve.

1) Velocity extrema lag light extrema by

$$T\ Mon \qquad \Delta\Phi = 0.044 \text{ periods},$$

$$SV\ Vul \qquad \Delta\Phi = 0.072 \text{ periods}.$$

This lag and its increase with period had been previously noted, esp. by Joy (1937).

2) The velocity curves are very nearly repetitive.

Concerning differential velocities the author makes the following comments.

a) No significant Δv between Fe II and Fe I.

b) Hydrogen shows positive residual of 50 km/s just before light maximum, *i.e.* there is a phase lag of 5 days in maximum positive velocity relative to Fe I.

c) H_α differs from other H lines by plus 20 km/s during decline of light.

d) Velocity residuals of H and metallic lines are correlated with line widths and the author suggests that both are produced by an extension and deepening of redward line wings.

Although the line profiles are generally symmetric, the H_α and H_β profiles show redward displaced cores near maximum light.

Emission lines of Ca II are strong in *T Mon* from phases $22^d\,28$ to $26^d\,895$ but are not apparent at other phases. Interpretation of the emission is complicated by the presence of an interstellar absorption line.

2. KRAFT, R. P.

Ref.: *P.A.S.P.*, **68**, 137 (1956).

Subject: Examination of plates of *X Cyg*, a classical cepheid of period 16 days.

The author concludes that the absorption lines of low excitation potential are doubled at $\Phi = 0^p.82$, *i.e.*, on the rising branch of the light curve. The separation of the Fe I line (λ 3923) is 38 km/s. The H_γ core is displaced

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42 km/s from the λ 3923 line. The undisplaced component is not observed and the author suggests, by analogy with *W Vir*, that it has been obliterated by emission.

The author comments that the luminosity expected from the shock producing the line-doubling is comparable with the total luminosity of the star. The atmospheric shock may therefore produce the hump on the light curve.

3. GRANDJEAN, J.

Ref.: *Mem. Acad. Roy. Belgique, Cl. Sci.*, Coll. No. 8, vol. **22**, fasc. 7 (1956).

Subject: A study of differential motions in the atmospheres of the classical cepheids η *Aql* and ζ *Gem* based on 2.9 Å/mm Mt. Wilson 100″ coude plates. Summarizes earlier work with the following comments:

« Tirer de toutes ces investigations une conclusion nette, n'est pas chose aisée.... L'aspect assez chaotique de ces observations est très explicable quand on pense que ces déterminations ont été faites à l'aide de spectrogrammes ayant une dispersion de 40 Å par mm à H_γ. A cette dispersion, la majorité des raies du spectre sont des « blends » non résolus et la variation d'intensité des composantes peut donner des déplacements apparents de la raie qui interprétés en termes d'effets Doppler, donnent des vitesses relatives n'ayant aucune réalité physique. Les travaux de JACOBSEN realisés à plus grande dispersion, 13 Å par mm à H_γ, ne sont guère plus convaincants. Néanmoins, ils excluent les grands écarts prétendument mis en évidence précédemment, car, à cette dispersion (3 fois plus élevée), de tels effets devraient apparaître nettement, or, il n'en est rien. Quant à des effets moins importants, rien de certain n'est établi; la dispersion est toujours faible, les erreurs probables relativement grandes, et ce que nous avons dit précédemment au sujet des « blends » reste vrai en partie. »

The author measured radial velocities of a large number of lines using a Zeiss comparator and concludes that small velocity gradients definitely exist (amounting to $(3 \div 4)$ km/s over the depth of formation of the lines measured). He comments that it is tempting to interpret these velocity differences as resulting from a slight phase lag of the higher layers relative to the lower but adds that an increase of velocity amplitude with height could also explain the differences.

On the basis of model atmospheres and calculated depths of line formation the author estimates that the velocity gradient is of the order of

$$1.5 \cdot 10^{-6} \text{ km/s km} .$$

Comment. – Recent theoretical work by R. G. TESKE indicates that line shifts dependent on line intensity can be produced in the spectra of stars even in the absence of real differential velocities.

4. JACOBSEN, T. S.

Ref.: *Pub. Dom. Ast. Obs.* **10,** No. 6 (1956).

Subject: A spectrographic study of η *Aql* and δ *Cep* based on plates of 20 Å/mm dispersion.

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The author finds essentially no velocity differences among photospheric absorption lines. Further, the core of H_δ seems to follow these lines closely. The Ca II *H* and *K* lines show large departure from the photospheric lines and also periodically show marked asymmetry.

In δ *Cep* the Ca II *K*-line has a larger amplitude but nearly the same phase as the photospheric lines. In η *Aql*, the amplitude of the *K*-line velocity curve is greater than that of the photospheric curve and the *K*-line curve is distorted and appears to be retarded by about $0^p.1$.

5. HERBIG, G. H.

Ref.: *Ap. J.*, **116**, 369 (1952)

Subject: Study of 10 Å/mm 100″ coude plates of *S Sge* ($P = 8.4$ days) with special emphasis on emission lines of Ca II.

Comments that Ca II emission is present in many classical Cepheids. In *S Sge* it occurs during the middle of ascending branch of light curve and is transitory, lasting two days or less.

The author remarks that there is considerable evidence that the emission lines in long period (red) variables originate below the layers of formation of the cores of the absorption lines. He notes the following points of similarity between the long period variable emission and that of *S Sge*.

1) The emission is diffuse and displaced to the violet relative to the absorption lines.

2) An increase of absolute strength of the emission with phase is shown for *S Sge* and is apparent in the long period variables from measured relative intensities.

3) In the long period variables $I(H) > I(K)$ of Ca II as opposed to the relative transition probabilities. This is an exaggeration of the tendency to equality $I(H) \simeq I(K)$ in *S Sge*.

4) Ca II *K* emission shows a greater violet displacement than Ca II *H* in both types of stars.

The most significant differences between the phenomena in the two types of stars is the greater strength in the long period variables and the fact that emission occurs during the descending branch of the light curve.

The author discusses self absorption in a simple two-layer atmosphere and indicates that a low-lying origin of the emission can account for the presence of the Ca II infrared emission in the absence of detectable *H* and *K* emission.

6. SCHWARZSCHILD, M. and B., ADAMS, W. S.

Ref.: *Ap. J.*, **108**, 207 (1948).

Subject: A study of the spectrum of η *Aql* ($P = 7.2$ days) based on 2.9 Å/mm plates.

Measurements of line cores show very little evidence for differential velocities. However:

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« ... the principal change in the spectrum of η *Aql* at minimum of light consists in a widening of many of the lines on the violet side [outward motion], thus producing a lack of symmetry, ... ».

A modification of standard curve-of-growth technique, using residual intensities in line cores, was employed. The results are similar to those of WALRAVEN for δ *Cep* as regards temperature and electron pressure variations.

The authors derived the (Lagrangian) variation of density following a particle finding the density to be some 4 times greater at the phase of maximum light and maximum outward velocity than at minimum light. This result could be very significant as it is the only available result of its kind.

The authors introduce the derived velocity and density amplitudes into the Rankine-Hugoniot equation for the isothermal case and state that the density amplitude is smaller by a factor 10 than is compatible with the velocity amplitude ($\Delta v = 10 \times$ velocity of sound). The authors justify the application of the isothermal R-H equation through noting that 1) the temperature variation is small relative to the density variation and 2) the atmospheric motion appears to have the form of a running wave since maximum density occurs at the time of maximum outward velocity.

Through the application of the linearized equation of motion the authors construct a composite model with a high temperature upper layer and show that the discrepancy can be eliminated.

Comments. – The quantitative results of this paper are derived with the aid of a number of approximations and assumptions. The amplitude of the density variations must be considered uncertain by a factor 2 or 3.

Further the use of linearized hydrodynamic equations to construct a composite pulsation model has not received adequate justification.

7. WALRAVEN, TH.

Ref.: *Pub. Astrom. Inst. Univ. of Amsterdam*, No. 8 (1948).

Subject: A spectrophotometric study of δ *Cep* based on 10 Å/mm plates.

The author did not measure velocities. His representative tracings show that blending of adjacent spectral lines is an exceedingly serious problem even at the dispersion employed. He employed the standard curve-of-growth procedure with the following results:

1) The hydrogen lines are greatly enhanced near maximum light, at which time they are too strong by a factor 10 on the basis of classical calculations using the excitation temperature derived from the metals.

2) Excitation and ionization temperatures vary by about (500 ÷ 1 000) °K during a cycle of pulsation depending on the spectral lines analysed. Electron pressure varies by a factor 10 during a cycle, reaching maximum with the temperature, *i.e.* near maximum light.

3) During increasing light, there is a pronounced relative weakening of lines of atoms and ions for which the 2-nd ionization potential is less than about 14 eV, suggesting an increase of Lyman radiation.

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8. BRUCK, H. A. and GREEN, H. E.

Ref.: *M. N.*, **376**, 101 (1941).

Subject: Study of radial velocities of *δ Cep* based on 13 Å/mm plates.

The authors quote references on earlier work concerned with differential velocities in classical cepheids. The early work was based largely on 40 Å/mm plates and was highly discordant. They note that of the 200 lines in the region λλ 4250÷4650, only 10% are sufficiently free of blends to merit measuring for velocities. Their paper concludes with the following:

« Cambridge four-prism spectrograms of *δ Cep* do not indicate any definite relative diplacements between lines of different atmospheric level, confirming thereby earlier negative results of PETRIE and JACOBSEN. There may exist small differences between the velocity curves of groups of neutral iron and ionized titanium lines and there is evidence for the existence of relative shifts in the case of the neutral calcium line λ 4425.6 ».

2. – Summaries of papers on RR Lyrae stars.

1. TIFFT, W. G. and H. J. SMITH.

Ref.: *Ap. J.*, **127**, 591 (1958).

Subject: A comprehensive study of *T Sext*, the first « C » type *RR Lyr* star to have been studied in detail.

The authors note the following characteristics of C-type variables;

1) Period about 6 hours.

2) Relatively sinusoidal light curve with an amplitude of less than 0.6 mag.

Photometric, spectrographic and radial-velocity data are presented and the following points are brought out.

1) The maximum velocity of approach lags the maximum light by 0.1 periods « in general agreement with results for the few *RR Lyr* stars of type *a* for which velocity curves are available. »

2) The light and color curves show real deviation from one cycle to another. The light curve fluctuates .03 mag.; the U-B color fluctuates by 0.1 mag. These two types of fluctuations appear to be uncorrelated.

3) A hump is evident on the ascending branch of the light curve. This hump is correlated with a hump on the (B-V) color curve but not the (U-B). We note that the (B-V) color is a measure of the slope of the continuous spectrum to the red of the Balmer limit while the (U-B) is sensitive to the Balmer discontinuity.

The authors estimate that the integral of the hump represents one percent of the total light emitted during a cycle. Therefore it contains about 10^{38} erg and is equivalent to the kinetic energy of a mass of $5 \cdot 10^{25}$ g moving 20 km/s. (The amplitude of the observed velocity-curve is about 30 km/s.) If spread over the entire surface of the star, this mass contains 15 g/cm² and has an optical thickness of the order of unity in the continuum. The authors suggest that the hump may be produced by loss of thermal energy following compression of the atmosphere. Present data are not sufficient to discuss the presence or absence of transitory emission lines of the type observed during rising light in *RR Lyr*.

The authors conclude with the following remarks:

« In summary, a qualitative description of the changes in *T Sext* can be formulated as follows: A reversing layer which produces the visible spectral lines has been thrown out from the photosphere by phase 0.9. It begins to fall back around phase 0.3. Meanwhile the light of the star has increased to maximum and begins to decline, largely as a function of the behavior of the underlying photospheric layers. At phase 0.6 a new pulsation wave is emerging from the interior, brightening the star. Collision between the top of this wave and the infalling layer liberates the energy needed to produce the hump at phase 0.85, which is superimposed on the over-all increase in light coming from the lower layers; and the cycle begins to repeat. Since the slope of the rising branch is rather sensitive to the timing and energy content of the collision and since the timing, in turn, depends on two essentially independent periodicities — namely, the pulsation frequency of the star and the interval required for the upper layer to execute a rise and fall under gravitational and possibly magnetic forces as well — we are not surprised to find variations between cycles, and these most pronounced on the rising branch of the light-curve. Indeed, such a picture might provide at least a partial explanation of the common occurrence of beat periods among intrinsic variables, on the grounds that the two independent periods normally would not be perfectly synchronized.

« Clearly, this rough model, although plausible, is quite preliminary. It points up the need for high-dispersion spectra of *T Sext* and for detailed atmospheric calculations. For example, if collision of layers occurs, a thin zone of high temperature must be present, which in turn could produce emission lines, as well as the observed strongly blue color of the hump. Higher-dispersion spectra taken near phase 0.85 would be useful in looking for the emission lines. In this connection, Balmer-series emission lines have been found by Struve (1947, 1948) and Sanford (1948) in *RR Lyr* at certain phases of its long period. It is of considerable interest that the emission lines are most pronounced near the long-period phase $\psi = 0.1$, at which phase, in turn the light curve, having lost its type *a* peak, shows the most pronounced *T Sext*-like hump on the rising branch (Walraven 1949). Further, within each cycle the appearance of emission lines coincides with the light-curve hump. Because of the over-all similarity of the two stars, these observations of *RR Lyr* lend some support to the inferential picture derived above for *T Sext* ».

2. Hardie, R. H.

Ref.: *Ap. J.*, **122**, 256 (1955).
Subject: A study of *RR Lyr* in three colors.

The equipment employed in this investigation is described in *Ap. J.*, **114**, 522 (1951).

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This paper presents photoelectric light curves adjusted to superposition 0.3 mag. above the minimum of light. The (B-V) color curve has the same form as the light curve. The (U-B) curve shows a short-lived maximum at the midpoint of the ascending branch of the light curve. This maximum, indicating a decrease of the Balmer discontinuity, coincides with an observed weakening of the hydrogen line-absorption and with the transitory emission observed at H_α.

3. SANFORD, R. F.

Ref.: *Ap. J.*, **109**, 208 (1949).

Subject: An examination of high-dispersion plates of *RR Lyr*.

The dispersion of the plates employed in this investigation (10 Å/mm, photographic; 20 Å/mm visual) is greater than that employed by STRUVE and BLAAUW (40 Å/mm), and distinct doubling of the hydrogen lines was found. Sanford interprets this doubling in the following terms:

« If the velocity variations of *RR Lyr* are caused by pulsations of its atmosphere, the behavior of the velocities from H and K and from H_α would seem to indicate that an atmospheric wave suddenly started outward with the maximum velocity of expansion at the time of maximum light and then slowed down, reaching maximum velocity of contraction at minimum light. Since the wave persists altogether for an interval of 1.C 60 for both H and K and H_α, two atmospheric layers may be forming two separate sets of absorption lines simultaneously. It is perhaps significant that the relative intensities (shortward/longward) of the two components during this stage increase with time, *i.e.*, the component belonging to maximum velocity tends to predominate in the first of the overlap interval, whereas the component accompanying the curve at minimum does so at the end of the overlap... ».

This line-doubling occurs during increasing light, as in *W Vir* stars. It is not detected in the metallic lines or the high members of the Balmer series.

Transitory emission at H_α develops at the midpoint of the ascending branch of the light curves and lasts 0.032 periods or about 30 minutes. The fading of this emission is followed by the formation, at the same wave length, of the new, violet-displaced, absorption line of H_α. The old, red-displaced component vanishes near phase .12 periods.

4. STRUVE, O. and BLAAUW, A.

Ref.: *Ap. J.*, **108**, 60 (1948).

Subject: A study of the radial velocity variations of *RR Lyr* based on 390 spectrograms of dispersion 40 Å/mm.

The authors states that:

« The purpose of this work was to investigate whether the velocity curve changes in conformity with the 41-day period of fluctuation in the character of the short-period light curve ($P = 0.567$ days) ».

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The authors conclude that the behavior of the velocity curve is quite analogous to the behavior of the light curve.

1) With respect to a uniform ephemeris the times of maximum velocity are advanced and retarded by .024 periods with a period of about 41 days.

2) Emission in hydrogen occurs during the midpoint of the rising branch of the light curve. This emission occurs during those cycles when the velocity curve shows nearly its greatest retardation.

3) *RR Lyr* shows only a weak correlation between velocity amplitude and retardation of the velocity curve. However in another paper (*Ap. J.*, **109,** 215 (1959)) STRUVE and VAN HOOF find for *XZ Cyg* an amplitude variation of some $(15 \div 20)\%$ such that maximum amplitude occurs in cycles of retarded maximum. Most of the amplitude variation is produced by an increase of the maximum of velocity.

Therefore it appears that retardation of the maximum of light and velocity is associated with *a*) increased amplitude of light and velocity and *b*) emission at hydrogen.

3. – Summaries of papers on W Virginis stars.

1. WALLERSTEIN, G.

Ref.: *Ap. J.*, **127,** 583 (1958).

Subject: A spectroscopic study of three Population II cepheids (two *W Vir* stars and one *RV Tau* star) based on plates of dispersions 18 Å/mm to 80 Å/mm.

Although these stars show no line splitting with the lower dispersions, the higher dispersion plates show doubling by about 50 km/s for two of the stars at their light maxima.

The *RV Tau* star M5 No. 84 shows alternating deep and shallow minima of the light curve. Wallerstein finds that, preceding the shallow minima, the maximum of outward velocity occurs 0.3 periods late and the velocity amplitude is reduced from 60 km/s (preceding deep minima) to 40 km/s. This result is in qualitative agreement with that of ABT for *U Mon.*

The writer comments that, for the *W Vir* and *RV Tau* stars, hydrogen « emission seems to have a marked maximum among stars of period 15-19 days. Emission is definitely weaker among the stars of period 20-30 days. »

The writer comments on the difference between the spectra of classical cepheids and those of Population II in the following terms:

« ... This [difference] is most apparent in the vicinity of maximum light, when the classical cepheids are close to spectral type F6 with abnormally strong hydrogen absorption lines. The Population II cepheids, on the other hand, have spectra of A5-F0 with either greatly weakened hydrogen lines or emission lines of hydrogen. The difference is so great for variables of period 15 days or more that a star can easily be separated into one of the two types using spectra of quite low dispersion... ».

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2. Abt, H. A.

Ref.: *Ap. J.*, **122**, 72 (1955).

Subject: A study of *U Mon* (*RV Tau* variable, $P \simeq 46$ days) based on photometric data and spectra of 11 Å/mm and 22 Å/mm dispersion.

This star is typical of *RV Tau* stars in showing alternating deep and shallow minima of the light curve. The light and velocity variations show a large degree of irregularity.

In other respects the behavior of this star is like that of *W Vir*. *I.e.* emission of hydrogen is present during increasing light. The radial velocity curve is discontinuous with a total amplitude of about 40 km/s and « just before each light-maximum a weak set of lines appears, displaced shortward with respect to the stronger lines. These lines quickly strengthen, move longward, and then fade during the next light-maximum... ».

The author discusses available data relevant to the question of systematic atmosphere-streaming. Table III of his paper, here reproduced, lists integrated radial velocities of globular clusters and mean velocities of the *W Vir* and *RV Tau* stars which are contained in these clusters.

Table III. – *Velocities of variables in globular clusters.*

Type	*NGC* (M)	Integrated velocity (km/s) (Mayall)	Wt. W_I	Variable No.	Mean velocity (km/s) (Joy)	Wt. W_M	Dif. (Mean-Integ.)	Weight
W Vir	5 272 (3)	— 150	24	154	— 153	14.0	— 3	8.84
	6 218(12)	— 32	4	1	— 42	5.6	— 10	2.33
	6 254(10)	+ 73	6	2	+ 67	7.0	— 6	3.23
RV Tau	5904 (5)	+ 45	13	42	+ 54	8.0	+ 9	4.95
	5904 (5)	+ 45	13	84	+ 50	8.0	+ 5	4.95
	6 779(56)	— 154	5	6	— 132	7.7	+ 22	3.03
	7089 (2)	— 3	17	11	— 4	10.8	— 2	6.60
Weighted mean	—	—	—	—	—	—	$+2 \pm 6$	—

It is known that the mean random velocities of non-variable stars within globular clusters are of the order of 5 km/s. Therefore, Abt's results

$$\Delta v = 2 \pm 6 \text{ km/s} ,$$

is consistent with the assumption of no systematic streaming within the atmospheres of Population II variables.

Application of the curve-of-growth analysis to the line intensities indicates,

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as for *W Vir*, an increase of random velocities, on the microscopic scale, with advancing phase. A random velocity of 3 km/s is characteristic of lines which have just developed and this increases to about 7 km/s as the lines shift toward the red with increasing age. The author suggests that this increase of width may be due to turbulence or dispersion of the pulsation velocity.

The author interprets the variations of line strength around maximum light as being due to variations of the continuous opacity of the atmospheric layers forming the lines. This assumption allows evaluation of the electron pressure and degree of ionization in the layers.

Comment. – The variations of line strength appear to be confined to the phases of line-splitting and, for the longward component, must be largely due to decreasing total mass in the layer as the shock moves upward.

3. ABT, H. A.

Ref.: *Ap. J., Supp.*, Ser. **1**, 63 (1954).

Subject: An analysis of the spectrum variations of *W Vir* ($P = 17.3$ days) based on 10.3 Å/mm coude plates.

The author combines color and light data with integrations of the velocity curve to derive a mean photospheric radius of $30 \cdot 10^6$ km and a pulsation semi-amplitude of about $10 \cdot 10^6$ km.

Examination of data on variable stars in globular clusters indicates that the median observed velocities of the variables are within 10 km/s of the velocity of the center of mass of the clusters. Hence systematic streaming motions at the region of line formation are probably less than 10 km/s.

Absorption lines show duplicity around maximum light with a separation of 55 km/s. (Private communication: Within each of the two sets of lines the differential velocities are small.)

The writer describes the appearance of the absorption spectrum in the following terms:

« When they first appear just before maximum light, the shortward absorption lines are very weak but have the appearance of an F2 supergiant. The luminosity class at all phases is about Ib — certainly not II or III. The new group of lines quickly strengthens, while the longward components fade, until at phase 0.2 they have nearly reached their full strength. The spectral type becomes later until at minimum light (phase 0.62) the metallic lines indicate G6, although the G band is either absent or very weak. After minimum light the lines begin to fade, and the spectral type quickly returns to F2. Between phases 0.825 and 0.1, when both absorption components are present and weak, both indicate a spectral type of F2, although there is a large difference in excitation temperature between the two spectra ».

In a private communication dated Nov. 3 (1954) the writer states:

« ... The measures in both *W Vir* and *U Mon* indicate that the Doppler velocity (from curves of growth) increases monotonically throughout each cycle. In so far as I can tell, the line widths in *W Vir* also increase correspondingly, so that at the phase of double lines, the longward (old) components are broader than the shortward ones. In *U Mon* the lines are not so well resolved but they seem to show the longward components broader... ».

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Concerning the hydrogen lines the writer states that « the behavior of the hydrogen lines is complex and shows large variations from cycle to cycle ». This behavior is shown in Fig. 6 of the writer's paper and, concisely, is the following. Shortward-displaced emission lines appear at light minimum (phase 0.6) and they have vanished by phase 0.1. The emission lines are replaced by narrow absorption lines which shift toward the red. These absorption lines persist, superimposed on the emission lines, until phase 0.1 of the following cycle.

The author notes the existence of very broad shallow absorption wings around maximum light and states that they have the same velocity as the shortward metallic lines and the *H* emission lines.

4. SANFORD, R. F.

Ref.: *Ap. J.*, **116**, 331 (1952).

Subject: A study of the spectrum and radial velocities of *W Vir* based on 10 Å/mm plates.

This paper presents the best existing velocity curve for *W Vir* and discusses the behavior of certain spectral lines.

Table II of the paper, reproduced hereafter, indicates that the relative intensities of the shortward (upward moving gas) and longward component change little while both are present.

TABLE II. – *Spectrograms of W Virginis with double absorption lines.*

Plate No. (Ce)	Phase (P)	Element and relative intensities: shortward to longward component ratio, $S:L$		
		Sr II	Ti II	Fe I
7102	0.94	2:3	2:3	2:1
6107	.95	2:1	2:2	2:trace
6211	.00	3:1	2:0.5	2:0
5647	.00	1:2	1:3	1:2
5617	.04	1:1	1:1	1:1
5618	0.10	1:1	1:1	1:1

Concerning the hydrogen emission, the writer states:

Emission lines of hydrogen prevail in the phase interval from light-maximum and for an interval of 0.1 *P* beyond, in which short interval *W Vir* is very little fainter than at maximum light (see Fig. 1). Each emission hydrogen line is divided into a strong shortward and relatively weak longward part by a superposed absorption line. This difference in intensity of components increases markedly from minimum to maximum of light, so that finally the longward parts are scarcely visible.

The hydrogen lines... are in both emission and absorption. The absorption lines give velocities which agree with the velocities from the absorption lines of the other elements if these lines are single, and with their longward components if they are double.

The superposed absorption components prevented satisfactory measures of the displacements of the hydrogen emission lines on the spectrograms themselves. However, microphotometer tracings which seemed fairly adequate for this purpose furnished the diplacements from which the velocities in Table I vere obtained. Although these velocities are liable to considerable error, they seem to give evidence of a variation as brought out in Fig. 3. The velocity trends downward from -70 km/s at phase $0.65P$ to -100 km/s at phase $0.95P$, with little or no change thereafter until the disappearance of the emission lines at phase $0.10P$. The average of the emission hydrogen lines is at least 2.3Å.

The paper closes with the following interpretation of the observation:

Absorption-line velocities of *W Vir* and the relation of their changes to changes of light may be explained quantitatively by means of shock-waves. Such a shock-wave is thought of as entering the reversing layer at its bottom and passing through its successive strata at the time when the reversing layer is falling in at its maximum velocity. This imparts a high outward velocity to successive strata, beginning at the bottom. That part of the reversing layer which has not been hit by the shock wave continues to rush rapidly inward. The Doppler effect separates the absorption lines of the inrushing from those of the outrushing gases. However, this would apparently require the relative intensities of the shortward to the longwaard components of the double lines to be weakest at the beginning and to grow stronger as the interval for double lines advances. This is not evidenced by the data in Table II... .

The explanation of the emission lines of *W Vir* is not clear. If these lines are a product of the shock wave, it would seem that the emission lines begin in layers below the reversing layer, perhaps after the shock-wave has passed by this lower level. If such a mechanism is present, it might account for the continuance of the emission lines until the shock-wave has completely traversed the reversing layer.

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