13 Supersymmetric dynamics

In the previous chapter, we learned how to build realistic particle physics models based on supersymmetry. There are already significant constraints on such theories, and experiments at the LHC will test whether these sorts of ideas are correct.

If supersymmetry is discovered, the question will become: how is supersymmetry broken? Supersymmetry breaking offers particular promise for explaining large hierarchies. Consider the non-renormalization theorems. Suppose that we have a model consisting of chiral fields and gauge interactions. If the superpotential is such that supersymmetry is unbroken at tree level, the non-renormalization theorems for the superpotential which we proved in Section 9.7 guarantee that supersymmetry is not broken to all orders of perturbation theory. But they do not necessarily guarantee that effects *smaller* than any power of the couplings will not break supersymmetry. So, if we denote the generic coupling constants by g^2, there might be effects of order, say, e^{-c/g^2} which break the symmetry. In the context of a theory like the MSSM, supposing that soft breakings are of this order might account for the wide disparity between the weak scale (correlated with the susy-breaking scale) and the Planck or unification scale.

So, one reason why the dynamics of supersymmetric theories is of interest is its role in aiding our understanding of dynamical supersymmetry breaking and perhaps in studying a whole new class of phenomena in nature. But there are yet other reasons to be interested, as was first clearly appreciated by Seiberg. Supersymmetric Lagrangians are far more tightly constrained than ordinary Lagrangians. It is often possible to make strong statements about the dynamics which would be difficult if not impossible for conventional field theories. We will see this includes phenomena such as electric–magnetic duality and confinement.

13.1 Criteria for supersymmetry breaking: the Witten index

We will consider a variety of theories, some of them strongly coupled. One might imagine that it is a hard problem to decide whether supersymmetry is broken. Even in weakly coupled theories, one might wonder whether one could establish reliably that supersymmetry is *not* broken since, unless one has solved the theory exactly, it would seem hard to assert that there is no tiny non-perturbative effect which does not break the symmetry. One thing we will learn in this chapter is that this is not, however, a particularly difficult problem. We will exploit several tools. One is known as the *Witten index*. Consider the field theory of interest in a finite box. At finite volume the supersymmetry charges

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are well defined, whether or not supersymmetry is spontaneously broken. Because of the supersymmetry algebra,

$$Q|B\rangle = \sqrt{E}|F\rangle, \quad Q|F\rangle = \sqrt{E}|B\rangle, \tag{13.1}$$

i.e. non-zero-energy states come in Fermi–Bose pairs. Zero-energy states are special; they need not be paired. In the infinite-volume limit, the question of supersymmetry breaking amounts to the question whether there are zero-energy states. To count these, Witten suggested evaluating

$$\Delta = \mathrm{Tr}\,(-1)^F\, e^{-\beta H}. \tag{13.2}$$

Non-zero-energy states do not contribute to the index. The exponential is present to provide an ultraviolet regulator: the Witten index Δ is *independent of* β. More strikingly, the index is independent of all the parameters of the theory. The only way in which Δ can change as some parameter is changed is by some zero-energy state acquiring non-zero energy or a non-zero-energy state acquiring zero energy. But, because of Eq. (13.1), whenever the number of zero-energy bosonic states changes, the number of zero-energy fermionic states changes by the same amount. The Witten index is thus topological in character, and it is from this that it derives its power as well as its applications in a number of areas of mathematics. What can we learn from this index? If $\Delta \neq 0$ then we can say with confidence that supersymmetry is not broken. If $\Delta = 0$, we do not know whether it is.

Let us consider an example: a supersymmetric gauge theory with gauge group $SU(2)$ and no chiral fields. Since Δ is independent of the parameters, we can consider the theory in a very tiny box, with very small coupling. We can evaluate Δ, somewhat heuristically, as follows. Work in the $A_0 = 0$ gauge. Consider, first, the bosonic degrees of freedom, the A_is, which are matrix valued. In order for the energy to be small, we need the A_is to be constant and to commute. So take A_i to lie in the third dimension in isospin space, and ignore the other bosonic degrees of freedom. One might try to remove these remaining variables by a gauge transformation $g = \exp(iA_i x^i)$, but g is only a sensible gauge transformation if it is single-valued, which means that $A_i^3 = 2\pi n/L$. Thus A_i^3 is a compact variable. This reduces the problem to the quantum mechanics of a rotor. Thus in the lowest state the wave function is a constant. Because the A_i^3s are non-zero, the lowest energy states will only involve the gluinos in the three direction. There are two, λ_1^3 and λ_2^3 (again independent of coordinates).

Now recall that in the $A_0 = 0$ gauge the states must be gauge invariant. One interesting gauge transformation is multiplication by σ_2. This flips the sign of A^3 and λ^3. If we assume that our Fock ground state is even under this transformation, the only invariant states are $|0\rangle$ and $\lambda_1^3\lambda_2^3|0\rangle$. So we find $\Delta = 2$. If we assume that the state is odd then we obtain $\Delta = -2$.

As we indicated, this argument is heuristic. A more detailed, but still heuristic, argument was provided by Witten in his paper on the index Δ. But Witten also provided a more rigorous proof, which yields the same result. For general $SU(N)$, one finds that $\Delta = N$.

This already establishes that a vast array of interesting supersymmetric field theories do not break supersymmetry, not only all the pure gauge theories but any theory with massive matter fields. This follows because Δ is independence of parameters. If the mass is finite, one can take it to be large; if it is sufficiently large we can ignore the matter fields and recover the pure gauge result. Later, we will understand the dynamics of these theories in some detail and will reproduce the result for the index. But we will also see that the limit of zero mass is subtle, and the index calculation is not directly relevant to that case.

13.2 Gaugino condensation in pure gauge theories

Our goal in this section is to understand the dynamics of a pure $SU(N)$ gauge theory with massless fermions in the adjoint representation. Without thinking about supersymmetry one might expect the following, from our experience with real QCD.

1. The theory has a mass gap, i.e. the lowest excitations of the theory are massive.
2. Gauginos, like quarks, condense, i.e.

$$\langle\lambda\lambda\rangle = c\Lambda^3 = ce^{-(8\pi^2/b_0 g^2)}. \tag{13.3}$$

Note that there is no Goldstone boson associated with the gluino (gaugino) condensate. The theory has no continuous global symmetry; the classical symmetry,

$$\lambda \to e^{i\alpha}\lambda, \tag{13.4}$$

is anomalous. However, a discrete subgroup,

$$\lambda \to e^{2\pi i/N}\lambda, \tag{13.5}$$

is free of anomalies. One can see this by considering instantons in this theory. The instanton has $2N$ zero modes; this would appear to preserve a Z_{2N} symmetry. But the transformation $\lambda \to -\lambda$ is actually equivalent to a Lorentz transformation (a rotation by 2π). Multi-instanton solutions also preserve this symmetry, and it is believed to be exact. So the gaugino condensate breaks the Z_N symmetry; there are N degenerate vacua. This neatly accounts for the N value of the index. Later we will show that, even though the theory is strongly coupled, we can demonstrate the existence of the condensate by a controlled semiclassical computation.

Gluino condensation implies a breakdown of the non-renormalization theorems at the non-perturbative level. Recall that the Lagrangian is

$$\mathcal{L} = \int d^2\theta\, SW_\alpha^2, \tag{13.6}$$

so $\langle\lambda\lambda\rangle$ gives rise to a superpotential, i.e.

$$\mathcal{L} = \int d^2\theta\, S\langle\lambda\lambda\rangle. \tag{13.7}$$

This is our first example of a non-perturbative correction to the superpotential. Note, however, that $\langle\lambda\lambda\rangle$ must depend on S, since it depends on g^2:

$$S\langle\lambda\lambda\rangle = e^{-3S/b_0}. \tag{13.8}$$

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So we actually have a superpotential for S:

$$W(S) = e^{-S/N}. \tag{13.9}$$

This superpotential violates the continuous shift symmetry which we used to prove the non-renormalization theorem, but it is compatible with the non-anomalous R symmetry,

$$S \to S + i\alpha N, \quad \lambda \to \lambda e^{i\alpha}. \tag{13.10}$$

Under this symmetry the superpotential transforms with charge 2.

13.3 Supersymmetric QCD

A rich set of theories for study is that collectively referred to as *supersymmetric QCD*. These are gauge theories with gauge group $SU(N)$, N_f flavor fields Q_f in the N representation and N_f flavor fields $\bar{Q}_f$ in the $\bar{N}$ representation; here $f = 1, \ldots, N_\mathrm{f}$. We will see that the dynamics is quite sensitive to the value of N_f. First, we will consider the theory without any classical superpotential for the quarks. In this case the theory has a large global symmetry. We can transform the Qs and $\bar{Q}$s by separate $SU(N_\mathrm{f})$ transformations. We can also multiply the Qs by a common phase and the $\bar{Q}$s by a separate common phase:

$$Q_f \to e^{i\alpha} Q_f, \quad \bar{Q}_f \to e^{i\beta} \bar{Q}_f. \tag{13.11}$$

Finally, the theory possesses an R symmetry, under which the Qs and $\bar{Q}$s are neutral. In terms of component fields, under this symmetry we have

$$\psi_Q \to e^{-i\alpha} \psi_Q, \quad \psi_{\bar{Q}} \to e^{-i\alpha} \psi_{\bar{Q}}, \quad \lambda^a \to e^{i\alpha} \lambda^a. \tag{13.12}$$

Now consider the question of anomalies. The $SU(N_\mathrm{f})$ symmetries are free of anomalies, as is the vector-like symmetry,

$$Q_f \to e^{i\alpha} Q_f, \quad \bar{Q}_f \to e^{-i\alpha} \bar{Q}_f. \tag{13.13}$$

The R symmetry and the axial $U(1)$ symmetry are both anomalous. But we can define a non-anomalous R by combining the two. The gauginos give a contribution to the anomaly proportional to N, so we need the fermions to carry an R-charge $-N/N_\mathrm{f}$. Since the bosons (and the chiral multiplets) carry an R-charge that is larger by 1, we have

$$Q_f(x, \theta) \to e^{i\alpha(N_\mathrm{f}-N)/N_\mathrm{f}} Q_f(x, \theta e^{-i\alpha}), \quad \bar{Q}_f(x, \theta) \to e^{i\alpha(N_\mathrm{f}-N)/N_\mathrm{f}} \bar{Q}_f(x, \theta e^{-i\alpha}). \tag{13.14}$$

So, the symmetry of the quantum theory is $SU(N_\mathrm{f})_\mathrm{L} \times SU(N_\mathrm{f})_\mathrm{R} \times U(1)_R \times U(1)_V$, where the vector symmetry $U(1)_V$ transforms the Q and $\bar{Q}$ fields by opposite phases.

We have seen that supersymmetric theories often have, classically, a large vacuum degeneracy and this is true of this theory. In the absence of a superpotential, the potential is completely determined by the D terms for the gauge fields. It is helpful to treat D as a matrix-valued field,

$$D = \sum T^a D^a. \tag{13.15}$$

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As a matrix, D can be expressed elegantly in terms of the scalar fields. We start with the identity

$$(T^a)_i^{\ j}(T^a)_k^{\ l} = \delta_i^l \delta_k^j - \frac{1}{N}\delta_i^j \delta_k^l. \tag{13.16}$$

One can derive this result in a number of ways. Consider propagators for fields (such as gauge bosons) in the adjoint representation of the gauge group. Take the group, first, to be $U(N)$. The propagator of the matrix-valued fields satisfies

$$\langle A_i^j A_k^l \rangle \propto \delta_i^l \delta_k^j. \tag{13.17}$$

But this is the same thing as

$$\langle A^a A^b (T^a)_i^j (T^b)_k^l \rangle. \tag{13.18}$$

So we obtain the identity without the $1/N$ terms. Now remembering that A must be traceless, we see that we need to subtract the trace as above. (This identity is important in understanding the $1/N$ expansion in QCD.) Thus a field ϕ in the fundamental representation makes a contribution

$$\delta D_i^j = \phi_i^* \phi^j - \frac{1}{N}\delta_i^j \phi_k^* \phi^k. \tag{13.19}$$

In the antifundamental representation the generators are $-T^{aT}$ (this follows from the fact that these generators are minus the complex conjugates of those in the fundamental representation, and the fact that the T^as are Hermitian). So the full D term is

$$D_i^j = \sum_f Q_i^* Q^j - \bar{Q}_i \bar{Q}^{*j} - \text{Tr terms}. \tag{13.20}$$

In this matrix form it is not difficult to look for supersymmetric solutions, i.e. solutions of $D_i^j = 0$. A simple strategy is first to construct

$$\hat{D}_i^j = \sum_f Q_i^* Q^j - \bar{Q}_i \bar{Q}^{*\,j} \tag{13.21}$$

and demand that $\hat{D}$ either vanish or be proportional to the identity. Let us start with the case $N_\mathrm{f} \leq N$. For definiteness, take $N = 3$, $N_\mathrm{f} = 2$; the general case is easy to work out. By a sequence of $SU(3)$ transformations, we can bring Q to the following form:

$$Q = \begin{pmatrix} v_{11} & v_{12} \\ 0 & v_{22} \\ 0 & 0 \end{pmatrix}. \tag{13.22}$$

By a sequence of $SU(N_\mathrm{f})$ transformations, we can bring this to the simpler form

$$Q = \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \\ 0 & 0 \end{pmatrix}. \tag{13.23}$$

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At this point we have used up our freedom to make further symmetry transformations on $\bar{Q}$. But it is easy to find the most general $\bar{Q}$ which makes the D terms vanish. The contribution of Q to D_i^j is simply

$$D = \mathrm{diag}(|v_1|^2, |v_2|^2). \tag{13.24}$$

So, in order that D vanish, $\bar{Q}$ must make an equal and opposite contribution. In order that there be no off-diagonal contributions, $\bar{Q}$ can have entries only on the diagonal, so

$$\bar{Q} = \begin{pmatrix} e^{i\alpha_1} v_1 & 0 \\ 0 & e^{i\alpha_2} v_2 \\ 0 & 0 \end{pmatrix}. \tag{13.25}$$

In general, in these *flat directions* – directions in field space in which the potential is flat – the gauge group is broken to $SU(N - N_{\rm f})$. The unbroken flavor group depends on the values of the v_is. We have exhibited $N_{\rm f}$ complex moduli above, but actually there are more, associated with the generators of the broken flavor symmetries $(SU(N_{\rm f}) \times U(1))$. Thus there are $N_{\rm f}^2 + 2N_{\rm f}$ complex moduli. Note that there are $2NN_{\rm f} - N_{\rm f}^2$ broken gauge generators, which gain mass by "eating" the components of $Q, \bar{Q}$ that are not moduli. Of the original $2NN_{\rm f}$ chiral fields this leaves precisely $N_{\rm f}^2 + 2N_{\rm f}$ massless fields, so we have correctly identified the number of moduli.

Our discussion, so far, does not look gauge invariant. But this is easily, and elegantly, rectified. The moduli can be written as the gauge-invariant combinations

$$M^f_{\bar{f}} = \bar{Q}_{\bar{f}}\, Q^f. \tag{13.26}$$

Expanding the fields Q and $\bar{Q}$ about their expectation values gives back the explicit form for the moduli in terms of the underlying gauge-invariant fields. This feature, we will see, is quite general.

The case $N_{\rm f} = N$ is similar to the case $N_{\rm f} < N$, but there is a significant new feature. In addition to the flat directions with $Q = \bar{Q}$ (up to phases), the potential also vanishes if $Q = v\mathrm{I}$, where I is the identity matrix. This possibility can also be described in a gauge-invariant way since now we have an additional pair of gauge invariant fields, which we will refer to as "baryons":

$$B = \epsilon^{i_1 \dots i_N} \epsilon_{a_1 \dots a_N} Q^{a_1}_{i_1} \cdots Q^{a_N}_{i_N}, \tag{13.27}$$

and similarly for $\bar{B}$.

In the case $N_{\rm f} > N$ there is a larger set of baryon-like objects, corresponding to additional flat directions. We will describe them in greater detail later. Before closing this section we should stress that for $N_{\rm f} \geq N - 1$ the gauge symmetry is completely broken. For large values of the moduli, the effective coupling of the theory is $g^2(v)$ since infrared physics cuts off at the scale of the gauge field masses. By taking v as sufficiently large that $g^2(v)$ is small, the theories can be analyzed by perturbative and semiclassical methods. Strong coupling is more challenging, but much can be understood. We will see that the dynamics naturally divides into three cases: $N_{\rm f} < N$, $N_{\rm f} = N$, and $N_{\rm f} > N$.

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13.4 $N_f < N$: a non-perturbative superpotential

Our problem now is to understand the dynamics of these theories. Away from the origin of the moduli spaces, this turns out to be a tractable problem. We consider first the case $N_\mathrm{f} < N$. Suppose that the v_is are large and roughly uniform in magnitude. Even here, we have to distinguish two cases. If $N_\mathrm{f} = N - 1$, the gauge group is completely broken and the low-energy dynamics consists of the set of chiral fields $M_{\bar{f}f}$. If $N_\mathrm{f} < N - 1$, there is an unbroken gauge group, $SU(N-N_\mathrm{f})$, with no matter fields (chiral fields) transforming under this group at low energies. The gauge theory is an asymptotically free theory, essentially like ordinary QCD with fermions in the adjoint representation. Such a theory is believed to have a mass gap of order the scale of the theory, Λ_{N-N_f}. Below this scale, again, the only light fields are the moduli $M^f_{\bar{f}}$. In both cases we can try to guess the form of the very-low-energy effective action for these fields from symmetry considerations.

We are particularly interested in whether there is a superpotential in this effective action. If not then the moduli have *exactly* no potential. In other words, even in the full quantum theory, they correspond to an exact, continuous, set of ground states. What features should this superpotential possess? Most important, it should respect the flavor symmetries of the original theory (because the fields M are gauge invariant, it automatically respects the gauge symmetry). Among these symmetries are the $SU(N_\mathrm{f}) \times SU(N_\mathrm{f})$ non-Abelian symmetry. The only invariant that we can construct from M is

$$\Phi = \det M. \tag{13.28}$$

The determinant is invariant because it transforms under $M \to VMU$ as $\det V \det U \det M$ and, for $SU(N_\mathrm{f})$ transformations, the determinant is unity. Under baryon number symmetry, M is invariant. But, under $U(1)_R$ symmetry the its transformation law is more complicated:

$$\Phi \to e^{2i\alpha(N_\mathrm{f}-N)}. \tag{13.29}$$

Under this R-symmetry, any would-be superpotential must transform with charge 2, so the form of the superpotential is unique:

$$W = \Lambda^{(3N-N_\mathrm{f})/(N-N_\mathrm{f})}\,\Phi^{-1/(N-N_\mathrm{f})}. \tag{13.30}$$

Here we have inserted a factor Λ, the scale of the theory, on dimensional grounds.

Our goal in the next two sections will be to understand the dynamical origin of this superpotential, known as the Affleck, Dine and Seiberg (ADS) superpotential. We will see that there is a distinct difference between the cases $N_\mathrm{f} = N - 1$ and $N_\mathrm{f} < N - 1$. First, though, consider the case $N = N_\mathrm{f}$. Then the field Φ has R-charge zero, and no superpotential is possible. So, no potential can be generated, perturbatively or non-perturbatively. Similarly, in the case $N_\mathrm{f} > N$ we cannot construct a gauge-invariant field which is also invariant under the $SU(N_\mathrm{f}) \times SU(N_\mathrm{f})$ flavor symmetry. This may not be obvious, since it would seem that we could again construct $\Phi = \det M$. But in this case $\Phi = 0$ in the flat directions.

From the perspective of ordinary, non-supersymmetric, field theories, what we have established here is quite surprising. Normally, we would expect that in an interacting theory, even if the potential vanished classically there would be quantum corrections. For

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theories with $N \geq N_f$, we have just argued that this is impossible. So this is a new feature of supersymmetric theories: there are often exact moduli spaces, even at the quantum level.

In the next few sections we will demonstrate that non-perturbative effects do indeed generate the superpotential of Eq. (13.30). The presence of the superpotential means that, at least at weak coupling (large v_i), there is no stable vacuum of the theory. At best, we can consider time-dependent, possibly cosmological, solutions. If we add a mass term for the quarks, however, we find an interesting result. If the masses are the same, we expect that all the v_is will be the same, $v_i = v$. Suppose that the mass term is small. Then the full superpotential, at low energies, is

$$W = m\bar{Q}Q + \Lambda^{(3N-N_f)/(N-N_f)}\Phi^{-1/(N-N_f)}. \tag{13.31}$$

Remembering that $\Phi \sim v^{2N_f}$, the equation for a supersymmetric minimum has the form

$$v^{2N/(N-N_f)} = \left(\frac{m}{\Lambda}\right)\Lambda^{2N/(N-N_f)}. \tag{13.32}$$

Note that v is a complex number; this equation has N roots

$$v = e^{2\pi ik/N}\Lambda\left(\frac{m}{\Lambda}\right)^{(N-N_f)/2N}. \tag{13.33}$$

What is the significance of these N solutions? The mass term breaks the $SU(N_f) \times SU(N_f)$ symmetry to the vector sum. It also breaks the $U(1)_R$. But it leaves unbroken a Z_N subgroup of the $U(1)$. In Eqs. (13.14), $\alpha = 2N_f/N$ is a symmetry of the mass term. So these N vacua are precisely those expected from the breaking of the Z_N subgroup. This Z_N is the same as that expected for a pure gauge theory, as one can see by thinking of the case where the mass of the Qs and $\bar{Q}$s is large.

13.4.1 The Λ-dependence of the superpotential

Previously, we proved a non-renormalization theorem for the gauge couplings by thinking of the gauge coupling itself as a background field S. This relied on the shift symmetry

$$S \to S + i\alpha.$$

This symmetry, however, is only a symmetry of perturbation theory. On the one hand, since the imaginary part a of S, couples to $F\tilde{F}$, instanton and other non-perturbative effects violate the symmetry. On the other hand the theory also has an anomalous chiral symmetry, the R symmetry, under which we can take all the scalar fields to be neutral. So the theory is symmetric under this R symmetry combined with a simultaneous shift

$$S \to S + i(N - N_f)\alpha. \tag{13.34}$$

Any superpotential must transform with charge 2 under this symmetry. The field Φ is neutral. But we have, for the Λ parameter,

$$\Lambda = \exp\left(-\frac{8\pi^2}{b_0 g^2}\right) = \exp\left(-\frac{8\pi^2}{3N - N_f}S\right) \tag{13.35}$$

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so it transforms as follows:

$$\Lambda^{(3N-N_{\rm f})/(N-N_{\rm f})} \to e^{2i\alpha}\Lambda^{(3N-N_{\rm f})/(N-N_{\rm f})}. \tag{13.36}$$

13.5 The superpotential in the case $N_{\rm f} < N - 1$

Consider first the case $N_{\rm f} < N-1$. At energies well below the scale v, the theory consists of a pure (supersymmetric) $SU(N-N_{\rm f})$ gauge theory and a number of neutral chiral multiplets. The chiral multiplets can couple to the gauge theory only through non-renormalizable operators. Because the moduli are neutral, there are no dimension-four couplings. There are possible dimension-five couplings; they are of the form

$$\delta\phi\, W_\alpha^2, \tag{13.37}$$

where $\delta\phi$ represents the fluctuations of the moduli fields about their expectation values; the coefficient of this operator will be of order $1/v$.

We can be more precise about the form of this coupling by noting that it must respect the various symmetries if it is written in terms of the original, unshifted fields (this is similar to our argument for the form of the superpotential). In particular, a coupling of the form

$$\mathcal{L}_{\rm coup} = (S + a \ln \Phi) W_\alpha^2 \tag{13.38}$$

respects all the symmetries: it clearly respects the $SU(N_{\rm f})$ symmetries, and it also respects the non-anomalous $U(1)_R$ symmetry, for a suitable choice of a, since

$$\ln\, \Phi \to \ln\, \Phi + (N - N_{\rm f})/N_{\rm f}\alpha. \tag{13.39}$$

It is not hard to see how this coupling is generated:

$$\Phi \approx v^N + v^{N-1}\phi. \tag{13.40}$$

Thus Im ϕ couples to $F\tilde{F}$ through the anomaly diagram, just like an axion. The real part couples to F^2. One can see this by a direct calculation or by noting that the masses of the heavy fields are proportional to v, so the gauge coupling of the $SU(N-N_{\rm f})$ theory depends on v:

$$\alpha^{-1}_{N-N_{\rm f}}(\mu) = \alpha_N^{-1}(v) + \frac{b_0^{(N-N_{\rm f})}}{4\pi} \ln \frac{\mu}{v}. \tag{13.41}$$

Since $\Phi \sim v^{N_{\rm f}}$, we see that we have precisely the correct coupling. It is easy to see which Feynman graphs generate the couplings to the real and imaginary parts.

But we have seen that in the $SU(N - N_{\rm f})$ theory, gaugino condensation gives rise to a superpotential for the coefficient of W_α^2; in this case, it is precisely

$$W = \frac{\Lambda^{(3N-N_{\rm f})/(N-N_{\rm f})}}{\Phi^{1/(N-N_{\rm f})}}. \tag{13.42}$$

So we have understood the origin of the superpotential in these theories.

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13.6 $N_f = N - 1$: the instanton-generated superpotential

In the case $N_f = N-1$, the superpotential is generated by a different mechanism: instantons. Before describing the actual computation we give some circumstantial evidence for this fact. Consider the instanton action. This is

$$\exp\left(\frac{-8\pi^2}{g^2(v)}\right). \tag{13.43}$$

Here we have assumed that the coupling is to be evaluated at the scale of the scalar vevs. The gauge group is, after all, completely broken so, provided that the computation is finite, this is the only relevant scale (we are also assuming that all the vevs are of the same order). Thus any superpotential we might compute behaves as

$$W \sim v^3 \left(\frac{\Lambda}{v}\right)^{2N+1} \sim \frac{\Lambda^{2N+1}}{v^{2N-2}}, \tag{13.44}$$

which is the behavior predicted by the symmetry arguments.

To actually compute the instanton contribution to the superpotential, we need to develop further than in Chapter 5 the instanton computation and the structure of the supersymmetry zero modes. The required techniques were developed by ’t Hooft, when he computed the baryon-number-violating terms in the effective action of the standard model; ’t Hooft started by noting that, in the presence of the Higgs field, *there is no instanton solution.* This can be seen by a simple scaling argument. Here the instanton solution will involve A^μ and ϕ. Suppose one has such a solution. Now simply do a rescaling of all lengths such that

$$x^\mu \to \rho x^\mu, \quad A^\mu \to \frac{1}{\rho}A^\mu, \quad \phi \to \phi \tag{13.45}$$

(because ϕ must tend to its expectation value at ∞, we cannot rescale it). Then the gauge kinetic terms are invariant but the scalar kinetic terms are not; $|D\phi|^2 \to \rho^2|D\phi|^2$. So the action is changed, and there is no solution.

However, the instanton configuration, while not a solution, is still distinguished by its topology; ’t Hooft argued that it makes sense to integrate over solutions of a given topology. This just means that we write down a configuration for each value of ρ, and integrate over ρ. For small ρ we can understand this in the following way. The non-zero modes of the instanton, before turning on the scalar vevs, all have eigenvalues of order $1/\rho$ or larger and can be ignored. There are also zero modes. Those associated with rotations and translations will remain at zero, even in the presence of the scalar, since they correspond to exact symmetries. But this is not the case for the dilatational zero mode; this mode is slightly lifted. The scaling argument above shows that the action is smallest at small ρ; we will see in a moment that the action of the interesting configurations vanishes as $\rho \to 0$. We know from our earlier studies of QCD, however, that renormalization of the coupling tends to make the action large at small ρ. Together, these effects yield a minimum of the action at small but finite ρ, giving a self-consistent justification of the approximation.

To proceed with the computation, we will use ’t Hooft’s notation for the instanton, which we introduced in Chapter 5. Recall that

$$A_\mu^a(x) = \frac{2\eta_{a\mu\nu}x_\nu}{x^2+\rho^2}. \tag{13.46}$$

It is straightforward to work out $F_{\mu\nu}$ (see the exercises):

$$F_{\mu\nu}^a = \frac{\eta_{a\mu\nu}}{(x^2+\rho^2)^2}. \tag{13.47}$$

We note that F is self-dual, since η is, so this is a solution of the Euclidean equations. A second-rank antisymmetric tensor $F_{\mu\nu}$ is a six-dimensional representation of $SO(4)$; under $SU(2) \times SU(2)$ it decomposes as $(3,1) + (1,3)$, where these are the self-dual and anti-self-dual parts of the tensor. The η symbol is essentially a Clebsch–Gordan coefficient, which describes a mapping of one $SU(2)$ subgroup of $SO(4)$ into $SU(2)$.

At large distances, the instanton is a gauge transformation of “nothing”. i.e. vanishing values for the fields. The gauge transformation is just

$$g_j^i = i\bar{\sigma}_j^{\mu i}\hat{x}^\mu. \tag{13.48}$$

This can be thought of as a mapping of S_3 into $SU(2)$; the winding number of the instanton just counts the number of times space is mapped onto the group.

In this form it is useful to note another way to describe the instanton solution. By an inversion of coordinates one can write

$$A_\mu^a = \frac{2}{g^2}\frac{\rho^2}{x^2+\rho^2}\eta_{a\mu\nu}\frac{x^\nu}{x^2}. \tag{13.49}$$

This *singular gauge* instanton is often useful since it falls off more rapidly at large x than the original instanton solution.

Now, for the doublets we solve the equation

$$D^2Q = D^2\bar{Q} = 0. \tag{13.50}$$

This has solutions

$$Q^i = \bar{Q}^{i\dagger} = i\bar{\sigma}_j^{\mu i}\hat{x}^\mu\left(\frac{1}{x^2+\rho^2}\right)^{1/2}\langle Q^j\rangle, \tag{13.51}$$

and similarly for $\bar{Q}$. Like the solution for A^μ, these solutions are “pure gauge” configurations as $r \to \infty$, i.e. they are gauge transformations by g of the constant vev. (Note, here and above, that the σ^μs are the Euclidean versions of the two-component Dirac matrices, $\sigma^\mu = (i, \vec{\sigma})$, $\bar{\sigma}^\mu = (i, -\vec{\sigma})$.)

The action of this configuration is

$$S(\rho) = \frac{1}{g^2}(8\pi^2 + 4\pi^2\rho^2v^2). \tag{13.52}$$

Some features of this result are worth noting.

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1. The integral over ρ now converges for large ρ, since it is exponentially damped.
2. Terms in the potential involving $|Q|^4$ make smaller contributions to the action, according to powers of ρ. Rescaling x as ρx, one sees that these terms are of order ρ^4. But ρ is at most of order $gv^{-1} = m_w$ (from item 1 above), so these terms are suppressed. This justifies their neglect in the equations of motion.

Our goal is to compute the instanton contribution to the effective action. We particularly want to see whether the instanton generates the conjectured non-perturbative superpotential. In order to compute the effective action, we need to ask about the fermion zero modes. Before turning on the vevs for the scalars, there are six zero modes. Two of these are generated by supersymmetry transformations of the instanton solution

$$\delta\lambda = \sigma_\alpha^{\mu\nu\beta} F^{\mu\nu} \epsilon_\beta, \tag{13.53}$$

so

$$\lambda_{\alpha a}^{\mathrm{SS}[\beta]} = \frac{8\sigma_\alpha^{\mu a\beta}}{(x^2+\rho^2)^2}. \tag{13.54}$$

Note that, because of the anti-self-duality of $\bar\sigma^{\mu\nu}$, two supersymmetry generators annihilate the lowest-order solution, i.e. there are only two supersymmetry zero modes. If we neglect the Higgs, the classical Yang–Mills action has a conformal (scale) symmetry. This is the origin of the zero mode associated with changes in ρ. in the classical solution. In the supersymmetric case, there is, apart from supersymmetry, an additional fermionic symmetry called superconformal invariance. In superspace the corresponding generators are

$$Q^{\mathrm{SC}} = \not{x} Q, \tag{13.55}$$

so

$$\lambda_{\alpha a}^{\mathrm{SC}[\beta]} = \frac{8\not{x}\sigma_\alpha^{\mu a\beta}}{(x^2+\rho^2)^2}. \tag{13.56}$$

There are also two matter-field zero modes, one for each of the quark doublets:

$$\psi_{Q\alpha}^i = \frac{\delta_\alpha^i}{(x^2+\rho^2)^{3/2}} = \psi_{\bar Q} \tag{13.57}$$

(in the last equation we treated $\bar Q$ as a doublet also; one can instead treat it as a 2^* representation by multiplying by ϵ_{ij}).

When we turn on the scalar vevs these modes are corrected. The superconformal symmetry is broken by the vevs and, not surprisingly, the superconformal zero modes are lifted. In fact, they pair with the two quark zero modes. We can compute this pairing by treating the Yukawa terms in the Lagrangian as a perturbation, replacing the scalar fields by their classical values. Expanding to second order, i.e. including

$$\int d^4x\, Q^* \psi_Q \lambda \int d^4x'\, \bar Q^* \psi_{\bar Q} \lambda \tag{13.58}$$

and expanding the fields in the lowest-order eigenmodes, the superconformal and matter-field zero modes can be absorbed by these terms. Note, in particular, that both Q_{cl} and

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λ^{SC} are odd under $x \to -x$ while the matter-field zero modes are even, so the integral is non-zero. The supersymmetry zero modes, being even, cannot be soaked up in this way.

The wave functions of the supersymmetry zero modes are altered in the presence of the Higgs fields, and they now have components in the ψ_Q^* and $\psi_{\bar{Q}}^*$ directions. For ψ_Q, for example, we need to solve the equation

$$D_\mu \bar{\sigma}^\mu \psi_Q^{SS*} = \lambda^{SS} Q^*. \tag{13.59}$$

This equation is easy to solve, starting with our solution of the scalar equation. If we simply take

$$\psi_Q^{SS} = D_\mu \sigma^\mu Q^*, \tag{13.60}$$

then, substituting back into the left-hand side of Eq. (13.59) we obtain

$$D^2 Q + \sigma_{\mu\nu} F^{\mu\nu} Q; \tag{13.61}$$

the first term vanishes for the classical solution, while the second is indeed just $\lambda^{SS} Q^*$.

With these ingredients we can compute the superpotential terms in the effective action. In particular, the non-perturbative superpotential predicts a non-zero term in the component form of the effective action proportional to

$$\frac{\partial^2 W}{\partial Q \partial \bar{Q}} = \frac{1}{v^4} \psi_Q \psi_{\bar{Q}}. \tag{13.62}$$

We can calculate this term by studying the corresponding Green's function. We need to be careful, now, about the various collective coordinates. We want to study the gauge-invariant correlation function

$$\langle \bar{Q}(x) \psi_Q(x) \psi_{\bar{Q}}(y) Q(y) \rangle \tag{13.63}$$

in the presence of the instanton. Since we are interested in the low-momentum limit of the effective action, we can take x and y to be widely separated. We need to integrate over the instanton location x_0 and the instanton orientation and scale size. Because the gauge fields are massive, we can take x and y both to be far from the instanton. Then, from our explicit solution for the supersymmetry zero modes, we obtain

$$\psi_Q(x) \propto \not{D} Q \propto \not{D} \frac{i\sigma^\mu \left(x^\mu - x_0^\mu\right)}{[(x - x_0)^2 + \rho^2]^{1/2}} \to g(x - x_0) S_F(x - x_0), \tag{13.64}$$

with a similar equation for $\psi_{\bar{Q}}$. The g and $g^\dagger$ factors are canceled by corresponding factors in Q and $\bar{Q}$, at large distances. Substituting these expressions into the path integral and integrating over x_0 gives a convolution, $v^2 \int d^4 x_0\, S_F(x - x_0) S_F(y - y_0)$. Extracting the external propagators, we obtain the effective action. Integrating over ρ gives a term of precisely the desired form. If we contract the gauge and spinor indices in a gauge and rotationally invariant manner, the integral over rotations just gives a constant factor. It requires some work to do all the bookkeeping correctly. The evaluation of the determinant is greatly facilitated by supersymmetry: there is a precise fermion–boson pairing of all the non-zero modes. In the exercises, you are asked to work out more details of this computation; further details can also be found in the references.

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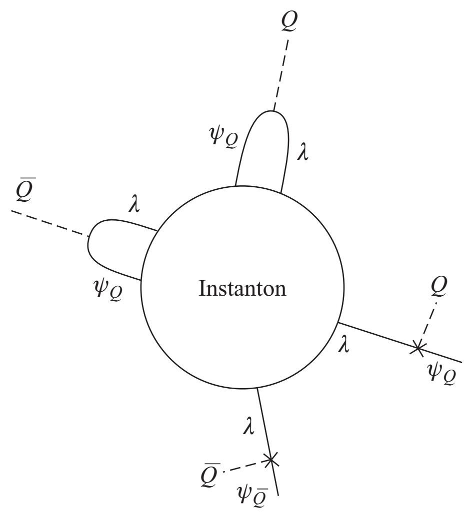

Fig. 13.1 Schematic description of the instanton computation of the superpotential. Four zero modes are tied together by the scalar vevs; two gluino zero modes turn into ψ zero modes as well.

Without working through all the details we can see the main features.

1. The perturbative lifting of the zero modes gives rise to a contribution proportional to v^2 (see Fig. 13.1).
2. The matter-field component of the supersymmetry zero modes studied above gives a contribution to the gauge-invariant correlation function:

$$v^4 \int d^4x_0\, S_f(x - x_0) S_f(y - y_0). \tag{13.65}$$

3. The integral over the gauge collective coordinates (equivalently the rotational collective coordinates) simply gives a constant, since we have computed a gauge- and rotationally invariant quantity.
4. The scale-size collective coordinate integral behaves as

$$W = A \int d\rho\, v^4 \exp - \left[\left(\frac{8\pi^2}{g^2(\rho)} + 4\pi^2\rho^2 v^2 \right) \right] \tag{13.66}$$

   where the power of ρ has been determined from dimensional analysis and A is a constant.
5. Extracting the constant requires careful attention to the normalization of the zero modes and to the Jacobians for the collective coordinates. However, the non-zero modes come in Fermi–Bose pairs, and their contribution to the functional integral cancels.
6. The final ρ integral gives

$$W = A' \frac{\Lambda^5}{v^2}, \tag{13.67}$$

   which is consistent with the expectations of the symmetry analysis.

This analysis generalizes straightforwardly to the case of general N_c.

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13.6.1 An application of the instanton result: gaugino condensation

The instanton calculation for the case $N_f = N - 1$ is a systematic weak-coupling computation of the superpotential which appears in the low-energy-effective action. Seiberg noted that this result, plus holomorphy, allows systematic study of the strongly coupled regime of other theories. To understand this, take $N = 2$ and add a mass term for the quark. In this case, for very small mass the superpotential is

$$W = m\bar{Q}Q + \frac{\Lambda^{2N+1}}{\bar{Q}Q}. \tag{13.68}$$

We can solve the equation for Q:

$$\bar{Q} = \begin{pmatrix} 0 \\ v \end{pmatrix}, \quad v = \left(\frac{\Lambda^5}{m}\right)^{1/4}. \tag{13.69}$$

Using this we can evaluate the expectation value of the superpotential at the minimum:

$$W(m, \Lambda) = \Lambda^{5/2} m^{1/2}. \tag{13.70}$$

Because W is holomorphic, this result also holds for large m. For large m, the low-energy theory is just a pure $SU(2)$ gauge theory. We expect for large m that the superpotential is $\langle\lambda\lambda\rangle = \Lambda_{\rm le}^3$. But this is equal to

$$W = \langle\lambda\lambda\rangle = m^3 \exp\left[-\frac{8\pi^2}{2g^2(m)}\right]. \tag{13.71}$$

The right-hand side is simply $\Lambda_{\rm le}^3$. We have, in fact, done a systematic, reliable computation of the gluino condensate in a strongly interacting gauge theory!

Suggested reading

Excellent treatments of supersymmetric dynamics appear in the text by Weinberg (1995), and in Michael Peskin's lectures (1997). We have already mentioned 't Hooft's original instanton paper (1976). The instanton computation of the superpotential is described in Affleck *et al*. (1984).

Exercises

(1) Verify that $\sigma_{\mu\nu}$ and $\bar{\sigma}_{\mu\nu}$ are self-dual and anti-self-dual, respectively. This means that $\mathrm{Tr}\,\sigma^a\sigma_{\mu\nu}$ is a self-dual tensor. Verify the connection to η; do the same thing for $\bar{\eta}$.

(2) Verify Eq. (13.47), which shows that F is self-dual and so solves the Euclidean Yang–Mills equations. Check that asymptotically the instanton potential is a gauge transform of "nothing."

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(3) Verify the solution Eq. (13.51) of the scalar field equation. Compute the action of this field configuration.

(4) Perform the zero-mode counting for the case of general N_c, $N_\mathrm{f} = N_\mathrm{c} - 1$. Show that, again, all but two zero modes pair with matter-field zero modes; two supersymmetry zero modes contain matter-field components which can give rise to the expected superpotential.