A REMARK ON THE EXISTENCE OF A DENUMERABLE BASE FOR A FAMILY OF FUNCTIONS

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A family F of functions is said to have a *denumerable base* if there exists a sequence of functions $\{f_n(x)\}$ (not necessarily $\in F$) such that any function $f \in F$ is the limit of a subsequence of $\{f_n(x)\}$. The *domain* X of a function $f(x)$ is the set of x's for which $f(x)$ is defined; we say $f(x)$ is a *function on* X. A *dyadic function* is a function taking only the values 0 and 1.

Let F be a family of dyadic functions on a set X.

PROPOSITION $(\mathfrak{m}, \mathfrak{n})$. *If* $\overline{\overline{F}} = \mathfrak{m}$ *and* $\overline{\overline{X}} = \mathfrak{n}$, *then the family* F *has a denumerable base.*

In an earlier paper I have shown that the proposition $(\aleph_1, \aleph_1)$ is true [**1**, p. 401, Theorem 3]. Hence, the continuum hypothesis implies the proposition $(\mathfrak{c}, \mathfrak{c})$ [*ibid.*, Corollary].

The problem is whether or not the proposition $(\mathfrak{c}, \mathfrak{c})$ can be proved independently (i.e., merely with the axiom of choice, but without any additional hypothesis such as the continuum hypothesis). We are going to prove a theorem which throws some light on this problem.

First, we need two lemmas (proofs omitted):

LEMMA A. *If* $\mathfrak{n}_1 > \mathfrak{n}_2$, *then proposition* $(\mathfrak{m}, \mathfrak{n}_1)$ *implies proposition* $(\mathfrak{m}, \mathfrak{n}_2)$.

LEMMA B. *If* $\aleph_\alpha < \mathfrak{c}$, *then proposition* $(\mathfrak{c}, \mathfrak{c})$ *implies proposition* $(\mathfrak{c}, \aleph_\alpha)$.

These will enable us to prove the following

THEOREM. *If there exists an* α *and a* β *such that*

$$\aleph_\alpha < \aleph_\beta, \quad \aleph_\beta \leqslant \mathfrak{c} < \aleph_{\omega_\beta}, \quad 2^{\aleph_\alpha} = \aleph_{\omega_\beta}, \tag{1}$$

then the proposition $(\mathfrak{c}, \aleph_\alpha)$ *is false.*

For example,

$$\alpha = 1, \beta = 2, \aleph_2 \leqslant \mathfrak{c} < \aleph_{\omega_2} = 2^{\aleph_1}.$$

Incidentally, the first relation in (1) is redundant: it follows from the third one by Koenig's theorem.

From this theorem, together with Lemma B, we have:

COROLLARY 1. *If* $\aleph_\alpha, \aleph_\beta$ *satisfying* (1) *exist, the proposition* $(\mathfrak{c}, \mathfrak{c})$ *is false.*

We have, in particular (special cases):

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COROLLARY 2. *The proposition* $(\mathfrak{c}, \mathfrak{c})$ *is false if any one of the following propositions* (2), (3), (4), *holds:*

$$2^{\aleph_1} = \aleph_{\omega_2} \quad and \quad \aleph_2 \leqslant \mathfrak{c} < \aleph_{\omega_2}; \tag{2}$$

$$2^{\aleph_2} = \aleph_{\omega_3} \quad and \quad \aleph_3 \leqslant \mathfrak{c} < \aleph_{\omega_3}; \tag{3}$$

$$2^{\aleph_n} = \aleph_{\omega_{n+1}} \qquad (n = 0, 1, 2, \ldots). \tag{4}$$

Note. In what follows, α, β are "constants"; γ, ξ are "variables."

Proof of Theorem. We assume (1), and we are going to construct a counterexample for the proposition $(\mathfrak{c}, \aleph_\alpha)$. Let G be the family of all dyadic functions on X, where $\overline{\overline{X}} = \aleph_\alpha$. Then $\overline{\overline{G}} = 2^{\aleph_\alpha}$, hence, by (1),

$$\overline{\overline{G}} = \aleph_{\omega_\beta}. \tag{5}$$

Given a sequence $\{\phi_n(x)\}$ of dyadic functions on X, let F_ϕ be the family of those functions which are limits of subsequences of $\{\phi_n(x)\}$. Furthermore, let Φ be the family of *all* sequences (of dyadic functions on X), and let $\mathscr{S}$ be the system of all families F_ϕ where $\phi \in \Phi$.

It follows that

$$\overline{\overline{F}}_\phi \leqslant \mathfrak{c}, \ \overline{\overline{\Phi}} = (2^{\aleph_\alpha})^{\aleph_0} = 2^{\aleph_\alpha} = \aleph_{\omega_\beta}, \tag{6}$$

and

$$\overline{\overline{\mathscr{S}}} \leqslant \overline{\overline{\Phi}} \quad \text{and} \quad \overline{\overline{G}} \leqslant \overline{\overline{\mathscr{S}}}. \tag{6'}$$

(The last inequality follows from the fact that every element of G corresponds to a one-element family F_ϕ whose base converges.)

Hence, from (1), (5), (6), and (6′), we have:

$$\overline{\overline{F}}_\phi \leqslant \mathfrak{c} < \aleph_{\omega_\beta}, \quad \overline{\overline{G}} = \overline{\overline{\mathscr{S}}} = \aleph_{\omega_\beta}. \tag{7}$$

Now, since F_ϕ is the "maximal" family with the base ϕ, every family of functions admitting a base is contained in some F_ϕ, i.e.,

(8) *If F has a base then, for some ϕ, $F \subset F_\phi \in \mathscr{S}$.*

On the assumption of (7) we are going to construct a family F° of power $\leqslant \aleph_\beta$ which is not contained in any $F_\phi \in \mathscr{S}$, which therefore, according to (8), has no base.

Let

$$F_1, F_2, \ldots F_\omega, \ldots F_\xi, \ldots |_{\omega_{\omega_\beta}}$$

be the elements of $\mathscr{S}$, ordered in a transfinite sequence.

We put

$$H_\gamma = \sum_{\xi < \omega_\gamma} F_\xi \qquad (\gamma < \omega_\beta), \tag{9}$$

and, for each $\gamma < \omega_\beta$, let h_γ be one element of the set $G - H_\gamma$:

(10) $$h_\gamma \in G - H_\gamma \qquad (\gamma < \omega_\beta).$$

This element h_γ exists because

$$\overline{\overline{H}}_\gamma \leqslant \mathfrak{c}\aleph_\gamma < \aleph_{\omega_\beta},$$

but $\overline{\overline{G}} = \aleph_{\omega_\beta}$ hence the set $G - H_\gamma$ is not empty.

Now let F° be the set of all these h_γ. Then $\overline{\overline{F^\circ}} \leqslant \aleph_\beta \leqslant \mathfrak{c}$. It follows from (10) that $h_\gamma \notin F_\xi$ for all $\xi < \omega_\gamma$. But $h_\gamma \in F^\circ$ by definition. Therefore F° cannot be contained in any F_ξ (for all $\xi < \omega_{\omega_\beta}$), i.e., in any $F_\phi \in \mathcal{S}$.

Thus, by (8), F° has no base, although

$$\overline{\overline{F^\circ}} \leqslant \mathfrak{c}.$$

Reference

1. F. Rothberger, *On families of real functions with a denumerable base*, Ann. Math., vol. 45 (1944), 397-406.

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