# LOGARITHMIC CAPACITY OF SETS AND DOUBLE TRIGONOMETRIC SERIES

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**1. Introduction.** It is the purpose of this paper to establish a closer connection between the logarithmic capacity of sets and double trigonometric series. In **(9)**, closed sets of logarithmic capacity zero were established as sets of uniqueness for a particular class of double trigonometric series under circular $(C, 1)$ summability. By slightly changing this class of series but still maintaining closed sets of logarithmic capacity zero as sets of uniqueness, it is shown in this paper that closed sets of positive logarithmic capacity form sets of multiplicity. Widening the class of series still further, it is shown here that closed sets of uniqueness and closed sets of logarithmic capacity zero also coincide for this new class under local uniform circular $(C, 1)$ summability.

The motivation for establishing these results arose from lectures on the uniqueness of one-dimensional trigonometric series delivered by Professor A. Beurling at the Institute for Advanced Study.

In this paper we are able, also, to obtain for planar sets a result analogous to one for linear sets given by Salem and Zygmund in **(8)**, where a necessary and sufficient condition that a linear set be of positive logarithmic capacity is given in terms of Fourier-Stieltjes series.

**2. Definitions and Notation.** Vectorial notation will be used whenever convenient and will be signified by capital letters thus:

$$P = (p, q), \quad X = (x, y), \quad \alpha X + \beta P = (\alpha x + \beta p, \alpha y + \beta q),$$
$$PX = px + qy, \quad |X| = (x^2 + y^2)^{\frac{1}{2}}.$$

Let $E$ be a bounded Borel set. Then under the usual definition **(5**, p. 48**)**, $E$ is said to be a set of positive logarithmic capacity if there exists a non-negative measure $\mu$ defined on the Borel sets in the plane such that $\mu(E) = 1$ and $\mu(A) = 0$ if $AE = 0$ and such that the potential

$$u(X) = \int_E \log|P - X|^{-1} d\mu(P) \tag{1}$$

has a positive upper bound. If no such measure exists for the set $E$, then $E$ is said to be a set of logarithmic capacity zero.

It is known **(2)** that if $E$ is a closed and bounded set of logarithmic capacity zero and $D$ is a domain, then $D - DE$ is a domain. Furthermore if $g(X)$ is harmonic and bounded in $D - DE$, then there exists a function $h(X)$ harmonic in $D$ and equal to $g(X)$ in $D - DE$ **(6**, p. 335**)**.

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A double trigonometric series

$$\sum_{M} a_M e^{iMX}, \tag{2}$$

where $M$ represents a lattice point $(m, n)$ and the $a_M$ are arbitrary complex numbers will be said to converge circularly at the point $X$ to the value $L(X)$ if the circular partial sums of rank $R$,

$$S_R(X) = \sum_{|M| \leqslant R} a_M e^{iMX}, \tag{3}$$

converge to the finite value $L(X)$. The series will be $(C, 1)$ circularly summable to $L(X)$ if the $(C, 1)$ circular means of rank $R$,

$$\sigma_R(X) = \sum_{|M| \leqslant R} a_M e^{iMX}\left(1 - \frac{|M|^2}{R^2}\right) = \frac{2}{R^2}\int_0^R S_r(X)\, r dr, \tag{4}$$

converge to the finite value $L(X)$.

In **(9)**, we called (2) a series of type $(U)$ if $a_M = o(1)$, that is if $a_M \to 0$ as $|M| \to \infty$, and if the partial sums

$$\sum_{1 \leqslant |M| \leqslant R} \frac{a_M}{|M|^2} e^{iMX}$$

converge uniformly. For the purpose of this paper it will be advantageous to widen the classes of series to be studied. We shall call (2) a series of class $(U')$ if

$$\sum_{M \neq 0} \frac{a_M}{|M|^2} e^{iMX} \tag{5}$$

is the Fourier series of a continuous periodic function. We call (2) a series of class $(B')$ if (5) is the Fourier series of a bounded function. For both of these classes no restriction is placed on the $a_M$.

The open disc of radius $t$ and center $P$ will be denoted in this paper by $D(P, t)$; the circumference of this disc, by $C(P, t)$. The fundamental semi-closed square

$$\{(x, y);\ -\pi < x \leqslant \pi,\ -\pi < y \leqslant \pi\}$$

will be designated by $\Omega$; the interior of $\Omega$ by $\Omega^\circ$.

We say that the series (2) is locally uniformly $(C, 1)$ circularly summable in a set $E$ if for every $P$ in $E$, there exists a $D(P, t)$, $t > 0$, such that $\sigma_R(X)$ defined by (4) tends uniformly to a finite limit for $X$ in $D(P, t)$.

Given a closed set $Z \subset \Omega$ we shall say that $Z$ is a set of uniqueness for a series of class $(U')$ under circular $(C, 1)$ summability if the fact that $\sum_M a_M e^{iMX}$ is a series of class $(U')$ for which $\sigma_R(X) \to 0$ in $\Omega - Z$ implies that $a_M = 0$ for all $M$.

Given a closed set $Z \subset \Omega$, we shall say that $Z$ is a set of uniqueness for series of class $(B')$ under local uniform circular $(C, 1)$ summability if the fact that $\sum_M a_M e^{iMX}$ is a series of class $(B')$ for which $\sigma_R(X) \to 0$ locally uniformly in $\Omega - Z$ implies that $a_M = 0$ for all $M$.

Let $E$ be a bounded Borel set and let $\mu$ be a non-negative measure defined on the Borel sets of the plane. If $\mu(E) = 1$ and if $\mu(A) = 0$ for all Borel sets $A$ with the property $AE = 0$, we say that $\mu$ is concentrated on $E$. Furthermore if $E$ is contained in $\Omega$, we can consider the Fourier-Stieltjes series of $\mu$, written

$$d\mu \sim \sum a_M e^{iMX} \tag{6}$$

where

$$a_M = \frac{1}{4\pi^2} \int_\Omega e^{-iMX} d\mu(X).$$

**3. Statement of main results.** We shall prove the following three theorems connecting the logarithmic capacity of sets and double trigonometric series.

THEOREM 1. *Let $E$ be a Borel set contained in the semi-closed square $\Omega$. Then a necessary and sufficient condition that $E$ be of positive logarithmic capacity is that there exists a non-negative measure $\mu$ concentrated on $E$ whose Fourier-Stieltjes series is of class $(B')$.*

THEOREM 2. *Let $Z$ be a closed set contained in the semi-closed square $\Omega$. Then a necessary and sufficient condition that $Z$ be a set of uniqueness for series of class $(U')$ under circular $(C, 1)$ summability is that $Z$ be of logarithmic capacity zero.*

THEOREM 3. *Let $Z$ be a closed set contained in the semi-closed square $\Omega$. Then a necessary and sufficient condition that $Z$ be a set of uniqueness for series of class $(B')$ under local uniform circular $(C, 1)$ summability is that $Z$ be of logarithmic capacity zero.*

Before proving these theorems, we should investigate the properties of Fourier-Stieltjes series and generalized Laplacians.

**4. Fourier-Stieltjes series.** Some of the notions in this section come from a course given by Professor Bochner at Princeton University.

Supposing $f(P)$ integrable on $C(X, t)$, we shall henceforth designate the mean-value of $f$ on this circle by $f_X(t)$, thus

$$f_X(t) = \frac{1}{2\pi} \int_0^{2\pi} f(x + t\cos\theta, y + t\sin\theta)\, d\theta. \tag{7}$$

Then by **(1)**, we have the following result:

LEMMA 1. *Let $f(x)$ be a function which is integrable on $\Omega$ and periodic of period $2\pi$ in each variable. Then the $(C, 1)$ circular mean of rank $R$ of the Fourier series of $f(X)$ is given by*

$$\sigma_R(X) = 2 \int_0^\infty f_X(t)\, J_2(tR)/t\, dt \tag{8}$$

*where $J_2(t)$ is the Bessel function of the first kind and order 2.*

REMARK 1. (8) can be replaced by the equality

$$\sigma_R(X) = \frac{1}{\pi}\int_{E_2} f(X+P)\,\frac{J_2(|P|R)}{|P|^2}\,dP \tag{9}$$

where $E_2$ is the plane and the expression on the right side of (9) is understood to be the Lebesgue integral over $E_2$, where $X$ in the integrand is a fixed point.

Remark 1 follows from the fact that for fixed $X$ and $R$ and for all $t \geqslant 0$ there is a constant $K$ such that

$$\int_{D(0,\,t+1)-D(0,\,t)} |f(X+P)|dP < K(t+1),$$

$$\frac{|J_2(|P|R)|}{|P|^2} < K \qquad \text{for } |P| \leqslant 1,$$

$$\frac{|J_2(|P|R)|}{|P|^2} < \frac{K}{|P|^{5/2}} \qquad \text{for } |P| > 1.$$

For then

$$\int_{D(0,T)} |f(X+P)|\,\frac{|J_2(|P|R)|}{|P|^2}\,dP \leqslant \sum_{i=0}^{[T]} \int_{D(0,\,i+1)-D(0,\,i)} |f(X+P)|\,\frac{|J_2(|P|R)|}{|P|^2}\,dP$$
$$\leqslant K^2 + \sum_{i=1}^{[T]} \frac{K^2(i+1)}{i^{5/2}} \leqslant K_1,$$

where $K_1$ is another constant independent of $T$.

Given a non-negative measure $\mu$ concentrated on a Borel set $E$ contained in the semi-closed square $\Omega$ we can form the $(C, 1)$ circular mean of rank $R$ of its Fourier-Stieltjes series. It is clear, however, that we have to extend $\mu$ so that it is defined on the whole plane before we can get an expression similar to the right side of (9) for the $(C, 1)$ circular mean of rank $R$.

We handle the problem of the extension of $\mu$ defined in $\Omega$ in the following manner. Let $\eta_M$ represent the point with the coordinates $(2\pi m, 2\pi n)$ where $m$ and $n$ represent any pair of integers positive, negative, or zero. Defining the point set $A + X$ to be the set of points $[P; P - X \text{ in } A]$, we have the double sequence of squares $\Omega_M = \Omega + \eta_M$. In particular, $\Omega_0 = \Omega$.

Now given a non-negative measure $\mu$ concentrated on a set $E \subset \Omega$, we call this measure $\mu_0$ and define a measure $\mu_M$ for every $M$ on the Borel sets of the plane by $\mu_M(A) = \mu(A - \eta_M)$. We thus see that $\mu_M$ is a non-negative measure concentrated on the set $E + \eta_M$. We then define a non-negative measure $\bar{\mu}$ on the bounded Borel sets of the plane by the formula

$$\bar{\mu}(A) = \sum_M \mu_M(A).$$

Noticing that

$$\bar{\mu}(A + \eta_{M_1}) = \sum_M \mu_M(A + \eta_{M_1}) = \sum_M \mu_{M-M_1}(A) = \bar{\mu}(A),$$

we call $\bar{\mu}$ the periodic extension of $\mu$. Henceforth the Fourier-Stieltjes series of $\bar{\mu}$ will be understood to be the Fourier-Stieltjes series of $\mu$ as defined in §2.

With this extension of the measure, we are now in a position to state and prove the following lemma:

LEMMA 2. *Let $\mu$ be a non-negative measure concentrated on a Borel set $E$ contained in $\Omega$ and let $\bar{\mu}$ be the periodic extension of $\mu$. Let $\sigma_R(X)$ be the $(C, 1)$ circular mean of rank $R$ of the Fourier-Stieltjes series of $\bar{\mu}$. Then*

$$\sigma_R(X) = \frac{1}{\pi}\int_{E_2} \frac{J_2(|P - X|R)}{|P - X|^2}\, d\bar{\mu}(P). \tag{10}$$

To prove the lemma, let us first observe that

$$\sigma_R(X) = \sum_{|M| \leqslant R} a_M e^{iMX}\left(1 - \frac{|M|^2}{R^2}\right) = \int_\Omega K_R(X - P)\, d\mu(P)$$

where

$$K_R(X) = \frac{1}{4\pi^2}\sum_{|M| \leqslant R} e^{iMX}\left(1 - \frac{|M|^2}{R^2}\right).$$

It is not difficult to see that the right side of (10) is a continuous function of $X$ and that the same is true for $\sigma_R(X)$. Therefore to prove (10), it is only necessary to show that if $A$ is any bounded Borel set then

$$\int_A dX \int_\Omega K_R(X - P) d\,\mu(P) = \pi^{-1}\int_A dX \int_{E_2} \frac{J_2(|P - X|R)}{|P - X|^2}\, d\bar{\mu}(P). \tag{11}$$

Now setting

$$\psi(B) = \pi^{-1}\int_B \frac{J_2(|X|R)}{|X|^2}\, dX$$

for any Borel set $B$, we see that $\psi$ is an additive function of a set defined on the Borel sets in the plane. Furthermore, we see that the right side of (11) is by Fubini's theorem equal to $\int_{E_2} \psi(A - P)\, d\bar{\mu}(P)$ which in turn is equal to $\int_{E_2} \bar{\mu}(A - P)\, d\psi(P)$. This last fact follows from the observations that

$$\bar{\mu}(D(P, 1)) = 0(|P|^{\frac{1}{4}}), \quad \int_{E_2} |P|^{\frac{1}{4}}\, \frac{|J_2(|P|R)|}{|P|^2}\, dP < \infty$$

and an application of (**10**, Lemma 1). But $\bar{\mu}(A - P)$ is for fixed $A$, a bounded periodic function of $P$. Consequently by Remark 1,

$$\begin{aligned}
\int_{E_2} \bar{\mu}(A - P)\, d\psi(P) &= \int_\Omega \bar{\mu}(A - P)\, K_R(P)\, dP \\
&= \sum_M \int_\Omega \mu(A - P - \eta_M)\, K_R(P)\, dP \\
&= \sum_M \int_{\Omega_M} \mu(A - X)\, K_R(X)\, dX \\
&= \int_{E_2} \mu(A - X)\, K_R(X)\, dX.
\end{aligned}$$

However, letting $\chi_A(X)$ be the characteristic function of $A$, we have that

$$\int_{E_2} \mu(A - X)\, K_R(X)\, dX = \int_{E_2} d\mu(P) \int_{E_2} K_R(X)\, \chi_A(X + P)\, dX$$
$$= \int_{\Omega} d\mu(P) \int_{E_2} K_R(X - P)\, \chi_A(X)\, dX$$
$$= \int_{\Omega} d\mu(P) \int_{A} K_R(X - P)\, dX,$$

which is the left side of (11), and the lemma is proved.

LEMMA 3. *Given a non-negative measure $\mu$ concentrated on a Borel set $E$ contained in $\Omega$, let $\bar{\mu}$ be its periodic extension. Suppose $\bar{\mu}[D(X_0, t_0)] = 0$. Then the Fourier-Stieltjes series of $\bar{\mu}$ is uniformly circularly summable $(C, 1)$ to zero in $D(X_0, \frac{1}{2}t_0)$.*

For by (10), we have, since $\bar{\mu}[D(X_0, t_0)] = 0$, that

$$\sigma_R(X) = \frac{1}{\pi} \int_{E_2 - D(X_0, t_0)} \frac{J_2(|P - X|R)}{|P - X|^2}\, d\bar{\mu}(P).$$

However, there is a constant $K$ such that

$$|J_2(u)| \leqslant K u^{-\frac{1}{2}} \qquad \text{for } u > 1.$$

Consequently, we see, for $R$ sufficiently large and $X$ in $D(X_0, \frac{1}{2}t_0)$, that

$$|\sigma_R(X)| \leqslant \frac{1}{\pi} \int_{E_2 - D(X_0, t_0)} \frac{K}{|P - X|^{5/2} R^{\frac{1}{2}}}\, d\bar{\mu}(P)$$
$$\leqslant \frac{1}{\pi R^{\frac{1}{2}}} \sum_{j=1}^{\infty} O(j^{-3/2}).$$

Therefore $\sigma_R(X) = O(R^{-\frac{1}{2}})$ uniformly for $X$ in $D(X_0, \frac{1}{2}t_0)$, and the lemma is proved.

**5. Generalized Laplacians.** Let us suppose that $F(X)$ is defined and integrable in $D(X_0, t)$ and let us set

$$F_{1,X_0}(t) = \frac{1}{\pi t^2} \int_{D(X_0, t)} F(P)\, dP.$$

We then say that $F(X)$ has a generalized Laplacian of the second kind at the point $X_0$, designated by $\Delta_2 F(X_0)$, equal to $\alpha_1$, if

$$\lim_{t \to 0} \frac{8[F_{1,X_0}(t) - F(X)]}{t^2} = \alpha_1.$$

For the purposes of this paper, it will be necessary to prove an extension of (**9**, Lemmas 1 and 2).

LEMMA 4. *Let $\sum a_M e^{iMX}$ be a double trigonometric series which is $(C, 1)$ circularly summable to zero at the point $X_0$. Furthermore, let*

$$- \sum_{M \neq 0} \frac{a_M}{|M|^2} e^{iMX}$$

*be uniformly circularly summable* $(C, 1)$ *to* $F(X) - \frac{1}{4}a_0|X|^2$ *in* $D(X_0, t_0)$, $t_0 > 0$. *Then* $\Delta_2 F(X_0) = 0$.

Setting

$$S_R(X) = \sum_{|M|\leqslant R} a_M e^{iMX} \text{ and } T_R(X) = -\sum_{1\leqslant|M|\leqslant R} \frac{a_M}{|M|^2} e^{iMX}\left(1 - \frac{|M|^2}{R^2}\right),$$

we observe that

(a) $$\frac{1}{\pi t^2}\int_{D(X_0,\,t)} T_R(X)\,dX = -2\sum_{1\leqslant|M|\leqslant R} a_M e^{iMX_0}\frac{J_1(|M|t)}{|M|^3 t}\left(1 - \frac{|M|^2}{R^2}\right),$$

(b) $S_R(X_0) = o(R^2)$,

(c) $$R^{-2}\sum_{1\leqslant|M|\leqslant R} a_M e^{iMX_0}\frac{J_1(|M|t)}{|M|t} = R^{-2}\int_0^R [S_u(X_0) - a_0]\frac{J_2(ut)}{u}\,du + R^{-2}[S_R(X_0) - a_0]\frac{J_1(Rt)}{Rt} = o(1) \text{ as } R\to\infty \text{ for fixed } t,$$

(d) $$\frac{1}{\pi t^2}\int_{D(X_0,\,t)} T_R(X)\,dX \to F_{1,X_0}(t) - \frac{a_0}{4}[|X|_0^2 + \tfrac{1}{2}t^2] \text{ for } 0 < t < t_0.$$

We conclude from (a), (b), (c), and (d) that

$$\frac{8}{t^2}[F_{1,X_0}(t) - F(X_0)] = a_0 + \lim_{R\to\infty} 8\sum_{1\leqslant|M|\leqslant R}\frac{a_M e^{iMX_0}}{|M|^2 t^2}\left[1 - \frac{2J_1(|M|t)}{|M|t}\right].$$

The lemma then follows from (**3**, Theorem 1).

**6. A particular set of Fourier coefficients.** Let us set $\Phi(X) = 2\pi \log |X|^{-1}$ for $X$ in $\Omega$ and then extend $\Phi(X)$ periodically; so that using the notation of §4, $\Phi(X) = 2\pi \log |X - \eta_M|^{-1}$ for $X$ in $\Omega_M$. We then have the following lemma:

LEMMA 5. *The Fourier coefficients* $1/\lambda_M$ of $\Phi(X)$, *with* $M \neq 0$, *have the following two properties*:

(i) $\frac{1}{\lambda_M} > 0$ *for all* $M$,

(ii) *There exists a constant* $K$ *independent of* $M$ *such that*

$$\left|\frac{1}{\lambda_M} - \frac{1}{|M|^2}\right| < K\left[\frac{1}{|M|^2(n^2+1)} + \frac{1}{|M|^2(m^2+1)}\right].$$

By means of Green's second identity, we observe that for $|M| \neq 0$,

$$(2\pi)^{-1}\int_{\Omega - D(0,\,\epsilon)} \log\frac{1}{|X|} e^{i(mx+ny)}dX = \frac{J_0(|M|\epsilon)}{|M|^2} - \int_{-\pi}^{\pi}\frac{\cos m\pi\, e^{iny} + \cos n\pi\, e^{imy}}{\pi^2 + y^2}\,dy + o(1).$$

Consequently, for $|M| \neq 0$,

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$$\frac{1}{\lambda_M} = \frac{1}{|M|^2} - \frac{2}{|M|^2}\int_0^{\pi} \frac{\cos m\pi \cos ny + \cos n\pi \cos my}{\pi^2 + y^2}\, dy.$$

Since

$$\int_0^{\pi} \frac{dy}{\pi^2 + y^2} = \frac{1}{4},$$

we find $\lambda_M > 0$. As two integrations by parts show, there is a constant $K$ such that

$$\left| \int_0^{\pi} \frac{\cos ny}{\pi^2 + y^2}\, dy \right| \leqslant \frac{K}{n^2 + 1}$$

for all $n$, and the lemma is proved.

**7. Proof of Theorem 1.** Let $E$ be a Borel set contained in the semi-closed square $\Omega$. Then a necessary and sufficient condition that $E$ have positive logarithmic capacity is that there exists a non-negative measure $\mu$ concentrated on $E$ such that

$$u(P) = \int_E \log |P - X|^{-1} d\mu(X) \tag{12}$$

is bounded above.

Using $\Phi(X)$ as defined in §6, we set

$$u_1(P) = \int_E \Phi(P - X)\, d\mu(X) \tag{13}$$

and observe that $u_1(P)$ is lower semi-continuous. Furthermore we observe that $u_1(P)$ is bounded above if and only if $u(P)$ is bounded above.

By (**4**, p. 84), if $E$ has positive logarithmic capacity $\mu$ can be chosen so that $u(P)$ is continuous. But it is clear that $u(P)$ is continuous if and only if $u_1(P)$ is continuous.

To prove the sufficiency condition, let us suppose $\mu$ is concentrated on $E$ and $d\mu \sim \sum a_M e^{iMX}$ and that

$$\sum_{M \neq 0} \frac{a_M}{|M|^2} e^{iMX}$$

is the Fourier series of a bounded function. Then it follows from Lemma 1 that

$$\left| \sum_{1 \leqslant |M| \leqslant R} \frac{a_M}{|M|^2} e^{iMX} \left(1 - \frac{|M|^2}{R^2}\right) \right| \leqslant K,$$

where $K$ is independent of $R$ and $X$. But since $a_M = O(1)$, we conclude that the circular partial sums of rank $R$ of

$$\sum_{M \neq 0} \frac{a_M}{|M|^2} e^{iMX}$$

are uniformly bounded. Furthermore the series

$$\sum_{M \neq 0} \frac{1}{|M|^2} \left[ \frac{1}{n^2 + 1} + \frac{1}{m^2 + 1} \right]$$

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is convergent. We thus have from Lemma 5 that the circular partial sums of rank $R$ of

$$\sum_{M\neq 0} \frac{a_M}{\lambda_M} e^{iMX}$$

are uniformly bounded.

Let $u(P)$ and $u_1(P)$ be given by (12) and (13) respectively. Then for $|M|\neq 0$,

$$\frac{1}{4\pi^2}\int_{\Omega} e^{-iMP} u_1(P)\, dP = \frac{1}{4\pi^2}\int_{E} e^{-iMX} d\mu(X) \int_{\Omega} e^{-iM(P-X)}\Phi(P-X)\, dP = \frac{a_M}{\lambda_M} 4\pi^2.$$

But then $u_1(P)$ is an essentially bounded function which is lower semi-continuous and consequently bounded above. $u(P)$ is therefore bounded above and the sufficiency condition is proved.

To prove the necessity, let $E$ be of positive logarithmic capacity. Let $\mu$ be a non-negative measure concentrated on $E$ chosen so that $u(P)$ given by (12) is continuous. Consequently $u_1(P)$ given by (13) is a continuous periodic function. In the same manner as before, we find that the Fourier series of $u_1(P) - (4\pi^2)^{-1}\int_\Omega u_1(P)\, dP$ is

$$\sum_{M\neq 0} \frac{4\pi^2 a_M}{\lambda_M} e^{iMX},$$

where the $a_M$ are the Fourier-Stieltjes coefficients of $\mu$. But the $(C, 1)$ circular means of rank $R$ of this series converge uniformly. It then follows from Lemma 5 that the $(C, 1)$ circular means of rank $R$ of

$$\sum_{M\neq 0} \frac{a_M}{|M|^2} e^{iMX}$$

converge uniformly. This latter series is consequently the Fourier series of a continuous periodic function and the necessary condition is proved.

It is to be noticed that we have also proved the following fact which we state as a remark.

Remark 2. Let $E$ be a Borel set contained in the semi-closed square $\Omega$. Then a necessary and sufficient condition that $E$ be of positive logarithmic capacity is that there exists a non-negative measure $\mu$ concentrated on $E$ whose Fourier-Stieltjes series is of class $(U')$.

**8. Proof of Theorem 2.** Suppose that $T = \sum a_M e^{iMX}$ is $(C, 1)$ circularly summable to zero in $\Omega - Z$ where $Z$ is a closed set of logarithmic capacity zero contained in the semi-closed square $\Omega$ and $T$ is a series of class $(U')$.

Set

$$F(X) - \tfrac{1}{4}a_0|X|^2 = -\lim_{R\to\infty} \sum_{1\leqslant |M|\leqslant R} \frac{a_M}{|M|^2}\left(1 - \frac{|M|^2}{R^2}\right) e^{iMX} \tag{14}$$

for all $X$ in the plane. Since $T$ is of class $(U')$, we have that the right side of (14) is uniformly convergent and consequently that

$$F(X) - \tfrac{1}{4}a_0|X|^2 = G(X) \tag{15}$$

where $G(X)$ is a continuous periodic function in the plane.

Take any bounded domain $D$ in the plane. Then by Lemma 4 and the properties of sets of logarithmic capacity zero, we have that there exists a closed and bounded set of logarithmic capacity zero $Z_1$ such that $\Delta_2 F(X) = 0$ in the domain $D - DZ_1$. But by (**7**, p. 14), $F(X)$ is then harmonic in $D - DZ_1$. Since $F(X)$ is continuous in the closure of $D$, we obtain by (**6**, p. 335) a function $H(X)$ equal to $F(X)$ in $D - DZ_1$ and harmonic in $D$. But $DZ_1$ is of measure zero, $F(X)$ is therefore harmonic in $D$ and consequently in the whole plane.

Furthermore, from (15), $F(X) = O(|X|^2)$. Therefore by (**11**, p. 19) $F(X)$ is a polynomial of at most degree 2, and the same is therefore true of $G(X)$. But $G(X)$ being continuous and periodic must then be a constant. Consequently

$$\sum_{1 \leqslant |M| \leqslant R} \frac{a_M}{|M|^2} e^{iMX} \left(1 - \frac{|M|^2}{R^2}\right) \to K$$

uniformly for all $X$, where $K$ is a constant. We conclude that $a_M = 0$ for $M \neq 0$, and then since our series was assumed $(C, 1)$ summable to zero in $\Omega - Z$, we have that $a_0 = 0$.

To show that $Z$ is not a set of uniqueness if $Z$ is a closed set of positive logarithmic capacity contained in the semi-closed square $\Omega$, take a non-negative measure $\mu$ concentrated on $Z$ with Fourier-Stieltjes series $\sum a_M e^{iMX}$ which is in class $(U')$. By Remark 2, this can always be done. By Lemma 3, $\sum a_M e^{iMX}$ is $(C, 1)$ circularly summable to zero in $\Omega - Z$. $Z$ is therefore not a set of uniqueness, and the theorem is proved.

**9. Proof of Theorem 3.** Let us prove the sufficiency first. Suppose that $S_R(X)$ is given by (3) and $\sigma_R(X)$ by (4), and suppose, further, that $\sigma_R(X) \to 0$ locally uniformly in $\Omega - Z$ where $Z$ is a closed set of logarithmic capacity zero contained in $\Omega$. Let $E_2$ designate the plane and $\tilde{Z} = \sum_M Z_M$ where $Z_M = Z + \eta_M$, $\eta_M$ as in §4. Then $\sigma_R(X) \to 0$ locally uniformly in $E_2 - \tilde{Z}$. It is furthermore clear that if $\sigma_R(X) \to 0$ uniformly in $D(X_0, t_0)$, then

$$- \sum_{1 \leqslant |M| \leqslant R} \frac{a_M}{|M|^2} e^{iMX}$$

converges uniformly in $D(X_0, t_0)$.

Setting

$$F(X) - \tfrac{1}{4} a_0 |X|^2 = - \lim_{R \to \infty} \sum_{1 \leqslant |M| \leqslant R} \frac{a_M}{|M|^2} e^{iMX} \qquad \text{in } E_2 - Z, \tag{16}$$

we see that

(a) $\Delta_2 F(X) = 0$ in $E_2 - \tilde{Z}$, by Lemma 4,

(b) $F(X)$ is continuous in $E_2 - \tilde{Z}$ by the discussion in the above paragraph,

(c) $F(X) - \frac{1}{4} a_0 |X|^2$ is bounded in $E_2 - \tilde{Z}$ since $\sum a_M e^{iMX}$ is a series of class $(B')$.

From the properties of $\tilde{Z}$, (**7**, p. 14), and (a), (b), and (c), we conclude that there is a function $F_1(X)$ harmonic in $E_2$ and equal to $F(X)$ in $E_2 - \tilde{Z}$.

Since $F_1(X) - \frac{1}{4}a_0|X|^2$ is bounded in $E_2 - \tilde{Z}$ and $\tilde{Z}$ is of measure zero, $F_1(X) - \frac{1}{4}a_0|X|^2$ is bounded in $E_2$. But then $F_1(X)$ is $O(|X|^2)$ and consequently a polynomial of degree at most 2. Therefore $F_1(X) - \frac{1}{4}a_0|X|^2$ is a bounded polynomial; hence $F(X) - \frac{1}{4}a_0|X|^2$ is constant in $\Omega - Z$.

From (16) and the fact that our original series was in class $(B')$, we have that for $M \neq 0$

$$-\frac{a_M}{|M|^2} = \int_{\Omega - Z} [F(X) - \tfrac{1}{4}a_0 X^2]\, e^{-iMX}\, dX.$$

We conclude first that $a_M = 0$ for $M \neq 0$ and then that $a_0 = 0$.

To prove the necessary condition of this theorem, let $Z$ contained in $\Omega$ be a closed set of positive logarithmic capacity, and let $\mu$ be the non-negative measure of Theorem 1 which is concentrated on $Z$ with Fourier-Stieltjes series $\sum a_M e^{iMX}$ which is in class $(B')$. By Lemma 3, this series is locally uniformly $(C, 1)$ circularly summable to zero in $\Omega - Z$. This completes the proof of the theorem.

## References

1. S. Bochner, *Summation of multiple Fourier series by spherical means*, Trans. Amer. Math. Soc., *40* (1936), 175–207.
2. M. Brelot, *Sur la structure des ensembles de capacité nulle*, C. R. Acad. Sci. Paris, *192* (1931), 206–208.
3. M. Cheng, *Uniqueness of multiple trigonometric series*, Ann. of Math. (2), *52* (1950), 403–416.
4. Ch. J. de la Vallée Poussin, *Le Potentiel logarithmique* (Paris, 1949).
5. O. Frostman, *Potentiel d'équilibre et capacité des ensembles* (Lund, 1935).
6. O. D. Kellogg, *Foundations of potential theory* (New York, 1929).
7. T. Rado, *Subharmonic functions*, Ergebnisse der Mathematik, *5*, no. 1 (Berlin, 1937).
8. R. Salem and A. Zygmund, *Capacity of sets and Fourier series*, Trans. Amer. Math. Soc., *59* (1946), 23–41.
9. V. L. Shapiro, *An extension of results in the uniqueness theory of double trigonometric series*, Duke Math. J., *20* (1953), 359–366.
10. ———, *Summability and uniqueness of double trigonometric integrals*, Trans. Amer. Math. Soc. (to be published).
11. G. Valiron, *Lectures on the general theory of integral functions* (Toulouse, 1923).

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