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Nagoya Math. J.
Vol. 193 (2009), 1–93

# LIMITS OF CHARACTERS OF WREATH PRODUCTS $\mathfrak{S}_n(T)$ OF A COMPACT GROUP $T$ WITH THE SYMMETRIC GROUPS AND CHARACTERS OF $\mathfrak{S}_\infty(T)$, I

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**Abstract.** In the first half of this paper, all the limits of irreducible characters of $G_n = \mathfrak{S}_n(T)$ as $n \to \infty$ are calculated. The set of all *continuous* limit functions on $G = \mathfrak{S}_\infty(T)$ is exactly equal to the set of all characters of $G$ determined in [HH6]. We give a necessary and sufficient condition for a series of irreducible characters of $G_n$ to have a continuous limit and also such a condition to have a discontinuous limit. In the second half, we study the limits of characters of certain induced representations of $G_n$ which are usually reducible. The limits turn out to be characters of $G$, and we analyse which of irreducible components are responsible to these limits.

## Introduction

In the present paper we first investigate limits of irreducible characters of wreath products $G_n := \mathfrak{S}_n(T)$ of a compact group $T$ with the symmetric groups $\mathfrak{S}_n$ and thereby capture characters of its wreath product with the infinite symmetric group $G := \mathfrak{S}_\infty(T)$. This constitutes an important step in our program to develop harmonic analysis on such infinite wreath product groups. Secondly we investigate limits of characters of induced representations of $G_n$ of certain standard type treated in [HH6] which are usually reducible (cf. Section 10 for definition).

**1.** The group of all finite permutations on a set $I$ is denoted by $\mathfrak{S}_I$. A permutation $\sigma$ on $I$ is called *finite* if its support, $\mathrm{supp}(\sigma) := \{i \in I\ ;\ \sigma(i) \neq i\}$, is finite. We call the permutation group $\mathfrak{S}_{\boldsymbol{N}}$ on the set of natural numbers $\boldsymbol{N}$ the *infinite symmetric group*. The index $\boldsymbol{N}$ is frequently replaced by $\infty$. The symmetric group $\mathfrak{S}_n$ of degree $n$ is naturally embedded in $\mathfrak{S}_\infty$ as the permutation group of the subset $\boldsymbol{I}_n := \{1, 2, \ldots, n\} \subset \boldsymbol{N}$.

Received April 23, 2007.
Accepted June 13, 2008.
2000 Mathematics Subject Classification: Primary 20C32; Secondary 20C15, 20E22, 22A25, 22C05, 43A35.

Let $T$ be a compact group. We consider wreath product group $\mathfrak{S}_I(T)$ of $T$ with permutation group $\mathfrak{S}_I$ as follows:

$$\mathfrak{S}_I(T) = D_I(T) \rtimes \mathfrak{S}_I\,, \quad D_I(T) = \prod_{i\in I}{}' \, T_i\,, \quad T_i = T \ \ (i \in I), \tag{0.1}$$

where the symbol $\prod'$ means the restricted direct product, and $\sigma \in \mathfrak{S}_I$ acts on $D_I(T)$ as

$$D_I(T) \ni d = (t_i)_{i\in I} \stackrel{\sigma}{\longmapsto} \sigma(d) = (t'_i)_{i\in I} \in D_I(T),\ t'_i = t_{\sigma^{-1}(i)}\ (i \in I). \tag{0.2}$$

Identifying groups $D_I(T)$ and $\mathfrak{S}_I$ with their images in semidirect product $\mathfrak{S}_I(T)$, we have $\sigma\, d\, \sigma^{-1} = \sigma(d)$. The groups $D_{\boldsymbol{I}_n}(T)$ and $\mathfrak{S}_{\boldsymbol{I}_n}(T)$ are simply denoted by $D_n(T)$ and $\mathfrak{S}_n(T)$ respectively. Then $G = \mathfrak{S}_\infty(T)$ is an inductive limit of $G_n = \mathfrak{S}_n(T) = D_n(T) \rtimes \mathfrak{S}_n$. Since $T$ is compact and hence so is $G_n$, the inductive system is an example of a countable LCG inductive system in [TSH]. We introduce in $G$ its inductive limit topology $\tau_{ind}$. Then $G$ equipped with $\tau_{ind}$ becomes a topological group (cf. Theorem 2.7 in [TSH]), but not locally compact if $T$ is not finite.

A natural subgroup of $G = \mathfrak{S}_\infty(T)$ is given as a wreath product of $T$ with the alternating group $\mathfrak{A}_\infty$ as $G' := \mathfrak{A}_\infty(T) = D_\infty(T) \rtimes \mathfrak{A}_\infty$. Moreover, in the case where $T$ is abelian, we put

$$P_I(d) = \prod_{i\in I} t_i \quad \text{for} \ \ d = (t_i)_{i\in I} \in D_I(T), \tag{0.3}$$

and take a subgroup $S$ of $T$, and define subgroup $\mathfrak{S}_I(T)^S$ of $\mathfrak{S}_I(T)$ as

$$\mathfrak{S}_I(T)^S = D_I(T)^S \rtimes \mathfrak{S}_I \ \ \text{with} \ \ D_I(T)^S := \{d = (t_i)_{i\in I}\,;\, P_I(d) \in S\}. \tag{0.4}$$

If $S = \{e_T\}$ is trivial ($e_T$ denoting the identity element of $T$), we simply write it as $\mathfrak{S}_I(T)^e$. These kinds of groups, $\mathfrak{S}_\infty(T)$ and $\mathfrak{S}_\infty(T)^S$ with $T$ abelian, contain the infinite Weyl groups of classical types: $W_{\mathrm{A}_\infty} = \mathfrak{S}_\infty$ of type $\mathrm{A}_\infty$, $W_{\mathrm{B}_\infty} = \mathfrak{S}_\infty(\boldsymbol{Z}_2)$ of type $\mathrm{B}_\infty/\mathrm{C}_\infty$, and $W_{\mathrm{D}_\infty} = \mathfrak{S}_\infty(\boldsymbol{Z}_2)^e$ of type $\mathrm{D}_\infty$, and moreover the inductive limits $\mathfrak{S}_\infty(\boldsymbol{Z}_r) = \lim_{n\to\infty} G(r,1,n)$ of complex reflection groups $G(r,1,n) = \mathfrak{S}_n(\boldsymbol{Z}_r)$ (cf. [Kaw], [Sho]).

**2.** Seen from the viewpoint of developing harmonic analysis on big groups, especially those which are not of type I, *characters* of such groups play an important role as fundamental objects. In general, for a topological group $G$, let $K_1(G)$ denote the set of invariant continuous positive definite functions $f$ on $G$ normalized as $f(e) = 1$ ($e$ denoting the identity element of $G$), and $E(G)$ the set of all extremal (or indecomposable) elements in $K_1(G)$. Then every $f \in E(G)$ gives canonically a character of a quasi-equivalence class of factor representations of $G$ of finite type ([HH4]), and is called itself a character of $G$ (see Section 2 below).

The first and the second authors have developed character theory of wreath product groups in a series of papers, which extends celebrated Thoma's theory for $\mathfrak{S}_\infty$ in [Tho]. When $T$ is finite, $\tau_{ind}$ in $G = \mathfrak{S}_\infty(T)$ is discrete. In this case, the characters of $G$ are given in [HH2]. When $T$ is infinite compact, $G$ equipped with $\tau_{ind}$ is not locally compact, and the subset $\{(d, \mathbf{1})\ ;\ d \in D_\infty(T)\} \cong D_\infty(T)$ is an open neighbourhood of the identity element $e$ of $G$, where $\mathbf{1}$ denotes the trivial permutation on $\boldsymbol{N}$. All the characters of $G$ are given in [HH5]–[HH6] with a general explicit character formula for $f_A \in E(G)$ associated with a parameter $A$ (see Theorem 2.3 below).

A nice realization of a finite factor representation of $G$ of Vershik-Kerov type corresponding to any character $f \in E(G)$ are constructed in [HHH1].

**3.** The purpose of the present paper is two-fold.

The first one is to show that all the characters of $G = \mathfrak{S}_\infty(T)$ are obtained as limits of characters of finite-dimensional irreducible unitary representations (= IURs) of $G_n = \mathfrak{S}_n(T)$. Furthermore we establish a necessary and sufficient condition on a sequence of IURs $\rho_n$ of $G_n$ for existence of a limit of their normalized characters $\widetilde{\chi}_{\rho_n} = \chi_{\rho_n}/\dim\rho_n$ as $n \to \infty$, and also determine explicitly the limits. Asymptotic frequencies of Young diagrams will appear in a more involved form than in [VK1] reflecting the effect of compact group $T$.

There exist also the cases where the limits obtained are nowhere continuous, and we clarify the situation in these cases too.

Moreover, for the subgroups $G' := \mathfrak{A}_\infty(T)$ and $G^S$ in the case of $T$ abelian, all the characters of them are obtained simply by restricting those of $G$ if $S$ is open in $T$ (see [HH6] and Theorem 2.4 below for $G^S$). We prove in Section 8 that all the characters of $G^S$ can be approximated by sequences of irreducible characters of $G_n^S := G_n \cap G^S$ as $n \to \infty$.

The second purpose of the present paper is to analyse the original method in [HH5]–[HH6] of getting the general character formula, from the present stand point of approximation by irreducible characters of $G_n$. The method in [HH5]–[HH6] is to use the induced representations $\Pi = \mathrm{Ind}_H^G \pi$ of IURs $\pi$ of a special kind of subgroups $H$ of $G$. Taking a diagonal matrix element of $\Pi$ and centralizing it with respect to $G_n$ and then we take the limit as $n \to \infty$ to get $f_A$. This method is briefly reviewed in Section 10, then we can translate it as to approximate characters $f_A$ of $G$ by those of induced representation $\Pi_n = \mathrm{Ind}_{H_n}^{G_n} \pi_n$ of $G_n$ which are highly reducible in almost all cases. We investigate which of irreducible components of $\Pi_n$ are responsible to the limit or to approximate the character $f_A$.

**4.** This paper is divided into three chapters. Chapter 1 consists of Sections 1–2 and is devoted to preliminary preparations. Chapter 2 consists of Sections 3–9 and is devoted to the first purpose, and Chapter 3 consists of Sections 10–14 and is devoted to the second purpose. Details of each section can be seen from the table of contents below.

In the present paper, which is Part I of our work under the same title, we largely use methods of explicit calculations in the theory of group representations. The subsequent Part II [HHH2] is devoted to an approach by way of probabilistic methods, which is an extension of Vershik-Kerov's ergodic method in [VK1].

# Chapter I. Review on the wreath product $\mathfrak{S}_\infty(T)$ and its characters

## §1. Structure of wreath product groups $\mathfrak{S}_\infty(T) = D_\infty(T) \rtimes \mathfrak{S}_\infty$

For a set $I$, the group of finite permutations on $I$ is denoted by $\mathfrak{S}_I$. Fix a compact group $T$, and take the wreath product group $\mathfrak{S}_I(T)$ of $T$ with the symmetric group $\mathfrak{S}_I$ as

$$\mathfrak{S}_I(T) = D_I(T) \rtimes \mathfrak{S}_I\,, \quad D_I(T) := \prod_{i\in I}{}' T_i\,, \quad T_i = T \quad (i \in I), \tag{1.1}$$

where $\prod'_{i\in I} T_i$ denotes the restricted direct product of copies $T_i$ of $T$, and $\sigma \in \mathfrak{S}_I$ acts on $d = (t_i)_{i\in I} \in D_I(T)$ as $\sigma(d) = (t_{\sigma^{-1}(i)})_{i\in I}$. When $I = I_n := \{1, 2, \ldots, n\}$ or $I = \boldsymbol{N}$, the suffices $I$ are replaced by $n$ or $\infty$ respectively.

Put $G_n = \mathfrak{S}_n(T)$ and $G = \mathfrak{S}_\infty(T)$. Then, $G$ is an inductive limit of the inductive system of compact groups $G_2 \hookrightarrow \cdots \hookrightarrow G_n \hookrightarrow G_{n+1} \hookrightarrow \cdots$, and we introduce in $G_\infty$ the inductive limit topology $\tau_{ind}$ from the compact $\tau_{G_n}$ of $G_n$. Then $G_\infty$ becomes a topological group and is not locally compact if $T$ is not finite. For an element $g = (d, \sigma) \in G$ with $d = (t_i)_{i\in\boldsymbol{N}}$, put

$$\mathrm{supp}(g) = \mathrm{supp}(d) \cup \mathrm{supp}(\sigma),$$
$$\mathrm{supp}(d) = \{i \in \boldsymbol{N}\,;\, t_i \neq e_T\}, \quad \mathrm{supp}(\sigma) = \{i \in \boldsymbol{N}\,;\, \sigma(i) \neq i\}.$$

An element $g = (d, \sigma) \in G = \mathfrak{S}_\infty(T)$ is called *basic* in the following two cases:

CASE 1: $\sigma$ is cyclic and $\mathrm{supp}(d) \subset \mathrm{supp}(\sigma)$;
CASE 2: $\sigma = \mathbf{1}$ and for $d = (t_i)_{i\in \boldsymbol{N}}$, $t_q \neq e_T$ only for one $q \in \boldsymbol{N}$.

The element $(d, \mathbf{1})$ in Case 2 is denoted by $\xi_q = \xi_q(t_q)$, and put $\mathrm{supp}(\xi_q) := \mathrm{supp}(d) = \{q\}$. For a cyclic permutation $\sigma = (i_1\ i_2\ \cdots\ i_\ell)$ of $\ell$ integers, we define its *length* as $\ell(\sigma) = \ell$, and for the identity permutation $\mathbf{1}$, put $\ell(\mathbf{1}) = 1$ for convenience. In this connection, $\xi_q$ is also denoted by $(t_q, (q))$ with a trivial cyclic permutation $(q)$ of length 1. In Cases 1 and 2, put $\ell(g) = \ell(\sigma)$ for $g = (d, \sigma)$, and $\ell(\xi_q) = 1$. For basic elements $g = (d, \sigma)$ and $\xi_q = (t_q, (q))$, their expressions in a form of matrices with entries from $\{0\} \cup T$ are displayed in [HH6, §3], and will be very helpful.

An arbitrary element $g = (d, \sigma) \in G$, is expressed as a product of basic elements as

$$g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r}g_1g_2\cdots g_m \tag{1.2}$$

with $g_j = (d_j, \sigma_j)$ in Case 1, in such a way that the supports of these components, $q_1, q_2, \ldots, q_r$, and $\mathrm{supp}(g_j) = \mathrm{supp}(\sigma_j)$ $(1 \le j \le m)$, are mutually disjoint. This expression of $g$ is unique up to the orders of $\xi_{q_k}$'s and $g_j$'s, and is called *standard decomposition* of $g$ as in [HH4], [HH6]. Note that $\ell(\xi_{q_k}) = 1$ for $1 \le k \le r$ and $\ell(g_j) = \ell(\sigma_j) \ge 2$ for $1 \le j \le m$, and that, for $\mathfrak{S}_\infty$-components, $\sigma = \sigma_1\sigma_2\cdots\sigma_m$ gives a cycle decomposition of $\sigma$.

To write down conjugacy class of $g = (d, \sigma)$, there appear products of components $t_i$ of $d = (t_i)$, where the orders of taking products are crucial when $T$ is not abelian. So we should fix notations well.

We denotes by $[t]$ the conjugacy class of $t \in T$, and by $T/\!\sim$ the set of all conjugacy classes of $T$, and $t \sim t'$ denotes that $t, t' \in T$ are mutually conjugate in $T$. For a basic component $g_j = (d_j, \sigma_j)$ of $g$, let $\sigma_j = (i_{j,1}\ i_{j,2}\ \cdots\ i_{j,\ell_j})$ and put $K_j := \mathrm{supp}(\sigma_j) = \{i_{j,1}, i_{j,2}, \ldots, i_{j,\ell_j}\}$ with $\ell_j = \ell(\sigma_j)$. For $d_j = (t_i)_{i\in K_j}$, we put

$$P_{\sigma_j}(d_j) := \left[t'_{\ell_j}t'_{\ell_j-1}\cdots t'_2t'_1\right] \in T/\!\sim \quad \text{with } t'_k = t_{i_{j,k}} \ \ (1 \le k \le \ell_j). \tag{1.3}$$

Note that the product $P_{\sigma_j}(d_j)$ is well-defined, because, for $t_1, t_2, \ldots, t_\ell \in T$, we have $t_1t_2\cdots t_\ell \sim t_kt_{k+1}\cdots t_\ell t_1\cdots t_{k-1}$ for any $k$, that is, the conjugacy class does not depend on any cyclic permutation of $(t_1, t_2, \ldots, t_\ell)$.

THEOREM 1.2. *Let $T$ be a compact group. Take an element $g \in G = \mathfrak{S}_\infty(T)$ and let its standard decomposition into basic elements be $g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r}g_1g_2\cdots g_m$ in* (1.2)*, with $\xi_{q_k} = (t_{q_k}, (q_k))$, and $g_j = (d_j, \sigma_j)$, $\sigma_j$ cyclic, $\mathrm{supp}(d_j) \subset \mathrm{supp}(\sigma_j)$. Then the conjugacy class $[g]$ of $g$ is determined by*

$$(1.4) \qquad [t_{q_k}] \in T/\sim \ (1 \le k \le r) \quad \text{and} \quad (P_{\sigma_j}(d_j), \ell(\sigma_j)) \ (1 \le j \le m),$$

*where $P_{\sigma_j}(d_j) \in T/\sim$ and $\ell(\sigma_j) \ge 2$.*

A factor representation of finite type is of type $\mathrm{I}_n$ or of type $\mathrm{II}_1$. For the group $G = \mathfrak{S}_\infty(T)$, finite-dimensional irreducible representations are all one-dimensional (of type $\mathrm{I}_1$) as shown in [HH6, §3], and their characters are contained in the general formula of characters given in Theorem 2.3 below. The problem of approximating these characters by sequences of irreducible characters of $G_n = \mathfrak{S}_n(T)$ as $n \to \infty$ is trivial in this case.

PROPOSITION 1.3. ([HH6, Lemma 3.4]) *A finite-dimensional continuous irreducible representation $\pi$ of $\mathfrak{S}_\infty(T) = D_\infty(T) \rtimes \mathfrak{S}_\infty$ is a one-dimensional character, and is given in the form $\pi = \pi_{\zeta,\varepsilon}$ with*

$$(1.5) \qquad \pi_{\zeta,\varepsilon}(g) = \zeta(P(d))\ (\mathrm{sgn}_{\mathfrak{S}})^{\varepsilon}(\sigma) \quad \text{for } g = (d, \sigma) \in \mathfrak{S}_\infty(T),$$

*where $\zeta$ is a one-dimensional character of $T$, $P(d)$ is a product of components $t_i$ of $d = (t_i)$, and $\mathrm{sgn}_{\mathfrak{S}}(\sigma)$ denotes the usual sign of $\sigma$ and $\varepsilon = 0, 1$. (Since $\zeta(P(d)) = \prod_{i \in \boldsymbol{N}} \zeta(t_i)$, the order of taking product of $t_i$'s for $P(d)$ has no meaning here, even if $T$ is not abelian.)*

## §2. Characters of finite type of the wreath product $\mathfrak{S}_\infty(T)$

### 2.1. Characterization of characters for $\mathfrak{S}_\infty(T)$

Let $G$ be a topological group. A unitary representation $\pi$ of $G$ is called *factorial* if the von Neumann algebra $\mathfrak{U}_\pi := \pi(G)''$ generated by $\pi(G)$ is a factor. If $\mathfrak{U}_\pi$ is of finite type, that is, type $\mathrm{I}_n$, $n < \infty$, or $\mathrm{II}_1$, then it has a unique trace $\phi_\pi : \mathfrak{U}_\pi \to \boldsymbol{C}$ everywhere defined and normalized as $\phi_\pi(I) = 1$ at the identity operator $I$. Put

$$f_\pi(g) := \phi_\pi\big(\pi(g)\big) \quad (g \in G).$$

Then it is a continuous positive definite function on $G$, invariant under inner automorphisms, and normalized as $f_\pi(e) = 1$ at the identity element $e \in G$.

This function $f_\pi$ is called the character of the factor representation $\pi$, and determines the quasi-equivalence class of the factor representation $\pi$.

The set of all characters $f_\pi$ is characterized as follows. Let $K(G)$ be the set of all continuous invariant positive definite functions on $G$, and put $K_1(G) = \{f \in K(G) \,;\, f(e) = 1\}$. The set $E(G)$ of all extremal points of the convex set $K_1(G)$ is exactly equal to the set of all characters $f_\pi$ (cf. [HH3, Theorem 1.6.2]). In this sense, an element in $E(G)$ is called a *character* of $G$.

Now, in the case of the wreath product group $G = \mathfrak{S}_\infty(T)$, a character of $G$ has another characterization which plays very important role in our present study.

DEFINITION 2.1. A positive definite function $f$ on $G = \mathfrak{S}_\infty(T)$ is called *factorizable* if, for $g_1, g_2 \in G$ such that $\mathrm{supp}(g_1) \cap \mathrm{supp}(g_2) = \emptyset$, there holds

$$f(g_1 g_2) = f(g_1) f(g_2). \tag{2.1}$$

This condition is equivalent to the following:

(FTP) For $g \in G$, let $g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r} g_1 g_2 \cdots g_m$ be its standard decomposition into basic components in (1.2), then

$$f(g) = \prod_{1\le k\le r} f(\xi_{q_k}) \cdot \prod_{1\le j\le m} f(g_j). \tag{2.2}$$

Let $F(G)$ be the set of all $f \in K_1(G)$ which are factorizable.

THEOREM 2.1. ([HH4, Theorem 12], [HH6, Theorem 4.2]) *An element $f \in K_1(G)$ is extremal if and only if it is factorizable. The set $E(G)$ of all characters coincides with $F(G)$.*

An invariant function is called also a *class function* because it is essentially a function on the set $G/\!\sim$ of conjugacy classes $[g]$ of $g \in G$. Taking into account Theorem 1.2, we see that, if an $f \in K_1(G)$ is factorizable, then the multiplicative factors $f(\xi_{q_k})$ and $f(g_j)$ are given respectively as follows.

For $t \in T$, let $\xi_q(t) = (t, (q))$ be a basic element of Case 2 with $t$ at $q \in \boldsymbol{N}$, and put

$$Y_1([t]) := f\big(\xi_q(t)\big) \quad \text{for } \xi_q(t) = (t, (q)). \tag{2.3}$$

For $t \in T$ and an $\ell \geq 2$, let $(d, \sigma)$ be a basic element in Case 1 such that $\sigma$ is a cycle with length $\ell(\sigma) = \ell$ and $P_\sigma(d) = [t]$, and put

$$(2.4) \qquad Y_\ell([t]) := f\big((d, \sigma)\big) \quad \text{for } \ell(\sigma) = \ell,\ P_\sigma(d) = [t].$$

For convenience we often write $Y_1([t])$ and $Y_\ell([t])$ simply by $Y_1(t)$ and $Y_\ell(t)$ respectively.

THEOREM 2.2. (i) *For an $f \in F(G)$, the functions $Y_\ell([t])$, $[t] \in T/\!\sim$, are well-defined for $\ell \geq 1$.*

(ii) *For $g \in G$, let $g = \xi_{q_1}\xi_{q_2} \cdots \xi_{q_r} g_1 g_2 \cdots g_m$ be its standard decomposition with $\xi_q = (t_q, (q))$, $g_j = (d_j, \sigma_j)$, $\ell_j = \ell(\sigma_j)$, then*

$$(2.5) \qquad f(g) = \prod_{1 \leq k \leq r} Y_1([t_q]) \cdot \prod_{1 \leq j \leq m} Y_{\ell_j}\big(P_{\sigma_j}(d_j)\big).$$

**2.2. Character formula for factor representations of finite type**

Let $\widehat{T}$ be the dual of $T$ consisting of all equivalence classes of continuous irreducible unitary representations (= IURs). For an IUR $\zeta$, its equivalence class is denoted by $[\zeta]$, and we identify the equivalence class $[\zeta]$ with its representative $\zeta$ if there is no fear of confusions. Thus $\zeta \in \widehat{T}$ is an IUR and denote by $\chi_\zeta$ its trace character: $\chi_\zeta(t) = \mathrm{tr}(\zeta(t))$ $(t \in T)$, then $\dim \zeta = \chi_\zeta(e_T)$.

For a $g \in G$, let its standard decomposition into basic components be

$$(2.6) \qquad g = \xi_{q_1}\xi_{q_2} \cdots \xi_{q_r} g_1 g_2 \cdots g_m\,,$$

where the supports of components, $q_1, q_2, \ldots, q_r$, and $\mathrm{supp}(g_j) := \mathrm{supp}(\sigma_j)$ $(1 \leq j \leq m)$, are mutually disjoint. Furthermore, $\xi_{q_k} = (t_{q_k}, (q_k))$, $t_{q_k} \neq e_T$, with $\ell(\xi_{q_k}) = 1$ for $1 \leq k \leq r$, and $\sigma_j$ is a cycle of length $\ell(\sigma_j) \geq 2$ and $\mathrm{supp}(d_j) \subset K_j = \mathrm{supp}(\sigma_j)$. For $d_j = (t_i)_{i \in K_j} \in D_{K_j}(T) \hookrightarrow D_\infty(T)$, put $P_{\sigma_j}(d_j)$ as in (1.3).

For one-dimensional characters of $\mathfrak{S}_\infty$, we introduce simple notation as

$$(2.7) \qquad \chi_\varepsilon(\sigma) := \mathrm{sgn}_{\mathfrak{S}}(\sigma)^\varepsilon \quad (\sigma \in \mathfrak{S}_\infty\,;\, \varepsilon = 0, 1).$$

As a parameter for characters of $G = \mathfrak{S}_\infty(T)$, we prepare a set of

$$(2.8) \qquad \alpha_{\zeta,\varepsilon}\ (\zeta \in \widehat{T},\ \varepsilon \in \{0, 1\}) \ \text{ and } \ \mu = (\mu_\zeta)_{\zeta \in \widehat{T}}\,,$$

of decreasing sequences $\alpha_{\zeta,\varepsilon}$ of non-negative real numbers

$$\alpha_{\zeta,\varepsilon} = (\alpha_{\zeta,\varepsilon,i})_{i\in\boldsymbol{N}}\,, \quad \alpha_{\zeta,\varepsilon,1} \geq \alpha_{\zeta,\varepsilon,2} \geq \alpha_{\zeta,\varepsilon,3} \geq \cdots \geq 0\,;$$

and a set of non-negative $\mu_\zeta \geq 0$ $(\zeta \in \widehat{T})$, which altogether satisfies the condition

$$\sum_{\zeta\in\widehat{T}} \sum_{\varepsilon\in\{0,1\}} \|\alpha_{\zeta,\varepsilon}\| + \|\mu\| = 1\,, \tag{2.9}$$

$$\text{with} \quad \|\alpha_{\zeta,\varepsilon}\| = \sum_{i\in\boldsymbol{N}} \alpha_{\zeta,\varepsilon,i}\,, \quad \|\mu\| = \sum_{\zeta\in\widehat{T}} \mu_\zeta\,.$$

THEOREM 2.3. ([HH4, Theorem 2], [HH6, Theorem 5.1]) *Let $G = \mathfrak{S}_\infty(T)$ be a wreath product group of a compact group $T$ with $\mathfrak{S}_\infty$. Then, for a parameter*

$$A = \Big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\;;\,\mu\Big)\,, \tag{2.10}$$

*in* (2.8)–(2.9)*, the following formula determines a character $f_A$ of $G$: for an element $g \in G$, let* (2.6) *be its standard decomposition, then*

$$f_A(g) = \prod_{1\leq k\leq r} \left\{ \sum_{\zeta\in\widehat{T}} \left( \sum_{\varepsilon\in\{0,1\}} \sum_{i\in\boldsymbol{N}} \frac{\alpha_{\zeta,\varepsilon,i}}{\dim\zeta} + \frac{\mu_\zeta}{\dim\zeta} \right) \chi_\zeta(t_{q_k}) \right\}$$
$$\times \prod_{1\leq j\leq m} \left\{ \sum_{\zeta\in\widehat{T}} \left( \sum_{\varepsilon\in\{0,1\}} \sum_{i\in\boldsymbol{N}} \left(\frac{\alpha_{\zeta,\varepsilon,i}}{\dim\zeta}\right)^{\ell(\sigma_j)} \chi_\varepsilon(\sigma_j) \right) \chi_\zeta\big(P_{\sigma_j}(d_j)\big) \right\},$$

*where $\chi_\varepsilon(\sigma_j) = \mathrm{sgn}_{\mathfrak{S}}(\sigma_j)^\varepsilon = (-1)^{\varepsilon(\ell(\sigma_j)-1)}$.*

*Conversely, any character of $G$ is given in the form of $f_A$.*

The functions $Y_1$ and $Y_\ell$, $\ell \geq 2$, corresponding to $f_A$, which we denote by $Y_\ell^A$ are given respectively as follows: for $t \in T$,

$$Y_1^A(t) = \sum_{\zeta\in\widehat{T}} \left( \sum_{\varepsilon\in\{0,1\}} \sum_{i\in\boldsymbol{N}} \frac{\alpha_{\zeta,\varepsilon,i}}{\dim\zeta} + \frac{\mu_\zeta}{\dim\zeta} \right) \chi_\zeta(t) \tag{2.11}$$

$$Y_\ell^A(t) = \sum_{\zeta\in\widehat{T}} \left( \sum_{\varepsilon\in\{0,1\}} \sum_{i\in\boldsymbol{N}} \left(\frac{\alpha_{\zeta,\varepsilon,i}}{\dim\zeta}\right)^{\ell} (-1)^{\varepsilon(\ell-1)} \right) \chi_\zeta(t) \tag{2.12}$$

*Remark* 2.1. When $T$ is not discrete or equivalently not finite, the equality condition (2.9) guarantees the continuity of the normalized function $f_A(g)$ at the identity element $g = e \in G$ since $f_A(e) = 1$, because $Y_1(e_T) = 1$ by (2.9).

By the same reason, the global character formula above is valid even when $t_{q_k} = e_T$ for some $q_k$, and in particular valid at the identity element $g = e$.

*Remark* 2.2. When $T$ is discrete or equivalently finite, if we discard the validity at $g = e$ of the above formula for $f_A(g)$, we can accept, in addition to the equality condition (2.9), the following inequality condition:

$$\sum_{\zeta\in\widehat{T}} \sum_{\varepsilon\in\{0,1\}} \|\alpha_{\zeta,\varepsilon}\| + \|\mu\| \leq 1\,. \tag{2.13}$$

In fact, we have a linear dependence on $T^* = T \setminus \{e_T\}$ as

$$1 = \chi_{\mathbf{1}_T} = -\sum_{\zeta\in\widehat{T}^*} (\dim\zeta)\,\chi_\zeta,$$

where $\widehat{T}^* := \widehat{T} \setminus \{\mathbf{1}_T\}$ with the trivial representation $\mathbf{1}_T$ of $T$. However in this case the parameter $A$ for a character $f_A$ is no more unique (cf. [HH6, **4.2**]).

### 2.3. Characters of canonical subgroups of $\mathfrak{S}_\infty(T)$ with $T$ abelian

When the compact group $T$ is abelian, take a subgroup $S$ of $T$, and define a subgroup $G^S = \mathfrak{S}_\infty(T)^S$ of $G = \mathfrak{S}_\infty(T)$ as follows:

$$G^S := \{g = (d,\sigma) \in \mathfrak{S}_\infty(T)\ ;\ P(d) \in S\}, \tag{2.14}$$

where $P(d) := \prod_{i\in\boldsymbol{N}} t_i$ for $d = (t_i)_{i\in\boldsymbol{N}}$. In case $S$ is not proper, or $S = T$, we have $G^S = G$.

With some additional discussions, the general character formula in Theorem 2.3 gives us in the case of compact abelian group $T$ the following character formula for $G^S = \mathfrak{S}_\infty(T)^S$. In this abelian case, $\widehat{T}$ is nothing but the dual group consisting of all continuous one-dimensional characters of $T$, and for each $\zeta \in \widehat{T}$, its character $\chi_\zeta$ is identified with $\zeta$ itself.

For a $g \in G^S$, let its standard decomposition in $G$ be as in (2.6), $g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r}g_1g_2\cdots g_m$, with $\xi_{q_k} = (t_{q_k}, (q_k))$, $t_{q_k} \neq e_T$, for $1 \leq k \leq r$, and $g_j = (d_j, \sigma_j)$ for $1 \le j \le m$. Note that each basic elements $\xi_{q_k}$, $g_j$ are not necessarily in the subgroup $G^S$. Put $K_j = \mathrm{supp}(\sigma_j)$, and for $d_j = (t_i)_{i\in K_j} \in D_{K_j}(T) \hookrightarrow D_\infty(T)$, put $\zeta(d_j) := \prod_{i\in K_j} \zeta(t_i)$.

THEOREM 2.4. ([HH6, Theorem 7.1]) *Assume a compact group $T$ be abelian, and take a subgroup $S$ of $T$, not necessarily closed. Let $G^S$ be the subgroup of the wreath product group $G = \mathfrak{S}_\infty(T)$ defined in* (2.14). *Then, for a parameter*

$$A = \Big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\,;\,\mu\Big)$$

*satisfying the condition* (2.9), *the following formula determines a character $f_A^S$ of $G^S$: for a $g \in G^S$, let its standard decomposition be as above, then*

$$f_A^S(g) = \prod_{1\le k\le r}\left\{\sum_{\zeta\in\widehat{T}}\left(\sum_{\varepsilon\in\{0,1\}}\sum_{i\in\boldsymbol{N}}\alpha_{\zeta,\varepsilon,i} + \mu_\zeta\right)\zeta(t_{q_k})\right\}$$
$$\times \prod_{1\le j\le m}\left\{\sum_{\zeta\in\widehat{T}}\left(\sum_{\varepsilon\in\{0,1\}}\sum_{i\in\boldsymbol{N}}(\alpha_{\zeta,\varepsilon,i})^{\ell(\sigma_j)}\cdot\chi_\varepsilon(\sigma_j)\right)\zeta(d_j)\right\},$$

*where $\chi_\varepsilon(\sigma_j) = \mathrm{sgn}_{\mathfrak{S}}(\sigma_j)^\varepsilon = (-1)^{\varepsilon(\ell(\sigma_j)-1)}$.*

*Conversely, if $S$ is open in $T$, or in particular if $T$ is finite, any character of $G^S$ is given in the form of $f_A^S$.*

This theorem says that each character $f_A$ of $G$ has its restriction $f_A^S = f|_{G^S}$ as a character of $G^S$, and conversely, if $S$ is open in $T$, any character of $G^S$ is obtained in this way by restriction.

The correspondence of the parameter $A \to f_A^S$ is no more one-one when $S$ is proper. Introduce a translation $R(\zeta_0)$ on $A$ by an element $\zeta_0 \in \widehat{T}$ as

$$R(\zeta_0)A := \big((\alpha'_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\,;\,R(\zeta_0)\mu\big)$$
$$\text{with}\quad \alpha'_{\zeta,\varepsilon} = \alpha_{\zeta{\zeta_0}^{-1},\varepsilon}\ \big((\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}\big);$$
$$R(\zeta_0)\mu = (\mu'_\zeta)_{\zeta\in\widehat{T}},\quad \mu'_\zeta = \mu_{\zeta{\zeta_0}^{-1}}. \tag{2.15}$$

PROPOSITION 2.5. ([HH6, Proposition 7.2]) *Two parameters of characters of $G = \mathfrak{S}_\infty(T) = D_\infty(T) \rtimes \mathfrak{S}_\infty$*

$$A = \big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\,;\,\mu\big) \quad \textit{and} \quad A' = \big((\alpha'_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\,;\,\mu'\big)$$

*determine the same function on $G^S$, that is, $f_A^S = f_{A'}^S$, if and only if $A' = R(\zeta^S)A$ for some $\zeta^S \in \widehat{T}$ which is trivial on $S$. In this case, as elements in $E(G)$ for the bigger group $G$, we have*

$$f_{A'}(g) = \pi_{\zeta^S,0}(g) \cdot f_A(g) \quad (g \in G),$$

*where $\pi_{\zeta,0}$ for $\zeta \in \widehat{T}$ denotes one-dimensional character of $G$ given in* (1.5) *with $\varepsilon = 0$: $\pi_{\zeta,0}(g) = \zeta(P(d))$ for $g = (d,\sigma) \in G = \mathfrak{S}_\infty(T)$.*

# Chapter II. Limits of irreducible characters of $\mathfrak{S}_n$ as $n \to \infty$

## §3. Construction of IURs of the wreath product group $\mathfrak{S}_n(T)$

For the semi-direct product group $G_n = \mathfrak{S}_n(T) = D_n(T) \rtimes \mathfrak{S}_n$ with $D_n(T) \cong T^n$, we can construct any IUR by a standard inducing-up method (for the case where $T$ is a finite group, see e.g. [JK, Chapter 4]). We explain it briefly to prepare an explicit calculation of irreducible characters in the next section.

### 3.1. Elementary IURs of $G_n = \mathfrak{S}_n(T) = D_n(T) \rtimes \mathfrak{S}_n$

Before going into details we define an elementary IUR of $G_n$ directly. Put $\boldsymbol{I}_n = \{1, 2, \ldots, n\}$ as before. First, for an IUR of $D_n(T) = \prod_{i \in \boldsymbol{I}_n} T_i$, $T_i = T$, we take IURs $\zeta_i$ of $T_i = T$ acting on $V(\zeta_i)$, and an outer tensor product $\eta := \boxtimes_{i \in \boldsymbol{I}_n} \zeta_i$ on the space $V(\eta) := \bigotimes_{i \in \boldsymbol{I}_n} V(\zeta_i)$, then we get an IUR of $D_n$ as

$$\eta(d) := \boxtimes_{i \in \boldsymbol{I}_n} \zeta_i(t_i) \quad \text{for } d = (t_i)_{i \in \boldsymbol{I}_n} \in D_n(T). \tag{3.1}$$

Moreover for $\sigma \in \mathfrak{S}_n$, consider an IUR $\sigma(\eta) := \boxtimes_{i \in \boldsymbol{I}_n} \zeta_{\sigma^{-1}(i)}$ of $D_n$ on $V\big(\sigma(\eta)\big) = \bigotimes_{i \in \boldsymbol{I}_n} V(\zeta_{\sigma^{-1}(i)})$, and an operator $I(\sigma) : V(\eta) \to V\big(\sigma(\eta)\big)$ given as

$$I(\sigma) : V(\eta) \ni \textstyle\bigotimes_{i \in \boldsymbol{I}_n} v_i \longrightarrow \bigotimes_{i \in \boldsymbol{I}_n} v_{\sigma^{-1}(i)} \in V\big(\sigma(\eta)\big), \tag{3.2}$$

for $v_i \in V(\zeta_i)$, $i \in \boldsymbol{I}_n$.

On the other hand, through the action of $\mathfrak{S}_n$ on $D_n$, we define an action of $\sigma$ on $\eta$ as a representation ${}^\sigma\eta$ on the same space $V(\eta)$ given through $\sigma^{-1} d \sigma = \sigma^{-1}(d) = (t_{\sigma(i)})_{i \in \boldsymbol{I}_n}$ by

$$({}^\sigma\eta)(d) := \eta(\sigma^{-1} d \sigma) = \boxtimes_{i \in \boldsymbol{I}_n} \zeta_i(t_{\sigma(i)}).$$

Then, $I(\sigma)$ intertwines the representation ${}^{\sigma}\eta$ on $V(\eta)$ with the one $\sigma(\eta)$ on $V\big(\sigma(\eta)\big)$, that is,

$$\sigma(\eta)(d)\cdot I(\sigma) = I(\sigma)\cdot {}^{\sigma}\eta(d) \quad (d\in D_n). \tag{3.3}$$

From this fact, we see that ${}^{\sigma}\eta$ is equivalent to $\eta$, or ${}^{\sigma}\eta\cong\eta$, if and only if $\zeta_i\cong\zeta_{\sigma(i)}$ $(i\in\boldsymbol{I}_n)$, where the symbol '$\cong$' denotes the equivalence of representations.

Fix an IUR $\zeta$ of $T$. We put $\eta_\zeta := \boxtimes_{i\in\boldsymbol{I}_n}\zeta_i$, $\zeta_i=\zeta$ $(i\in\boldsymbol{I}_n)$, and, for $(d,\sigma)\in D_n(T)\rtimes\mathfrak{S}_n = G_n$,

$$\rho_\zeta\big((d,\sigma)\big) := \eta_\zeta(d)I(\sigma). \tag{3.4}$$

Then $\sigma(\eta_\zeta)=\eta_\zeta$ for $\sigma\in\mathfrak{S}_n$, and $\rho_\zeta$ gives an IUR of $G_n=\mathfrak{S}_n(T)$.

LEMMA 3.1. *Assume that an IUR $\pi$ of $G_n=\mathfrak{S}_n(T)$ is still irreducible when it is restricted to the subgroup $D_n$. Then $\pi$ is equivalent to $\rho_\zeta$ or to $\rho_\zeta\cdot\mathrm{sgn}_{\mathfrak{S}_n}$ for some $\zeta\in\widehat{T}$, where $\mathrm{sgn}_{\mathfrak{S}_n}$ denotes the sign character of $\mathfrak{S}_n$.*

Now take an IUR $\xi$ of the group $\mathfrak{S}_n\cong G_n/D_n$ and consider it as a representation of $G_n$, trivial on $D_n$. We define an IUR of $G_n$, denoted by $\pi_{\zeta,\xi}$ (or by $\eta_\zeta\boxdot\xi$ in accordance with the later notation), by taking the tensor product $\rho_\zeta\otimes\xi$, acting on $V(\rho_\zeta)\otimes V(\xi)$, as

$$\pi_{\zeta,\xi}\big((d,\sigma)\big) := \rho_\zeta\big((d,\sigma)\big)\otimes\xi(\sigma) \quad \big((d,\sigma)\in G_n\big).$$

LEMMA 3.2. *Assume that an IUR $\pi$ of $G_n=\mathfrak{S}_n(T)$ contains an IUR equivalent to $\eta_\zeta$ for some $\zeta\in\widehat{T}$, when it is restricted to the subgroup $D_n$. Then there exists an IUR $\xi\in\widehat{\mathfrak{S}_n}$ such that $\pi$ is equivalent to the tensor product $\pi_{\zeta,\xi}=\rho_\zeta\otimes\xi$.*

### 3.2. Construction of general IURs of $G_n$

Take the dual space $\widehat{D_n}\cong(\widehat{T})^n$ of $D_n=D_n(T)$. Then $\mathfrak{S}_n$ acts on it as $({}^{\sigma}\eta)(d)=\eta(\sigma^{-1}(d))$ for $d=(t_i)_{i\in\boldsymbol{I}_n}$ and $\sigma(d):=(t_{\sigma^{-1}(i)})$. Firstly fix an element $[\eta]\in\widehat{D_n}$, where $[\eta]$ denotes the equivalence class of $\eta$. Then take the stationary subgroup $S_{[\eta]}$ of it in $\mathfrak{S}_n$:

$$S_{[\eta]} = \{\sigma\in\mathfrak{S}_n\ ;\ [{}^{\sigma}\eta]=[\eta]\} = \{\sigma\in\mathfrak{S}_n\ ;\ {}^{\sigma}\eta\cong\eta\}.$$

Note that $\eta$ is, modulo unitary equivalence, of the form $\eta = \boxtimes_{1\le i\le n}\zeta_i$ with $[\zeta_i] \in \widehat{T}$. Let $\widehat{T}'$ be the set of different $[\zeta_i]$'s. Hereafter we denote the condition $[\zeta] \in \widehat{T}'$ simply by $\zeta \in \widehat{T}'$ to simplify the notation. Let

$$\boldsymbol{I}_n = \bigsqcup_{\zeta\in\widehat{T}'} I_{n,\zeta}$$

be a partition of $\boldsymbol{I}_n$ such that $[\zeta_i] = [\zeta]$ if and only if $i \in I_{n,\zeta}$. Then, arranging the order of components of the tensor product, we see that $\eta$ is unitary equivalent to

$$\mathop{\boxtimes}_{\zeta\in\widehat{T}'} \zeta^{I_{n,\zeta}}, \quad \zeta^{I_{n,\zeta}} := \mathop{\boxtimes}_{i\in I_{n,\zeta}} \zeta_i \quad \text{with } \zeta_i = \zeta \ \ (i \in I_{n,\zeta}).$$

Replace $\eta$ by this standard representation. Then the stationary subgroup $S_{[\eta]}$ of $[\eta] \in (\widehat{T}')^n$ ($\hookrightarrow \widehat{D_n}$) in $\mathfrak{S}_n$ is given as $S_{[\eta]} = \prod_{\zeta\in\widehat{T}'} \mathfrak{S}_{I_{n,\zeta}}$, and the stationary subgroup $H_n$ in the group $G_n = D_n \rtimes \mathfrak{S}_n$ is given as $H_n = D_n \rtimes S_{[\eta]} = \prod_{\zeta\in\widehat{T}'} \mathfrak{S}_{I_{n,\zeta}}(T)$. For each component $\mathfrak{S}_{I_{n,\zeta}}(T)$, taking into account Lemma 3.1, we have an elementary IUR $\rho_\zeta$ similarly as (3.1)–(3.4), and then taking an outer tensor product, we have an IUR of $H_n$ as $\rho := \boxtimes_{\zeta\in\widehat{T}'} \rho_\zeta$.

Secondly, taking into account Lemma 3.2, we take an IUR $\xi$ of the group $S_{[\eta]} = \prod_{\zeta\in\widehat{T}'} \mathfrak{S}_{I_{n,\zeta}} \cong H_n/D_n$ and consider it as a representation of $H_n$, then take the tensor product $\pi = \rho \otimes \xi$ acting on $V(\eta) \otimes V(\xi)$, which we denote by $\eta \boxdot \xi$:

$$\pi((d,\sigma)) = (\eta \boxdot \xi)((d,\sigma)) := \rho((d,\sigma)) \otimes \xi(\sigma) \qquad ((d,\sigma) \in D_n \rtimes S_{[\eta]} = H_n). \tag{3.5}$$

Note that the representation $\pi = \eta \boxdot \xi$ consists of IURs of the type in Lemma 3.2 for each components $\mathfrak{S}_{I_{n,\zeta}}(T)$, $\zeta \in \widehat{T}'$.

The induced representation $\Pi = \mathrm{Ind}_{H_n}^{G_n} \pi$ from $H_n = D_n \rtimes S_{[\eta]}$ to $G_n$ is realized as follows: for the representation space $V(\Pi)$, take the space of continuous $V(\pi)$-valued functions $\varphi$ on $G_n$ satisfying $\varphi(hg) = \pi(h)\big(\varphi(g)\big)$ ($h \in H_n$, $g \in G_n$) with the space $V(\pi)$ of $\pi$, and introduce $L^2(H_n\backslash G_n)$-norm as

$$\|\varphi\|^2 = \int_{H_n\backslash G_n} \|\varphi(g)\|^2_{V(\pi)}\, d\mu_{H_n\backslash G_n}(\widehat{g}),$$

where $\widehat{g}$ denotes the coset $H_n g$ and $d\mu_{H_n\backslash G_n}$ denotes the normalized invariant measure on $H_n\backslash G_n$. Note that, for the normalized Haar measure $d\mu_{G_n}$ on $G_n$, we have $\mu_{G_n}(H_n) = |H_n\backslash G_n|^{-1} = |S[\eta]\backslash\mathfrak{S}_n|^{-1}$, and that, according to $G_n = D_n \rtimes \mathfrak{S}_n$, the Haar measure $d\mu_{G_n}$ is written as follows, with

$g = (d, \sigma)$ and the normalized Haar measure $d\mu_{D_n}$ on $D_n$, for a continuous function $\psi$ on $G_n$,

$$\int_{G_n} \psi(g)\, d\mu_{G_n}(g) = \frac{1}{|\mathfrak{S}_n|} \sum_{\sigma \in \mathfrak{S}_n} \int_{D_n} \psi\big((d, \sigma)\big)\, d\mu_{D_n}(d).$$

We get a Hilbert space $V(\Pi)$, because actually $\dim \Pi < \infty$. In fact, $H_n \backslash G_n \cong S_{[\eta]} \backslash \mathfrak{S}_n$, and so $\dim \Pi = \dim \pi \times |S_{[\eta]} \backslash \mathfrak{S}_n| < \infty$.

The representation is given as $(\Pi(g_0)\varphi)(g) = \varphi(gg_0)$ $(g_0, g \in G_n)$.

We denote $\Pi$ also by $\Pi^{\eta,\xi}$ when the dependence on $(\eta, \xi)$ should be specified.

THEOREM 3.3. (i) *For an IUR* $\pi = \eta \boxdot \xi$ *of* $H_n$ *of the form in* (3.5), *the induced representation* $\Pi = \Pi^{\eta,\xi} = \mathrm{Ind}_{H_n}^{G_n} \pi$ *of* $G_n$ *is irreducible.*

(ii) *Any irreducible representation of* $G_n$ *is equivalent to one of the induced representations* $\Pi^{\eta,\xi}$ *with* $[\eta] \in \widehat{D_n}$, $[\xi] \in \widehat{S_{[\eta]}}$.

*Proof.* (i) Denote by $\mathcal{I}(\Pi)$ the space of intertwining operators for $\Pi$. Then we should prove that $\dim \mathcal{I}(\Pi) = 1$. Since $\dim \Pi < \infty$, any linear transformation $L$ on $V(\Pi)$ is expressed as follows as an integral operator with a kernel $K'(g, g')$ $(g, g' \in G_n)$ taking values in $\mathcal{L}(V(\pi))$, the space of linear transformations on $V(\pi)$, and satisfies the homogeneity condition

$$K'(hg, h'g') = \pi(h) K'(g, g') \pi(h')^{-1} \quad (g, g' \in G_n,\ h, h' \in H_n), \tag{3.6}$$

$$\begin{aligned} L\varphi(g) &= \int_{H_n \backslash G_n} K'(g, g') \varphi(g')\, d\mu_{H_n \backslash G_n}(\widehat{g'}) \\ &= \int_{G_n} K'(g, g') \varphi(g')\, d\mu_{G_n}(g') \quad (g \in G_n). \end{aligned} \tag{3.7}$$

The condition $L \in \mathcal{I}(\Pi)$ or $\Pi(g_0)\, L = L\, \Pi(g_0)$ $(g_0 \in G_n)$ is equivalent to

$$K'(gg_0, g'g_0) = K'(g, g') \quad (g, g', g_0 \in G_n). \tag{3.8}$$

Put $K(g) = K'(g, e)$ with the identity element $e \in G_n$, then $K'(g, g') = K(gg'^{-1})$, and $K(hgh') = \pi(h) K(g) \pi(h')$ $(h, h' \in H_n,\ g \in G_n)$.

Now take a representative $\tau \in \mathfrak{S}_n$ of a coset in $H_n \backslash G_n / H_n \cong S_{[\eta]} \backslash \mathfrak{S}_n / S_{[\eta]}$,

$$\pi(h) K(\tau) = K(\tau) \pi(\tau^{-1} h \tau) \quad (h \in H_n \cap \tau H_n \tau^{-1}). \tag{3.9}$$

Let $h = (d, \sigma) \in H_n = D_n \rtimes S_{[\eta]}$, then,

$$\tag{3.10} \begin{gathered} \tau^{-1} h \tau = (\tau^{-1}(d), \tau^{-1}\sigma\tau), \\ \pi(\tau^{-1} h \tau) = \rho\big(\tau^{-1}(d), \tau^{-1}\sigma\tau\big) \otimes \xi(\tau^{-1}\sigma\tau). \end{gathered}$$

Suppose that $\tau$ represents a double coset different from $H_n$, then two representations $\eta(d)$ and $\eta(\tau^{-1}(d)) = {}^{\tau}\eta(d)$ of $D_n \subset H_n \cap \tau^{-1} H_n \tau$ are not mutually equivalent. Therefore, from the intertwining property (3.9) of $K(\tau)$, we have $K(\tau) = 0$.

For the representative $\tau = \mathbf{1}$ of $H_n$, we see that $K(\tau)$ is a multiple of the identity operator thanks to the irreducibility of $\pi$.

Altogether we get $\dim \mathcal{I}(\Pi) = 1$, and the irreducibility of $\Pi$.

(ii) Let $\Pi'$ be an IUR of $G_n$ on a Hilbert space $V(\Pi')$. Restrict $\Pi'$ to the subgroup $D_n \cong T^n$, then we have an IUR $\eta$ acting on its subspace $V(\eta)$, and $\eta$ is, modulo unitary equivalence, of the form $\eta = \boxtimes_{1 \le i \le n} \zeta_i$ with $[\zeta_i] \in \widehat{T}$. Let $\widehat{T}'$ be the set of different $[\zeta_i]$'s. Let a partition $\boldsymbol{I}_n = \bigsqcup_{\zeta \in \widehat{T}'} I_{n,\zeta}$ be as above, and take $\eta$, modulo unitary equivalence, as a standard representation

$$\eta = \mathop{\boxtimes}_{\zeta \in \widehat{T}'} \zeta^{I_{n,\zeta}}, \quad \zeta^{I_{n,\zeta}} := \mathop{\boxtimes}_{i \in I_{n,\zeta}} \zeta_i \quad \text{with } \ \zeta_i = \zeta \ \ (i \in I_{n,\zeta}).$$

Then a $g \in G_n$ acts on the normal subgroup $D_n$, and accordingly on $\eta$ by $({}^{g}\eta)(d) := \eta(g^{-1} d g)$. The stationary subgroup $S_{[\eta]}$ of $[\eta] \in (\widehat{T}')^n$ $(\hookrightarrow \widehat{D_n})$ in $\mathfrak{S}_n$ is $S_{[\eta]} = \prod_{\zeta \in \widehat{T}'} \mathfrak{S}_{I_{n,\zeta}}$, and the stationary subgroup $H_n$ in the group $G_n = D_n \rtimes \mathfrak{S}_n$ is $H_n = D_n \rtimes S_{[\eta]} = \prod_{\zeta \in \widehat{T}'} \mathfrak{S}_{I_{n,\zeta}}(T)$.

Now take the span of $\Pi'(H_n)V(\eta) = \Pi'(S_{[\eta]})V(\eta)$ and pick up an irreducible subspace $V(\pi)$ under $H_n$, where $\pi$ is the IUR of $H_n$ acting on it. Then, discussing for each component $\mathfrak{S}_{I_{n,\zeta}}(T)$, we see by Lemma 3.2 that, modulo equivalence, $\pi$ can be assumed to be of the form $\eta \boxdot \xi$ with an IUR $\xi$ of $S_{[\eta]} = \prod_{\zeta \in \widehat{T}'} \mathfrak{S}_{I_{n,\zeta}}$.

Let $P$ be the orthogonal projection of $V(\Pi')$ onto $V(\pi)$. For $v \in V(\Pi')$, consider a $V(\pi)$-valued function $\varphi = \Phi(v)$ given as

$$\tag{3.11} \varphi(g) = P\big(\Pi'(g)v\big) \quad (g \in G_n).$$

Then, $\varphi$ belongs to the space $V(\Pi)$ of the induced representation $\Pi = \mathrm{Ind}_{H_n}^{G_n} \pi$. In fact, $\varphi$ is continuous in $g$, and

$$\varphi(hg) = P\big(\Pi'(hg)v\big) = P\big(\Pi'(h)\Pi'(g)v\big) = \pi(h)P\big(\Pi'(g)v\big) = \pi(h)\big(\varphi(g)\big).$$

Thus we have a linear map $\Phi : V(\Pi') \ni v \mapsto \varphi \in V(\Pi)$. It intertwines the representations. In fact,

$$\Pi'(g_0)v \longrightarrow P\big(\Pi'(g)\Pi'(g_0)v\big) = P\big(\Pi'(gg_0)v\big) = \varphi(gg_0) = \big(\Pi(g_0)\varphi\big)(g).$$

Moreover $\Phi$ is injective. In fact, suppose $\Phi(v) = \varphi = 0$. Then, for any $w \in V(\pi)$,

$$0 = \langle \varphi(g), w\rangle = \langle \Pi'(g)v, w\rangle = \langle v, \Pi'(g^{-1})w\rangle \quad (g \in G_n).$$

Hence $v \perp \Pi'(G_n)V(\pi)$, and so $v = 0$ since $\Pi'$ is irreducible.

We know from the part (i) that $\Pi = \mathrm{Ind}_{H_n}^{G_n} \pi$ is irreducible. Hence $\Phi$ should be surjective, and so gives an equivalence of $\Pi'$ to $\Pi$. ∎

Take an IUR $\eta$ of $D_n$, and an IUR $\xi$ of the stationary subgroup $S_{[\eta]}$. Then a $\kappa \in \mathfrak{S}_n$ acts on $\eta$ as $({}^{\kappa}\eta)(d) = \eta(\kappa^{-1}(d))$, and $S_{[{}^{\kappa}\eta]} = \kappa S_{[\eta]}\kappa^{-1}$. Define an IUR ${}^{\kappa}\xi$ of $S_{[{}^{\kappa}\eta]}$ by

$$({}^{\kappa}\xi)(\sigma) := \xi(\kappa^{-1}\sigma\kappa) \quad (\sigma \in S_{[{}^{\kappa}\eta]}).$$

THEOREM 3.4. *Two IURs $\Pi^{\eta,\xi}$ and $\Pi^{\eta',\xi'}$ are mutually equivalent if and only if there exists a $\kappa \in \mathfrak{S}_n$ such that $[\eta'] = [{}^{\kappa}\eta]$ and $[\xi'] = [{}^{\kappa}\xi]$.*

## §4. Characters of IURs of $\mathfrak{S}_n(T)$ with $T$ a compact group

In the previous section, we constructed IURs of $G_n = \mathfrak{S}_n(T) = D_n(T) \rtimes \mathfrak{S}_n$ as induced representation from standard subgroups. Using this construction, we calculate explicitly characters of IURs of $G_n$ and express them in the form which fits to the later calculations of their limits as $n \to \infty$. For the case where $T$ is a finite group, a combinatorial construction of irreducible characters is given in [Mac, Chapter I, Appendix A].

### 4.1. A formula for characters of induced representations

Before getting into explicit calculations, we prepare some generality. For a continuous positive definite function $f$ on a topological group $G$ and a compact subgroup $G' \subset G$, we defined in [HH4], [HH6] a *centralization* of $f$ with respect to $G'$ as

$$f^{G'}(g) := \int_{g' \in G'} f(g' g g'^{-1})\, d\mu_{G'}(g'), \tag{4.1}$$

where $d\mu_{G'}$ denotes the normalized Haar measure on $G'$.

LEMMA 4.1. *For an IUR* $\Pi = \mathrm{Ind}_{H_n}^{G_n} \pi$ *of* $G_n$*, let* $\widetilde{\chi}_\pi = \chi_\pi / \dim \pi$ *(resp.* $\widetilde{\chi}_\Pi = \chi_\Pi / \dim \Pi$*) be the normalized character of* $\pi$ *(resp.* $\Pi$*). Then* $\widetilde{\chi}_\Pi$ *is the centralization of the trivial extension of* $\widetilde{\chi}_\pi$ *to* $G_n$*, which equals, by definition, to* $\widetilde{\chi}_\pi$ *on* $H_n$*, and to* $0$ *outside* $H_n$*.*

From this we see that $\widetilde{\chi}_\Pi(g) \neq 0$ only when $g$ is conjugate to some $h \in H_n$ in $G_n$, and that for an $h \in H_n$,

$$\widetilde{\chi}_\Pi(h) = \int_{G_n} \widetilde{\chi}_\pi(g' h g'^{\,-1})\, d\mu_{G_n}(g'), \tag{4.2}$$

where we put $\widetilde{\chi}_\pi(g' h g'^{\,-1}) = 0$ if $g' h g'^{\,-1} \notin H_n$.

### 4.2. Characters of elementary IURs $\pi_{\zeta,\xi} = \rho_\zeta \otimes \xi$ of $G_n = D_n \rtimes \mathfrak{S}_n(T)$

For a finite-dimensional representation $\pi$, its character $\mathrm{tr}\big(\pi(g)\big)$, $g \in G_n$, is denoted by $\chi_\pi(g)$, and its normalized character by $\widetilde{\chi}_\pi$. Then, for the tensor product representation $\pi_{\zeta,\xi} = \rho_\zeta \otimes \xi$, we have

$$\chi_{\pi_{\zeta,\xi}} = \chi_{\rho_\zeta} \cdot \chi_\xi, \quad \widetilde{\chi}_{\pi_{\zeta,\xi}} = \widetilde{\chi}_{\rho_\zeta} \cdot \widetilde{\chi}_\xi.$$

Therefore we calculate $\chi_{\rho_\zeta}(g)$, $g \in G_n$. The representation space $V(\rho_\zeta)$ is nothing but the one $V(\eta_\zeta) = \bigotimes_{i \in \boldsymbol{I}_n} V(\zeta_i)$ with $\zeta_i = \zeta$ $(i \in \boldsymbol{I}_n)$.

For $d = (t_i)_{i \in \boldsymbol{I}_n}$, we have $\rho_\zeta(d) = \eta_\zeta(d) = \boxtimes_{i \in \boldsymbol{I}_n} \zeta_i(t_i)$ and $\chi_{\rho_\zeta}(d) = \prod_{i \in \boldsymbol{I}_n} \chi_\zeta(t_i)$.

For $(d, \sigma) \in D_n \rtimes \mathfrak{S}_n$, we have for $v_i \in V(\zeta)$, $i \in \boldsymbol{I}_n$,

$$\begin{aligned} &\rho_\zeta\big((d,\sigma)\big)(v_1 \otimes v_2 \otimes \cdots \otimes v_n) \\ &\quad = \zeta(t_1) v_{\sigma^{-1}(1)} \otimes \zeta(t_2) v_{\sigma^{-1}(2)} \otimes \cdots \otimes \zeta(t_n) v_{\sigma^{-1}(n)}. \end{aligned}$$

Choose a complete orthonormal system $\{w_p\,;\, 1 \le p \le \dim \zeta\}$ of $V(\zeta)$, and put $i' = \sigma^{-1}(i)$ for $i \in \boldsymbol{I}_n$, then

$$\begin{aligned} &\rho_\zeta\big((d,\sigma)\big)(w_{p_1} \otimes w_{p_2} \otimes \cdots \otimes w_{p_n}) \\ &\quad = \zeta(t_1) w_{p_{1'}} \otimes \zeta(t_2) w_{p_{2'}} \otimes \cdots \otimes \zeta(t_n) w_{p_{n'}}. \end{aligned}$$

Calculate the sum of diagonal matrix elements

$$\begin{aligned} &\langle \rho_\zeta\big((d,\sigma)\big)(w_{p_1} \otimes w_{p_2} \otimes \cdots \otimes w_{p_n}), w_{p_1} \otimes w_{p_2} \otimes \cdots \otimes w_{p_n} \rangle \\ &\quad = \prod\nolimits_{i \in \boldsymbol{I}_n} \langle \zeta(t_i) w_{p_{i'}}, w_{p_i} \rangle, \end{aligned}$$

over $p_1, p_2, \ldots, p_n \in \{1 \le j \le \dim\zeta\}$.

To do so, instead of the standard decomposition in (2.6) into basic elements of $g = (d, \sigma) \in G_n \subset G$, we take a *standard decomposition in the wider sense* as follows. For the part $\sigma$ of $g$, take its cycle decomposition as $\sigma = \sigma_1\sigma_2\cdots\sigma_m$, and take basic elements $g_j = (d_j, \sigma_j)$, where $d_j = (t_i)_{i\in K_j}$ with $K_j = \mathrm{supp}(\sigma_j)$. Put $Q^0 = \boldsymbol{I}_n \setminus \mathrm{supp}(\sigma)$, and let $\xi_q = (t_q, (q))$ for $q \in Q^0$ be an element of $D_n \subset G_n$ with element $t_q \in T$ at $q$ and $e_T$ at other $i \in \boldsymbol{I}_n$. Then we have an expression of $g$ as

$$g = (d, \sigma) = \prod\nolimits_{q\in Q^0} \xi_q \cdot g_1 g_2 \cdots g_m. \tag{4.3}$$

Note that the difference from the standard decomposition (2.6) is that we accept in (4.3) here $\xi_q$'s with $t_q = e_T$, and so we call it standard decomposition in a wider sense.

To calculate the above sum of diagonal matrix elements, we avoid superficial complications of suffices with many levels, and can proceed with typical examples. For example, if $\sigma_1 = (1\ 2\ \cdots\ \ell)$, then $1' = \sigma_1^{-1}(1) = \ell, 2' = \sigma_1^{-1}(2) = 1, \ldots, \ell' = \sigma_1^{-1}(\ell) = \ell - 1$. Put the matrix elements of $\zeta(t)$ as $\zeta_{ab}(t) = \langle\zeta(t)w_b, w_a\rangle$, then the partial summation over $p_i$, $i \in K_1 = \{1, 2, \ldots, \ell\}$ gives us

$$\begin{aligned}\sum_{1\le p_1\le\dim\zeta} \cdots \sum_{1\le p_\ell\le\dim\zeta} &\zeta_{p_1p_\ell}(t_1)\zeta_{p_2p_1}(t_2)\cdots\zeta_{p_\ell p_{\ell-1}}(t_\ell) \\ = \sum_{1\le p_\ell\le\dim\zeta} &\zeta_{p_\ell p_\ell}(t_\ell t_{\ell-1}\cdots t_2t_1) = \chi_\zeta\big(P_{\sigma_1}(d_1)\big).\end{aligned}$$

Moreover, for $q \in Q^0 = \boldsymbol{I}_n \setminus \mathrm{supp}(\sigma)$, we have $q' = \sigma^{-1}(q) = q$, and the partial sum over $1 \le p_q \le \dim\zeta$ gives us $\sum_{1\le p_q\le\dim\zeta} \zeta_{p_qp_q}(t_q) = \chi_\zeta(t_q)$. Thus, altogether we obtain the following character formula.

Lemma 4.2. *For $g = (d, \sigma) \in G_n = D_n \rtimes \mathfrak{S}_n$, let $g = \prod_{q\in Q^0} \xi_q \cdot g_1g_2\cdots g_m$ be its standard decomposition in the wider sense, with $Q^0 = \boldsymbol{I}_n \setminus \mathrm{supp}(\sigma)$, then*

$$\chi_{\rho_\zeta}\big((d, \sigma)\big) = \prod_{q\in Q^0} \chi_\zeta(t_q) \cdot \prod_{1\le j\le m} \chi_\zeta\big(P_{\sigma_j}(d_j)\big), \tag{4.4}$$

$$\chi_{\pi_{\zeta,\xi}}\big((d, \sigma)\big) = \prod_{q\in Q^0} \chi_\zeta(t_q) \cdot \prod_{1\le j\le m} \chi_\zeta\big(P_{\sigma_j}(d_j)\big) \times \chi_\xi(\sigma). \tag{4.5}$$

The dimension of $\pi_{\zeta,\xi} = \rho_\zeta \otimes \xi$ is given as $\dim\bigl(\pi_{\zeta,\xi}\bigr) = \dim \rho_\zeta \times \dim \xi = \bigl(\dim \zeta\bigr)^n \times \dim \xi$. On the other hand,

$$n = |Q^0| + |\operatorname{supp}(\sigma)| = |Q^0| + \sum_{1\le j\le m} |\operatorname{supp}(\sigma_j)| = |Q^0| + \sum_{1\le j\le m} \ell(\sigma_j).$$

Thus we get the normalized character $\widetilde{\chi}_{\pi_{\zeta,\xi}}$ of the elementary IUR $\pi_{\zeta,\xi}$ as follows.

LEMMA 4.3. (i) *For $g = (d,\sigma) \in G_n = D_n \rtimes \mathfrak{S}_n$, let $g = \prod_{q\in Q^0} \xi_q \cdot g_1 g_2 \cdots g_m$ be its standard decomposition in the wider sense, with $Q^0 = \boldsymbol{I}_n \setminus \operatorname{supp}(\sigma)$ and $g_j = (d_j, \sigma_j)$, then*

$$(4.6)\qquad \widetilde{\chi}_{\pi_{\zeta,\xi}}\bigl((d,\sigma)\bigr) = \prod_{q\in Q^0} \frac{\chi_\zeta(t_q)}{\dim \zeta} \cdot \prod_{1\le j\le m} \frac{\chi_\zeta\bigl(P_{\sigma_j}(d_j)\bigr)}{(\dim \zeta)^{\ell(\sigma_j)}} \times \widetilde{\chi}_\xi(\sigma).$$

(ii) *Let $g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r} g_1 g_2 \cdots g_m$ be the standard decomposition in (2.6) of $g$ into a product of basic elements, so that $\xi_{q_k} = (t_{q_k}, (q_k))$ with $t_{q_k} \ne e_T$. Put $Q = \{q_1, q_2, \ldots, q_r\}$, then*

$$(4.7)\qquad \widetilde{\chi}_{\pi_{\zeta,\xi}}\bigl((d,\sigma)\bigr) = \prod_{q\in Q} \frac{\chi_\zeta(t_q)}{\dim \zeta} \cdot \prod_{1\le j\le m} \frac{\chi_\zeta\bigl(P_{\sigma_j}(d_j)\bigr)}{(\dim \zeta)^{\ell(\sigma_j)}} \times \widetilde{\chi}_\xi(\sigma).$$

### 4.3. Twist by $\tau \in \mathfrak{S}_n(\sigma)$ of irreducible character: $\widetilde{\chi}_\pi(\tau h \tau^{-1})$

We follow the notations in the part (ii) of the proof of Theorem 3.3. For the partition $\boldsymbol{I}_n = \bigsqcup_{\zeta\in\widehat{T}'} I_{n,\zeta}$, we put

$$(4.8)\qquad H_n = D_n \rtimes S_{[\eta]}, \quad D_n = D_n(T) = D_{\boldsymbol{I}_n}(T), \quad S_{[\eta]} = \prod\nolimits_{\zeta\in\widehat{T}'} \mathfrak{S}_{I_{n,\zeta}},$$

$$H_n = \prod\nolimits_{\zeta\in\widehat{T}'} H_{n,\zeta}, \quad H_{n,\zeta} = \mathfrak{S}_{I_{n,\zeta}}(T) = D_{I_{n,\zeta}}(T) \rtimes \mathfrak{S}_{I_{n,\zeta}},$$

$$(4.9)\qquad \pi = \eta \boxdot \xi, \quad \eta = \mathop{\boxtimes}_{\zeta\in\widehat{T}'} \zeta^{I_{n,\zeta}}, \quad \xi = \mathop{\boxtimes}_{\zeta\in\widehat{T}'} \pi(\lambda^{n,\zeta}),$$

where $\lambda^{n,\zeta}$ is a Young diagram of size $|I_{n,\zeta}|$ and $\pi(\lambda^{n,\zeta})$ denotes the IUR of $\mathfrak{S}_{I_{n,\zeta}}$ determined by the Young diagram $\lambda^{n,\zeta}$. We put $\Pi = \Pi_n = \operatorname{Ind}_{H_n}^{G_n} \pi$.

Denote by $\widetilde{\chi}(\lambda^{n,\zeta};\sigma')$ ($\sigma' \in \mathfrak{S}_{I_{n,\zeta}}$) the normalized character of $\pi(\lambda^{n,\zeta})$. If $\sigma' = \sigma'_1\sigma'_2\cdots\sigma'_s$ is a cycle decomposition of $\sigma'$, then $\widetilde{\chi}(\lambda^{n,\zeta};\sigma')$ is determined by the set $\{\ell'_p = \ell(\sigma'_p)\ ;\ 1 \le p \le s\}$ of lengths, and so it is also denoted by $\widetilde{\chi}(\lambda^{n,\zeta};(\ell'_p)_{1\le p\le s})$, that is,

$$(4.10)\qquad \widetilde{\chi}(\lambda^{n,\zeta};(\ell'_p)_{1\le p\le s}) := \widetilde{\chi}(\lambda^{n,\zeta};\sigma') \quad \text{if } \sigma' = \sigma'_1\sigma'_2\cdots\sigma'_s,\ \ell'_p = \ell(\sigma'_p).$$

Now take an $h = (d, \sigma) \in H_n \subset G_n$, and let its standard decomposition into basic elements be as in (2.6)

$$h = (d, \sigma) = \xi_{q_1}\xi_{q_2} \cdots \xi_{q_r} g_1 g_2 \cdots g_m, \tag{4.11}$$

$$\text{with } \xi_q = (t_q, (q)) \;\; (q \in Q), \quad g_j = (d_j, \sigma_j) \;\; (j \in J),$$

$$Q = \{q_1, q_2, \ldots, q_r\}, \quad J = \{1, 2, \ldots, m\}, \tag{4.12}$$

where the supports $Q$ and $K_j := \mathrm{supp}(g_j) := \mathrm{supp}(\sigma_j)$ $(j \in J)$ are disjoint subsets of $\boldsymbol{I}_n$, and $\sigma_j$ a cycle of length $\ell(\sigma_j) \geq 2$ and $\mathrm{supp}(d_j) \subset K_j$ for $j \in J$. For $\mathfrak{S}_n$-components, $\sigma = \sigma_1\sigma_2 \cdots \sigma_m$ is a cycle decomposition of $\sigma$. For $d_j = (t_i)_{i \in K_j} \in D_{K_j}(T) \hookrightarrow D_n(T)$, put $P_{\sigma_j}(d_j)$ as in (1.3). Put $S(h) := \{g' \in G_n \,;\, g'hg'^{-1} \in H_n\}$. Then, since $D_n \subset S(h)$, the set $S(h)$ is a union of

$$D_n\tau \quad \text{with } \tau \in \mathfrak{S}_n(\sigma) := \Big\{\tau \in \mathfrak{S}_n \,;\, \tau\sigma\tau^{-1} \in S_{[\eta]} = \prod\nolimits_{\zeta \in \widehat{T}'} \mathfrak{S}_{I_{n,\zeta}}\Big\}.$$

Moreover, since $D_n \subset H_n$, the integral in (4.2) is rewritten as

$$\widetilde{\chi}_{\Pi_n}(h) = \frac{1}{|\mathfrak{S}_n|} \sum_{\tau \in \mathfrak{S}_n(\sigma)} X_\pi(\tau h \tau^{-1}), \tag{4.13}$$

$$\text{with} \quad X_\pi(h') := \int_{D_n} \widetilde{\chi}_\pi(\widetilde{d} h' \widetilde{d}^{-1})\, d\mu_{D_n}(\widetilde{d}) = \widetilde{\chi}_\pi(h') \;\; (h' \in H_n),$$

where $d\mu_{D_n}(\widetilde{d})$ $(\widetilde{d} \in D_n)$ denotes the normalized Haar measure on $D_n$. Here we prefer the notation $X_\pi$ rather than $\widetilde{\chi}_\pi$ for simplicity. Put $h' = \tau h \tau^{-1}$, then

$$h' = (d', \sigma') = \xi'_{q_1}\xi'_{q_2} \cdots \xi'_{q_r} g'_1 g'_2 \cdots g'_m, \;\; d' = \tau(d), \; \sigma' = \tau\sigma\tau^{-1}, \tag{4.14}$$

$$\begin{cases} \xi'_{q_k} = \tau\xi_{q_k}\tau^{-1} = (t_{q_k}, (\tau(q_k))) : \;\; \text{element } t_{q_k} \text{ at } i = \tau(q_k) \in \boldsymbol{N}, \\ g'_j = \tau g_j \tau^{-1} = (d'_j, \sigma'_j), \; d'_j = (t'_i)_{i \in \tau(K_j)}, \; t'_i = t_{\tau^{-1}(i)}, \; \sigma'_j := \tau\sigma_j\tau^{-1}. \end{cases}$$

Here $\sigma' = \sigma'_1\sigma'_2 \cdots \sigma'_m$ is a cycle decomposition of $\sigma'$. Since $h' \in H_n = \prod_{\zeta \in \widehat{T}'} H_{n,\zeta}$, there exist partitions of $Q = \{q_1, q_2, \ldots, q_r\}$ and $J = \{1, 2, \ldots, m\}$ as

$$\begin{aligned} \mathcal{Q} &= (Q_\zeta)_{\zeta \in \widehat{T}'} \quad \text{with } Q = \bigsqcup\nolimits_{\zeta \in \widehat{T}'} Q_\zeta\,; \\ \mathcal{J} &= (J_\zeta)_{\zeta \in \widehat{T}'} \quad \text{with } J = \bigsqcup\nolimits_{\zeta \in \widehat{T}'} J_\zeta\,, \end{aligned} \tag{4.15}$$

such that for $H_{n,\zeta} = D_{I_{n,\zeta}}(T) \rtimes \mathfrak{S}_{I_{n,\zeta}}$

$$\begin{aligned} &\xi'_q \in H_{n,\zeta} \ \text{ or } \ \mathrm{supp}(\xi'_q) = \tau(q) \in I_{n,\zeta} && \text{if } q \in Q_\zeta\,; \\ &g'_j \in H_{n,\zeta} \ \text{ or } \ \mathrm{supp}(g'_j) = \tau(K_j) \subset I_{n,\zeta} && \text{if } j \in J_\zeta\,. \end{aligned} \tag{4.16}$$

LEMMA 4.4. *Let $\tau \in \mathfrak{S}_n(\sigma)$. Then there exist partitions $\mathcal{Q} = (Q_\zeta)_{\zeta \in \widehat{T}'}$ of $Q$ and $\mathcal{J} = (J_\zeta)_{\zeta \in \widehat{T}'}$ of $J$ such that* (4.16) *holds. Then $X_\pi(\tau h \tau^{-1})$ is given with $\ell_j = \ell(\sigma_j)$ as*

$$\begin{aligned} &X_\pi(\tau h \tau^{-1}) \\ &\quad = \prod_{\zeta \in \widehat{T}'} \left( \prod_{q \in Q_\zeta} \frac{\chi_\zeta(t_q)}{\dim \zeta} \times \prod_{j \in J_\zeta} \frac{\chi_\zeta\big(P_{\sigma_j}(d_j)\big)}{(\dim \zeta)^{\ell(\sigma_j)}} \times \widetilde{\chi}\big(\lambda^{n,\zeta}; (\ell_j)_{j \in J_\zeta}\big) \right). \end{aligned} \tag{4.17}$$

Put $\mathcal{I}_n = (I_{n,\zeta})_{\zeta \in \widehat{T}}$ and $\Lambda^n = (\lambda^{n,\zeta})_{\zeta \in \widehat{T}}$, where the size $|\lambda^{n,\zeta}|$ of Young diagram of $\lambda^{n,\zeta}$ is $|I_{n,\zeta}|$ and, except a finite number of $\zeta \in \widehat{T}$, $I_{n,\zeta} = \emptyset$ and accordingly $\lambda^{n,\zeta} = \emptyset$. We denote the function on the right hand side of (4.17) also as

$$\begin{aligned} &X\big(\Lambda^n; \mathcal{Q}, \mathcal{J}; h\big) \\ &\quad := \prod_{\zeta \in \widehat{T}} \left( \prod_{q \in Q_\zeta} \frac{\chi_\zeta(t_q)}{\dim \zeta} \times \prod_{j \in J_\zeta} \frac{\chi_\zeta\big(P_{\sigma_j}(d_j)\big)}{(\dim \zeta)^{\ell(\sigma_j)}} \times \widetilde{\chi}\big(\lambda^{n,\zeta}; (\ell_j)_{j \in J_\zeta}\big) \right). \end{aligned} \tag{4.18}$$

### 4.4. Character formula for IURs of $\mathfrak{S}_{\boldsymbol{n}}(\boldsymbol{T})$

Now we come to the calculation of the sum of $X_\pi(\tau h \tau^{-1})$ over $\tau \in \mathfrak{S}_n(\sigma)$ in (4.13). We divide the sum into partial sums depending on a pair $(\mathcal{Q}, \mathcal{J})$ of partitions of $Q = \{q_1, q_2, \ldots, q_r\}$ and $J = \{1, 2, \ldots, m\}$ in such a way that $\tau$ satisfies

CONDITION ON $\tau \in \mathfrak{S}_n(\sigma)$:

$$\tau(q) \in I_{n,\zeta} \ \ (q \in Q_\zeta); \quad \tau(K_j) \subset I_{n,\zeta} \ \ (j \in J_\zeta). \tag{4.19}$$

This kind of $\tau$ exists for a pair $(\mathcal{Q}, \mathcal{J})$ if and only if the following inequalities hold:

$$|I_{n,\zeta}| \geq |Q_\zeta| + \sum_{j \in J_\zeta} |K_j| = |Q_\zeta| + \sum_{j \in J_\zeta} \ell(\sigma_j) \quad (\forall\, \zeta \in \widehat{T}). \tag{4.20}$$

For any $\tau$ satisfying (4.19), the value $X_\pi(\tau h\tau^{-1})$ coincides with the term in (4.18). So we should calculate the number $N(\mathcal{I}_n;\mathcal{Q},\mathcal{J})$ of $\tau\in\mathfrak{S}_n(\sigma)$ satisfying (4.19), where $\mathcal{I}_n := (I_{n,\zeta})_{\zeta\in\widehat{T}}$.

Since $\tau(i)\in I_{n,\zeta}$ can be chosen arbitrary inside $I_{n,\zeta}$ for any $i\in Q_\zeta\sqcup\big(\bigsqcup_{j\in J_\zeta}K_j\big)$, and then, for any $i\in \boldsymbol{I}_n\setminus\big(Q\sqcup\mathrm{supp}(\sigma)\big)$, $\tau(i)$ can be chosen freely, we get the following, because $\mathrm{supp}(\sigma)=\bigsqcup_{j\in J}K_j$ and

$$|\boldsymbol{I}_n\setminus\big(Q\sqcup\mathrm{supp}(\sigma)\big)| = n-\sum_{\zeta\in\widehat{T}}|Q_\zeta|-\sum_{j\in J}|K_j| = n-|Q|-|\,\mathrm{supp}(\sigma)|:$$

$$\begin{aligned}(4.21)\qquad & N(\mathcal{I}_n;\mathcal{Q},\mathcal{J})\\ &=\prod_{\zeta\in\widehat{T}}|I_{n,\zeta}|\big(|I_{n,\zeta}|-1\big)\cdots\Big(|I_{n,\zeta}|-|Q_\zeta|-\sum_{j\in J_\zeta}|K_j|+1\Big)\\ &\qquad\times(n-|Q|-|\,\mathrm{supp}(\sigma)|)!\end{aligned}$$

$$(4.22)\qquad \sum_{\mathcal{Q},\mathcal{J}}N(\mathcal{I}_n;\mathcal{Q},\mathcal{J})=n!\quad\text{for}\ \ n\geq|Q|+|\,\mathrm{supp}(\sigma)|.$$

Here even for a pair $(\mathcal{Q},\mathcal{J})$ which does not satisfy the inequality condition (4.20), the above formula for $N(\mathcal{I}_n;\mathcal{Q},\mathcal{J})$ is valid, since it says $N(\mathcal{I}_n;\mathcal{Q},\mathcal{J})=0$.

After these calculations we obtain finally the normalized character of an IUR of the wreath product $G_n=\mathfrak{S}_n(T)=D_n(T)\rtimes\mathfrak{S}_n$ of a compact group $T$ with the $n$-th symmetric group $\mathfrak{S}_n$ as follows.

THEOREM 4.5. (i) *Let $\mathcal{I}_n=(I_{n,\zeta})_{\zeta\in\widehat{T}}$ be a partition of $\boldsymbol{I}_n$, and $\Lambda^n=(\lambda^{n,\zeta})_{\zeta\in\widehat{T}}$ be a set of Young diagrams such that $\lambda^{n,\zeta}$ determines an IUR $\pi(\lambda^{n,\zeta})$ of $\mathfrak{S}_{I_{n,\zeta}}\cong\mathfrak{S}_{|I_{n,\zeta}|}$, where the size of $\lambda^{n,\zeta}$, denoted by $|\lambda^{n,\zeta}|$, is equal to $|I_{n,\zeta}|$. Put*

$$(4.23)\qquad \eta=\boxtimes_{\zeta\in\widehat{T}}\zeta^{I_{n,\zeta}},\qquad \xi=\boxtimes_{\zeta\in\widehat{T}}\pi(\lambda^{n,\zeta}),\qquad \pi=\eta\boxdot\xi.$$

*Then $\pi$ is an IUR of $H_n=D_n(T)\rtimes S_{[\eta]}$ with $S_{[\eta]}=\prod_{\zeta\in\widehat{T}}\mathfrak{S}_{I_{n,\zeta}}$, and the induced representation $\Pi_n=\mathrm{Ind}_{H_n}^{G_n}\pi$ is irreducible. Every IUR of $G_n$ is equivalent to an induced representation of this type.*

(ii) *Take a $g=(d,\sigma)\in G_n$ which is conjugate to an element in $H_n$. Let its standard decomposition be as in (4.11), and correspondingly define $Q=\{q_1,q_2,\ldots,q_r\}$ and $J=\{1,2,\ldots,m\}$ as in (4.12). Then the value*

$\widetilde{\chi}_{\Pi_n}(g)$ *of the normalized character* $\widetilde{\chi}_{\Pi_n}$ *of the IUR* $\Pi_n$ *of* $G_n$ *is given as follows*:

$$\widetilde{\chi}_{\Pi_n}(g) = \sum_{\mathcal{Q},\mathcal{J}} c(\mathcal{I}_n; \mathcal{Q}, \mathcal{J})\, X\big(\Lambda^n; \mathcal{Q}, \mathcal{J}; g\big), \tag{4.24}$$

*with*
$$c(\mathcal{I}_n; \mathcal{Q}, \mathcal{J}) = \frac{N(\mathcal{I}_n; \mathcal{Q}, \mathcal{J})}{n!} = \frac{\prod_{\zeta\in\widehat{T}} |I_{n,\zeta}|\big(|I_{n,\zeta}|-1\big)\cdots\Big(|I_{n,\zeta}| - |Q_\zeta| - \sum_{j\in J_\zeta}|K_j| + 1\Big)}{n(n-1)(n-2)\cdots\big(n - |Q| - |\operatorname{supp}(\sigma)| + 1\big)},$$

$$X\big(\Lambda^n; \mathcal{Q}, \mathcal{J}; g\big) = \prod_{\zeta\in\widehat{T}} \left( \prod_{q\in Q_\zeta} \frac{\chi_\zeta(t_q)}{\dim\zeta} \times \prod_{j\in J_\zeta} \frac{\chi_\zeta\big(P_{\sigma_j}(d_j)\big)}{(\dim\zeta)^{\ell(\sigma_j)}} \times \widetilde{\chi}\big(\lambda^{n,\zeta}; (\ell(\sigma_j))_{j\in J_\zeta}\big) \right),$$

*where* $\mathcal{Q} = (Q_\zeta)_{\zeta\in\widehat{T}}$ *and* $\mathcal{J} = (J_\zeta)_{\zeta\in\widehat{T}}$ *run over partitions of* $Q$ *and* $J$ *respectively, and* $\widetilde{\chi}\big(\lambda^{n,\zeta}; *\big)$ *denotes the normalized character of IUR* $\pi(\lambda^{n,\zeta})$ *of* $\mathfrak{S}_{I_{n,\zeta}}$ *as in* (4.10).

*Here, except a finite number of* $\zeta\in\widehat{T}$, $Q_\zeta = \emptyset$ *and* $J_\zeta = \emptyset$, *and then the corresponding* $\zeta$*-th factor in* $\prod_{\zeta\in\widehat{T}} \bullet$ *should be understood as equal to* 1.

(iii) *For a* $g = (d,\sigma) \in G_n$ *which is not conjugate to any element in* $H_n$, *the character vanishes*: $\widetilde{\chi}_{\Pi_n}(g) = 0$. *The above character formula* (4.24) *is also valid for* $g$ *in the sense that there is no pair of partitions* $(\mathcal{Q},\mathcal{J})$ *for which* $|I_{n,\zeta}| - |Q_\zeta| - \sum_{j\in J_\zeta}|K_j| \geq 0$ $(\forall\,\zeta\in\widehat{T})$, *or no* $\mathcal{J}$ *for which* $|I_{n,\zeta}| \geq \sum_{j\in J_\zeta}|K_j|$ $(\forall\,\zeta\in\widehat{T})$.

*Remark* 4.1. The character formula in Theorem 4.5 (ii) is given for $g = (d,\sigma)$ such that $\operatorname{supp}(d)\backslash\operatorname{supp}(\sigma) = Q$, but it is valid even for $g' = (d',\sigma)$ with the same $\sigma$ but with a $D_n$-part $d'$ satisfying a weaker condition:

$$\operatorname{supp}(d') \setminus \operatorname{supp}(\sigma) \subset Q. \tag{4.25}$$

This validity can be seen from the continuity of the both sides of (4.24).

*Remark* 4.2. The partition $\mathcal{I}_n$ of $I_n$ is used to construct IUR $\Pi_n$, but in the character formula for $\widetilde{\chi}_{\Pi_n}$, the property of the partition $\mathcal{I}_n$ itself has disappeared, and there remains only the partition $\mathcal{N}_n = (n_\zeta)_{\zeta\in\widehat{T}}$ of $n = |I_n|$ given as $n = \sum_{\zeta\in\widehat{T}} n_\zeta$, $n_\zeta = |I_{n,\zeta}|$.

It is worth noting here that many related interesting things are going on in the case where $T$ is finite (see e.g. [AK], [Kaw] and [Sho]).

## §5. Towards limits of irreducible characters of $\mathfrak{S}_n(T)$

### 5.1. Conditions for existence of limits of irreducible characters

Let us study asymptotic behavior as $n \to \infty$ of the characters of IURs $\Pi = \Pi_n$ of $\mathfrak{S}_n(T)$ parametrized by a pair of a partition $\mathcal{I}_n = (I_{n,\zeta})_{\zeta \in \widehat{T}}$ of $\boldsymbol{I}_n$ and a set of Young diagrams $\Lambda^n = (\lambda^{n,\zeta})_{\zeta \in \widehat{T}}$ corresponding to $\mathcal{I}_n$ by (4.23). The present purpose is to determine a necessary and sufficient condition for pointwise convergence according as both $\mathcal{I}_n$'s and $\Lambda^n$'s are increasing so that $I_{n,\zeta} \subseteq I_{n+1,\zeta} \subseteq \cdots$ (and $\boldsymbol{I}_n = \bigsqcup_{\zeta \in \widehat{T}} I_{n,\zeta} \nearrow \boldsymbol{N}$) and that, for $n \geq n_0$, $\Lambda^n \nearrow \Lambda^{n+1}$. Here by definition

$$(5.1) \qquad \Lambda^n = (\lambda^{n,\zeta})_{\zeta \in \widehat{T}} \nearrow \Lambda^{n+1} = (\lambda^{n+1,\zeta})_{\zeta \in \widehat{T}}$$

means that, for one $\zeta'$ only, $\lambda^{n,\zeta'} \nearrow \lambda^{n+1,\zeta'}$ and, for other $\zeta \neq \zeta'$, $\lambda^{n,\zeta} = \lambda^{n+1,\zeta}$, and in turn, $\lambda^{n,\zeta'} \nearrow \lambda^{n+1,\zeta'}$ means that the Young diagram of $\lambda^{n,\zeta'}$ increases to that of $\lambda^{n+1,\zeta'}$ by adding one box to some possible position.

Admitting the empty set as a Young diagram of size 0, we put $\emptyset = (\emptyset_\zeta)_{\zeta \in \widehat{T}}$. In the second part [HHH2] of our present work, we will treat in detail the spaces of paths

$$(\Lambda^n)_{n \geq 0} : \Lambda^0 = \emptyset \nearrow \Lambda^1 \nearrow \cdots \nearrow \Lambda^n \nearrow \Lambda^{n+1} \nearrow \cdots$$

of infinite lengths on the set of Young diagrams and probability measures on it, and discuss problems of limits of irreducible characters from the stand point of stochastic processes in discrete time. However in this first part of our present work we do not get into details in this direction, and restrict ourselves to discuss these limit problems by explicit calculations in the theory of group representations.

In this section we proceed step by step to get limits of irreducible characters, and in the next section we obtain explicitly the limits and also determine a necessary and sufficient condition on a sequence of irreducible characters of $\mathfrak{S}_n(T)$ to have a limit as $n \to \infty$.

Note that, in the case where the compact group $T$ is finite, the limit group $G = D_\infty \rtimes \mathfrak{S}_\infty$ with the inductive limit topology $\tau_{ind}$ is discrete, and so the continuity of the limit functions is not a problem. However, in the case where the compact group $T$ is infinite, $G$ with $\tau_{ind}$ is no more locally compact, and the continuity of limit functions is an important point to study. In this case, the group $D_\infty = D_\infty(T) = \lim_{n\to\infty} D_n(T)$ embedded into $G$ is an open subgroup, and accordingly positive definite functions obtained as limits of irreducible characters of $G_n$ as $n \to \infty$ are continuous on $G$ if and only if so are they on $D_\infty$.

Take an $h = (d, \sigma) \in \mathfrak{S}_{n_0}(T)$ and consider it as an element of each $\mathfrak{S}_n(T) \hookleftarrow \mathfrak{S}_{n_0}(T)$, $n \geq n_0$.

STEP 1. Case of $h = (d, \mathbf{1})$, $d = (t_i)_{i \in \boldsymbol{I}_n} \in D_n$, with $\mathbf{1} \in \mathfrak{S}_n$ the identity element.

In this case, $Q = \mathrm{supp}(d) = \{q \in \boldsymbol{N}\,;\, t_q \neq e_T\}$ and $J = \emptyset$, and so the parameter $\mathcal{J}$ does not exist, and the character formula (4.24) in Theorem 4.5 reduces at $h = (d, \mathbf{1})$ as follows:

$$
(5.2)\qquad \begin{cases} \widetilde{\chi}_{\Pi_n}\big((d,\mathbf{1})\big) = \displaystyle\sum_{\mathcal{Q}} c(\mathcal{I}_n; \mathcal{Q})\, X\big(\mathcal{Q}; (d, \mathbf{1})\big), \\[2ex] c(\mathcal{I}_n; \mathcal{Q}) = \dfrac{\prod_{\zeta \in \widehat{T}} |I_{n,\zeta}|\big(|I_{n,\zeta}| - 1\big) \cdots \big(|I_{n,\zeta}| - |Q_\zeta| + 1\big)}{n(n-1)(n-2) \cdots \big(n - |Q| + 1\big)}, \\[2ex] X\big(\mathcal{Q}; (d, \mathbf{1})\big) = \displaystyle\prod_{\zeta \in \widehat{T}} \left( \prod_{q \in Q_\zeta} \frac{\chi_\zeta(t_q)}{\dim \zeta} \right) \quad (d \in D_n). \end{cases}
$$

Note that $\mathcal{Q} = (Q_\zeta)_{\zeta \in \widehat{T}}$ runs over all partitions of $Q$ into subsets indexed by $\widehat{T}$, and that

$$
(5.3)\qquad c(\mathcal{I}_n; \mathcal{Q}) \asymp \prod_{\zeta \in \widehat{T}} \left( \frac{|I_{n,\zeta}|}{n} \right)^{|Q_\zeta|} \quad \text{(asymptotically equivalent as } n \to \infty\text{)},
$$

$$
(5.4)\qquad \sum_{\mathcal{Q}} c(\mathcal{I}_n; \mathcal{Q}) = 1, \qquad \sum_{\mathcal{Q}} \prod_{\zeta \in \widehat{T}} \left( \frac{|I_{n,\zeta}|}{n} \right)^{|Q_\zeta|} = 1 \quad \text{for any } n \geq |Q|.
$$

Note that the first equality in (5.4) is for the case where we take a number of $|Q|$ elements without repetitions, and the second one is for the case with repetitions.

Note further that for a $d \in D_\infty$, there exists $n_0 > 0$ such that $d$ is contained in $D_n$ and so $h = (d, \mathbf{1}) \in G_n$, for $n \geq n_0$.

LEMMA 5.1. *If there exists the pointwise limit* $\lim_{n \to \infty} \widetilde{\chi}_{\Pi_n}((d, \mathbf{1}))$ *on* $D_\infty$*, then the ratio* $|I_{n,\zeta}|/n$ *converges for every* $\zeta \in \widehat{T}$*, or equivalently there holds*:

$$
(5.5)\qquad (\text{CONDITION I})\ \textit{the following limits exist: } B_\zeta := \lim_{n \to \infty} \frac{|I_{n,\zeta}|}{n} \quad (\zeta \in \widehat{T}).
$$

*Proof.* For a finite subset $Q \subset \boldsymbol{N}$, define a compact subgroup $D_Q$ of $D_\infty$ and its open subset $D'_Q$ as

$$(5.6) \qquad \begin{aligned} D_Q &:= \{d = (t_i)_{i \in \boldsymbol{N}} \in D_\infty \,;\, t_i = e_T \ (i \notin Q)\}, \\ D'_Q &:= \{d = (t_i)_{i \in \boldsymbol{N}} \in D_Q \,;\, t_i \neq e_T \ (i \in Q)\}. \end{aligned}$$

Then (5.2) is valid not only on $D'_Q$ but also on the compact group $D_Q$. Let $L^2(D_Q)$ be the Hilbert space of square-summable functions with respect to the normalized Haar measure on $D_Q$. Take $n$ sufficiently large so that $D_Q \subset D_n$. Put $f(\mathcal{Q}; d) = X\big(\mathcal{Q}; (d, \mathbf{1})\big)$, $d \in D_Q$. Then they are mutually orthogonal in $L^2(D_Q)$, and have the lengths $\|f(\mathcal{Q})\| = \prod_{\zeta \in \widehat{T}} (\dim \zeta)^{-|Q_\zeta|} = (\dim \zeta)^{-|Q|}$.

On the other hand, the functions $\widetilde{\chi}_{\Pi_n}$, $n \geq n_0$, on $D_Q$ are uniformly bounded by 1. Hence, if the sequence $\widetilde{\chi}_{\Pi_n}$ converges pointwise on $D_Q$, then the inner product in $L^2(D_Q)$

$$\langle \widetilde{\chi}_{\Pi_n}, f(\mathcal{Q}) \rangle = c(\mathcal{I}_n; \mathcal{Q})\, (\dim \zeta)^{-2|Q|}$$

will converge. This gives Condition I. ∎

Under Condition I, we have $B_\zeta \geq 0$, and $\sum_{\zeta \in \widehat{T}} B_\zeta \leq 1$.

Assume $T$ be finite. Then the dual $\widehat{T}$ is automatically finite, and we get in the inequality above the equality: $\sum_{\zeta \in \widehat{T}} B_\zeta = 1$.

Assume $T$ be infinite. Then we consider the following equality condition:

$$(5.7) \quad (\text{Condition E}) \qquad \sum_{\zeta \in \widehat{T}} B_\zeta = 1.$$

Lemma 5.2. *Assume* Condition I. *Then, under* Condition E, *the limit positive definite function* $\lim_{n\to\infty} \widetilde{\chi}_{\Pi_n}((d, \mathbf{1}))$ *on* $D_\infty \hookrightarrow G$ *is given by*

$$(5.8) \qquad F(d) = \sum_{\mathcal{Q}} \prod_{\zeta \in \widehat{T}} \left( \prod_{q \in Q_\zeta} B_\zeta \frac{\chi_\zeta(t_q)}{\dim \zeta} \right) = \prod_{q \in Q} F_1(t_q),$$

$$F_1(t) := \sum_{\zeta \in \widehat{T}} B_\zeta \frac{\chi_\zeta(t)}{\dim \zeta} \quad (t \in T),$$

*where* $Q = \mathrm{supp}(d) = \{q \in \boldsymbol{N} \,;\, t_q \neq e_T\}$. *The convergence is uniform on each* $D_k$.

*If $T$ is infinite, the limit function $F$ is continuous on $D_\infty$ under* Condition E.

*Proof.* Take a finite subset $\mathcal{F}$ of partitions $\mathcal{Q} = (Q_\zeta)_{\zeta\in\widehat{T}}$ of $Q$. Then,

$$\left|\widetilde{\chi}_{\Pi_n}\big((d,\mathbf{1})\big) - F(d)\right| \le J_1(n;\mathcal{F}) + J_2(n;\mathcal{F}) + J_3(\mathcal{F}),$$

where, with $c(\mathcal{Q}) := \lim_{n\to\infty} c(\mathcal{I}_n;\mathcal{Q}) = \prod_{\zeta\in\widehat{T}} B_\zeta^{|Q_\zeta|}$,

$$\begin{aligned} J_1(n;\mathcal{F}) &= \left|\sum_{\mathcal{Q}\in\mathcal{F}} \Big(c(\mathcal{I}_n;\mathcal{Q}) - c(\mathcal{Q})\Big) \prod_{\zeta\in\widehat{T}} \prod_{q\in Q_\zeta} \frac{\chi_\zeta(t_q)}{\dim\zeta}\right| \\ &\le \sum_{\mathcal{Q}\in\mathcal{F}} \Big|c(\mathcal{I}_n;\mathcal{Q}) - c(\mathcal{Q})\Big| =: J_1^0(n;\mathcal{F}), \end{aligned}$$

and, under the similar evaluations,

$$J_2(n;\mathcal{F}) = \left|\sum_{\mathcal{Q}\notin\mathcal{F}} c(\mathcal{I}_n;\mathcal{Q}) \prod_{\zeta\in\widehat{T}} \prod_{q\in Q_\zeta} \frac{\chi_\zeta(t_q)}{\dim\zeta}\right| \le \sum_{\mathcal{Q}\notin\mathcal{F}} c(\mathcal{I}_n;\mathcal{Q}) =: J_2^0(n;\mathcal{F}),$$

$$J_3(\mathcal{F}) = \left|\sum_{\mathcal{Q}\notin\mathcal{F}} \prod_{\zeta\in\widehat{T}} \prod_{q\in Q_\zeta} B_\zeta \frac{\chi_\zeta(t_q)}{\dim\zeta}\right| \le \sum_{\mathcal{Q}\notin\mathcal{F}} c(\mathcal{Q}) =: J_3^0(\mathcal{F}).$$

Note that

$$\sum_{\mathcal{Q}} c(\mathcal{I}_n;\mathcal{Q}) = 1, \quad \sum_{\mathcal{Q}} c(\mathcal{Q}) = \Big(\sum_{\zeta\in\widehat{T}} B_\zeta\Big)^{|Q|} = 1, \tag{5.9}$$

then we have

$$\sum_{\mathcal{Q}\notin\mathcal{F}} c(\mathcal{I}_n;\mathcal{Q}) = 1 - \sum_{\mathcal{Q}\in\mathcal{F}} c(\mathcal{I}_n;\mathcal{Q}) = \sum_{\mathcal{Q}} c(\mathcal{Q}) - \sum_{\mathcal{Q}\in\mathcal{F}} c(\mathcal{I}_n;\mathcal{Q}) \tag{5.10}$$

$$= \sum_{\mathcal{Q}\in\mathcal{F}} \big(c(\mathcal{Q}) - c(\mathcal{I}_n;\mathcal{Q})\big) + \sum_{\mathcal{Q}\notin\mathcal{F}} c(\mathcal{Q}), \tag{5.11}$$

and so, $J_2^0(n;\mathcal{F}) \le J_1^0(n;\mathcal{F}) + J_3^0(\mathcal{F})$. Hence

$$\left|\widetilde{\chi}_{\Pi_n}\big((d,\mathbf{1})\big) - F(d)\right| \le 2J_1^0(n;\mathcal{F}) + 2J_3^0(\mathcal{F}).$$

Now, for a given small $\varepsilon > 0$, take a finite set $\mathcal{F}$ sufficiently large so that $J_3^0(\mathcal{F}) < \varepsilon$. Then, choose $n$ sufficiently large so that $J_1^0(n; \mathcal{F}) < \varepsilon$. So, we have

$$\left|\widetilde{\chi}_{\Pi_n}\big((d, \mathbf{1})\big) - F(d)\right| \leq 4\varepsilon.$$

This proves that $\widetilde{\chi}_{\Pi_n}\big((d, \mathbf{1})\big)$ converges to $F(d)$ uniformly on $D_Q$.

The continuity of $F$ on $D_\infty$ is easy to see. ∎

In the case of infinite compact group $T$, we have the following interesting case of discontinuous limit functions. For a finite subset $Q$ of $\boldsymbol{N}$, define a compact subgroup $D_Q$ of $D_\infty$ and its open subset $D'_Q$ as in (5.6). Assume Condition I holds, and using $B_\zeta = \lim_{n\to\infty} |I_{n,\zeta}|/n$, we put $F(d)$ for $d \in D_\infty$ as in Lemma 5.2, if $\mathrm{supp}(d) = Q$,

$$F(d) := \prod_{q\in Q} F_1(t_q), \quad F_1(t) := \sum_{\zeta\in\widehat{T}} B_\zeta \frac{\chi_\zeta(t)}{\dim\zeta} \quad (t \in T). \tag{5.12}$$

Then the sum expressing $F_1(t)$ is uniformly convergent on $T$, and $F(d)$, $d \in D'_Q$, has a continuous extension $f_Q$ onto $D_Q$, with the same factorized form.

Assume Condition E does not hold, or $\sum_{\zeta\in\widehat{T}} B_\zeta < 1$. In this case, contrary to the case of Lemma 5.2 under Condition E, the function $F$, presumably supposed to be the pointwise limit, is nowhere continuous on $D_\infty$. In fact, the system of functions $f_Q$ on $D_Q$ are not consistent with inclusion relations $D_Q \supset D_{Q^0}$ with $Q \supset Q^0$ as seen in Remark 5.1 below. However we presume that $F$ is the pointwise limit of $\widetilde{\chi}_{\Pi_n}\big((d, \mathbf{1})\big)$ in this case too. Here we only show that $F$ is a kind of weak limit in the following lemma (also cf. Example 6.1).

Lemma 5.3. *Let $T$ be an infinite compact group. Assume that* Condition I *in* (5.5) *holds, but* Condition E *in* (5.7) *does not hold. For each finite subset $Q$ of $\boldsymbol{N}$, take a compact subgroup $D_Q$ of $D_\infty$. Then the restriction $\widetilde{\chi}_{\Pi_n}\big((d, \mathbf{1})\big)$, $d \in D_Q$, of irreducible character $\widetilde{\chi}_{\Pi_n}$ converges weakly in $L^2(D_Q)$ to a positive definite function*

$$f_Q(d) := \prod_{q\in Q} F_1(t_q) \quad (d = (t_q)_{q\in Q} \in D_Q).$$

*Proof.* For a finite subset $Q$, the function $f_Q$ on $D_Q$ is continuous and positive definite. Then, Condition I says that $\langle\widetilde{\chi}_{\Pi_n}\big((d, \mathbf{1})\big) - f_Q(d),$

$f(\mathcal{Q};d)\rangle \to 0$ as $n \to \infty$ in $L^2(D_Q)$. On the other hand, for the subspace $L^2(D_Q)^0$ consisting of functions invariant under inner automorphisms, $\{f(\mathcal{Q})\}$ gives a complete orthogonal system. Therefore $\langle \widetilde{\chi}_{\Pi_n} - f_Q, f\rangle \to 0$ for any $f \in L^2(D_Q)^0$, that is, $\widetilde{\chi}_{\Pi_n}$ converges weakly to $f_Q$. □

We also wonder if $\widetilde{\chi}_{\Pi_n}\big((d,\mathbf{1})\big)$ converges strongly to $f_Q(d)$ in $L^2(D_Q)$. If so, $\widetilde{\chi}_{\Pi_n}\big((d,\mathbf{1})\big)$ converges on $D_\infty$ to $F$ '*almost everywhere*' pointwise. The strong convergence is equivalent to $\|\widetilde{\chi}_{\Pi_n}\|^2 \to \|f_Q\|^2$, that is,

$$\sum_{\mathcal{Q}} c(\mathcal{I}_n, \mathcal{Q})^2 (\dim\zeta)^{-2|Q|} \longrightarrow \sum_{\mathcal{Q}} \Big(\prod_{\zeta\in\widehat{T}} B_\zeta^{|Q_\zeta|}\Big)^2 (\dim\zeta)^{-2|Q|},$$

$$\text{or} \qquad \sum_{\mathcal{Q}} c(\mathcal{I}_n, \mathcal{Q})^2 \longrightarrow \Big(\sum_{\zeta\in\widehat{T}} B_\zeta^2\Big)^{|Q|}.$$

*Remark* 5.1. In the situation of Lemma 5.3, we have $B := \sum_{\zeta\in\widehat{T}} B_\zeta < 1$, since CONDITION E does not hold. Note that the system of functions $f_Q$ is not consistent with the inclusion of the groups $D_Q \supset D_{Q^0}$ as $Q \supset Q^0$, In fact, take a $q_0 \in Q$ and put $Q^0 = Q \setminus \{q_0\}$, and let an open subset $D'_Q$ of $D_Q$ be as above. Then, when $t_{q_0} \to e_T$ in $d = (t_i)_{i\in\boldsymbol{N}} \in D'_Q$, we have $d \to d' := (t'_i)_{i\in\boldsymbol{N}} \in D'_{Q^0}$ with $t'_{q_0} = e_T$ and $t'_i = t_i$ $(i \neq q_0)$, and for $t_{q_0} \to e_T$,

$$f_Q(d) \to \Big(\sum_{\zeta\in\widehat{T}} B_\zeta\Big) \cdot \prod_{q\in Q^0} F_1(t_q) = B \cdot f_{Q^0}(d') \neq f_{Q^0}(d').$$

For each compact subgroup $D_Q \subset D_\infty$, the weak limit function $F|_{D_Q}$ in $L^2(D_Q)$, a measurable positive definite function on $D_Q$, has a continuous version $f_Q$. However, on the whole $D_\infty$ which is no more locally compact, $F$ is a measurable positive definite function which is nowhere continuous and has no continuous version.

*Remark* 5.2. A reason why we have inconsistency at the boundary $D_Q \setminus D'_Q$ of $D_Q$ can be seen as follows. By Remark 4.1, the formula (5.2) for the character value $\widetilde{\chi}_{\Pi_n}\big((d,\mathbf{1})\big)$ is valid not only on $D'_Q$ but also on the compact group $D_Q$.

Take a $q^0 \in Q$ and a boundary point $d^0 \in D_Q \setminus D'_Q$ given as $d^0 = (t_q)_{q\in Q}$ with $t_{q^0} = e_T$ and $t_q \neq e_T$ $(q \in Q^0 = Q \setminus \{q^0\})$, where, outside $Q$, $t_i = e_T$

for all $i \notin Q$ and these are omitted. Then $d^0 \in D'_{Q^0}$, and the proper formula for $D'_{Q^0}$ gives us

$$\widetilde{\chi}_{\Pi_n}\big((d^0,\mathbf{1})\big) = \sum_{\mathcal{Q}^0} c(\mathcal{I}_n;\mathcal{Q}^0)\, X\big(\mathcal{Q}^0;(d^0,\mathbf{1})\big),$$

$$c(\mathcal{I}_n;\mathcal{Q}^0) = \frac{\prod_{\zeta\in\widehat{T}} |I_{n,\zeta}|\big(|I_{n,\zeta}|-1\big)\cdots\big(|I_{n,\zeta}|-|Q^0_\zeta|+1\big)}{n(n-1)(n-2)\cdots\big(n-|Q^0|+1\big)},$$

$$X\big(\mathcal{Q}^0;(d^0,\mathbf{1})\big) = \prod_{\zeta\in\widehat{T}}\left(\prod_{q\in Q^0_\zeta}\frac{\chi_\zeta(t_q)}{\dim\zeta}\right),$$

where $\mathcal{Q}^0 = (Q^0_\zeta)_{\zeta\in\widehat{T}}$ runs over all partitions of $Q^0$.

Now compare this expression with the expression of $\widetilde{\chi}_{\Pi_n}(d^0)$ in the formula (5.2), extended from $D'_Q$ to $D_Q$. In the above expression for $(d^0,\mathbf{1}) \in D'_{Q^0}$, the monomial terms $X\big(\mathcal{Q}^0;(d^0,\mathbf{1})\big)$ are mutually orthogonal. On the other hand, in the extended formula (5.2), each monomial term $X\big(\mathcal{Q}^0;(d^0,\mathbf{1})\big)$ appears in many places corresponding to different partitions $\mathcal{Q}$ so as to realize $B = \sum_{\zeta\in\widehat{T}} B_\zeta < 1$.

The situation is similar to the following simple example. Consider for each $n$ an expression of $b_n = 1$ as

$$b_n = \sum_{1\le k\le n} a_{n,k}, \quad b_n = 1, \quad a_{n,k} := \frac{1}{n} \quad \Big(n \text{ times of } \frac{1}{n}\Big).$$

Then, starting from $\lim_{n\to\infty} b_n = 1 \neq 0 = \sum_{1\le k<\infty} \lim_{n\to\infty} a_{n,k}$.

STEP 2. Limits of the coefficients $c(\mathcal{I}_n;\mathcal{Q},\mathcal{J})$.

Assume the above limit condition CONDITION I. Since

$$\sum_{\zeta\in\widehat{T}} |Q_\zeta| + \sum_{j\in J_\zeta} \ell(\sigma_j) = |\operatorname{supp}(h)|$$

for a fixed $h = (d,\sigma)$, we get from the formula of $c(\mathcal{I}_n;\mathcal{Q},\mathcal{J})$ in (4.24)

$$\lim_{n\to\infty} c(\mathcal{I}_n;\mathcal{Q},\mathcal{J}) = c(\mathcal{Q},\mathcal{J}) := \prod_{\zeta\in\widehat{T}}\Big({B_\zeta}^{|Q_\zeta|}\cdot\prod_{j\in J_\zeta}{B_\zeta}^{\ell(\sigma_j)}\Big). \tag{5.13}$$

STEP 3. Limits of $c(\mathcal{I}_n;\mathcal{Q},\mathcal{J})\, X\big(\Lambda^n;\mathcal{Q},\mathcal{J};h\big)$ in case $B_\zeta = \lim_{n\to\infty} |I_{n,\zeta}|/n = 0$.

For a pair of partitions $(\mathcal{Q},\mathcal{J})$, assume that $M_\zeta := |Q_\zeta| + \sum_{j\in J_\zeta} |K_j| > 0$. Then, from the above formula, $\lim_{n\to\infty} c(\mathcal{I}_n;\mathcal{Q},\mathcal{J}) = \prod_{\zeta\in\widehat{T}} B_\zeta^{M_\zeta} = 0$. On the other hand, $|X(\Lambda^n;\mathcal{Q},\mathcal{J};h)| \le 1$, whence

$$c(\mathcal{I}_n;\mathcal{Q},\mathcal{J})\, X(\Lambda^n;\mathcal{Q},\mathcal{J};h) \longrightarrow 0.$$

Put $\widehat{T}^+ = \{\zeta \in \widehat{T}\ ;\ B_\zeta > 0\}$. Then the subset $\widehat{T}^+ \subset \widehat{T}$ is at most countably infinite. For $\zeta \in \widehat{T}$, put $I_{\infty,\zeta} = \lim_{n\to\infty} I_{n,\zeta} = \bigcup_{n\ge 1} I_{n,\zeta}$. Then $I_{\infty,\zeta}$ is infinite for $\zeta \in \widehat{T}^+$. We have $\bigsqcup_{\zeta\in\widehat{T}^+} I_{\infty,\zeta} \subset \boldsymbol{N}$, whereas $\bigsqcup_{\zeta\in\widehat{T}} I_{\infty,\zeta} = \boldsymbol{N}$. Even in the case where $T$ is infinite (and compact), there exists at most a countably infinite number of $\zeta$ for which $I_{\infty,\zeta} \ne \emptyset$.

Step 4. By Step 3, we need to consider only such $(\mathcal{Q},\mathcal{J})$ that $M_\zeta = |Q_\zeta| + \sum_{j\in J_\zeta} |K_j| > 0$ only for $\zeta \in \widehat{T}^+$. We fix such a $(\mathcal{Q},\mathcal{J})$. Then, by Step 2, the coefficient $c(\mathcal{I}_n;\mathcal{Q},\mathcal{J})$ converges to a positive constant as $n \to \infty$, and so the point to study is the convergence of the factor $X(\Lambda^n;\mathcal{Q},\mathcal{J};h)$. It is given as

$$\prod_{\zeta\in\widehat{T}^+} \left( \prod_{q\in Q_\zeta} \frac{\chi_\zeta(t_q)}{\dim\zeta} \times \prod_{j\in J_\zeta} \frac{\chi_\zeta\big(P_{\sigma_j}(d_j)\big)}{(\dim\zeta)^{\ell(\sigma_j)}} \times \widetilde{\chi}\big(\lambda^{n,\zeta};(\ell(\sigma_j))_{j\in J_\zeta}\big) \right). \tag{5.14}$$

The moving factor in it is only the normalized character of IUR $\pi(\lambda^{n,\zeta})$ of $\mathfrak{S}_{I_{n,\zeta}}$:

$$\widetilde{\chi}\big(\lambda^{n,\zeta};(\ell(\sigma_j))_{j\in J_\zeta}\big) = \widetilde{\chi}\big(\lambda^{n,\zeta};\textstyle\prod_{j\in J_\zeta}\sigma_j\big). \tag{5.15}$$

This term moves along with a weakly increasing sequence of Young diagrams $\lambda^{n,\zeta}$ together with $I_{n,\zeta} \nearrow I_{\infty,\zeta}$ such that $|I_{n,\zeta}| = |\lambda^{n,\zeta}|$.

Since $I_{\infty,\zeta}$ is infinite, we have an order-preserving bijection

$$\Phi : I_{\infty,\zeta} \longrightarrow \boldsymbol{N} \quad \text{such that} \quad \Phi(I_{n,\zeta}) = \boldsymbol{I}_{N_n} \tag{5.16}$$

with $N_n = |I_{n,\zeta}| \nearrow \infty$, and through $\Phi$ an isomorphism $\varphi$ such that

$$\varphi : \mathfrak{S}_{I_{\infty,\zeta}} \ni \sigma \longrightarrow \varphi(\sigma) = \Phi\circ\sigma\circ\Phi^{-1} \in \mathfrak{S}_{\boldsymbol{N}} = \mathfrak{S}_\infty \tag{5.17}$$

and accordingly a consistent sequence of isomorphisms $\mathfrak{S}_{I_{n,\zeta}} \to \mathfrak{S}_{N_n}$ with $N_n \nearrow \infty$. Correspondingly, the Young diagram $\lambda^{n,\zeta}$ for $\mathfrak{S}_{I_{n,\zeta}}$ is mapped to such a one $\Phi(\lambda^{n,\zeta})$ for $\mathfrak{S}_{N_n}$, and the IUR $\pi(\lambda^{n,\zeta})$ of $\Phi(\lambda^{n,\zeta})$ to an IUR $\pi(\Phi(\lambda^{n,\zeta}))$ of $\mathfrak{S}_{N_n}$.

Thus, we arrive at a situation where, for an increasing sequence of pairs of symmetric groups $\mathfrak{S}_{N_n}$ and their IURs $\pi(\Phi(\lambda^{n,\zeta}))$, we look for a necessary and sufficient condition for the existence of the pointwise limits of the normalized characters $\widetilde{\chi}(\Phi(\lambda^{n,\zeta}))$ of $\pi(\Phi(\lambda^{n,\zeta}))$, and then calculate the limit of irreducible characters.

This problem has been settled in [VK1] and [VK2]. Only a difference from the present situation is that here $\mathfrak{S}_{N_n}$ is weakly increasing up to $\mathfrak{S}_\infty$, but there $\mathfrak{S}_n$ goes up to $\mathfrak{S}_\infty$ one by one as $n \to n+1 \to n+2 \to \cdots$.

### 5.2. Known results on limits of irreducible characters for $\mathfrak{S}_n \nearrow \mathfrak{S}_\infty$

The infinite symmetric group $\mathfrak{S}_\infty$ is an inductive limit of the $n$-th symmetric group $\mathfrak{S}_n$. An equivalence class of IURs of the latter is parametrized by a Young diagram

$$\lambda^{(n)} = (\lambda_1^{(n)}, \lambda_2^{(n)}, \dots, \lambda_n^{(n)}), \quad \lambda_1^{(n)} \ge \lambda_2^{(n)} \ge \cdots \ge \lambda_n^{(n)} \ge 0, \tag{5.18}$$

of size $n = |\lambda^{(n)}|$, which is by definition the number of boxes in $\lambda^{(n)}$. The normalized character corresponding to it is denoted by $\widetilde{\chi}(\lambda^{(n)};\sigma)$ $(\sigma \in \mathfrak{S}_n)$. Denote by $r_k(\lambda^{(n)})$ and $c_k(\lambda^{(n)})$ the length of $k$-th row and $k$-th column of $\lambda^{(n)}$. Then $r_k(\lambda^{(n)}) = \lambda_k^{(n)}$ and

$$\sum_{1\le k\le n} r_k(\lambda^{(n)}) = n, \qquad \sum_{1\le k\le n} c_k(\lambda^{(n)}) = n. \tag{5.19}$$

Consider the case of an increasing sequence of Young diagrams $\lambda^{(n)}$ as $\boldsymbol{I}_n \nearrow \boldsymbol{N}$. In [VK2, Theorem 1], the following is asserted (cf. also Theorems 1 and 2 in [VK1]):

THEOREM 5.4. ([VK1], [VK2]) *The point-wise limit* $\lim_{n\to\infty} \widetilde{\chi}(\lambda^{(n)};\sigma)$ *exists for each* $\sigma \in \mathfrak{S}_\infty$ *if and only if there exist limits of relative lengths of the rows and columns as*

$$\lim_{n\to\infty} \frac{r_k(\lambda^{(n)})}{n} = \alpha_k, \quad \lim_{n\to\infty} \frac{c_k(\lambda^{(n)})}{n} = \beta_k \quad (k = 1, 2, \dots). \tag{5.20}$$

*In this case, the limit is the character of* $\mathfrak{S}_\infty$ *corresponding to the Thoma parameter* $\alpha = (\alpha_k)_{k\ge1}$, $\beta = (\beta_k)_{k\ge1}$.

Let $b_k := r_k(\lambda^{(n)}) - k$, $a_k := c_k(\lambda^{(n)}) - k$ be the lengths of $k$-th row and $k$-th column of $\lambda^{(n)}$ counted off diagonal. Suppose $a_k \ge 0$, $b_k \ge 0$, only for $1 \le k \le r$. Then $a_1 > a_2 > \cdots > a_r \ge 0$, $b_1 > b_2 > \cdots > b_r \ge 0$, $\sum_i a_i + \sum_i b_i = n - r$, and

$$\begin{pmatrix} a_r & a_{r-1} & \cdots & a_1 \\ b_r & b_{r-1} & \cdots & b_1 \end{pmatrix} \tag{5.21}$$

is taken as a parameter for the irreducible character $\chi_{\pi(\lambda^n)}$ of $\mathfrak{S}_n$ and called 'Characteristik' (of rank $r$) by Frobenius in his paper [Frob], where irreducible characters of symmetric groups have been first studied (the numbering of $a_i$ and that of $b_i$ are reversed here from that in [Frob]). Frobenius dimension formula is given as

$$\dim \pi(\lambda^{(n)}) = \frac{n!\,\Delta(a_1, a_2, \ldots, a_r)\,\Delta(b_1, b_2, \ldots, b_r)}{\prod_{1\le i\le r} a_i! \cdot \prod_{1\le j\le r} b_j! \cdot \prod_{1\le i,j\le r}(a_i + b_j + 1)}, \tag{5.22}$$

where $\Delta$ denotes the difference products. The ratios $b_k/n$, $a_k/n$ have the same limits as $\alpha_k$, $\beta_k$ in (5.20).

A basis of the above theorem is an asymptotic evaluation of the normalized irreducible character $\widetilde{\chi}(\lambda^{(n)}; \sigma)$ as $n \to \infty$. For instance, let $\sigma^{(\ell)} \in \mathfrak{S}_n$ be a cycle of length $\ell$. Then we have the following evaluation deduced from a formula due to F. Murnaghan ([Mur], [VK1]):

$$\widetilde{\chi}\big(\lambda^{(n)}; \sigma^{(\ell)}\big) = \sum_k \left(\frac{b_k}{n}\right)^{\ell} + (-1)^{\ell-1} \sum_k \left(\frac{a_k}{n}\right)^{\ell} + O\left(\frac{1}{n}\right). \tag{5.23}$$

### 5.3. Thoma's character formula revisited

We revue here Thoma's character formula for $\mathfrak{S}_\infty$. In his paper [Tho], a character of $\mathfrak{S}_\infty$ is, by definition, an extremal element in $K_1(\mathfrak{S}_\infty)$ or an element in $E(\mathfrak{S}_\infty)$ (cf. **2.1**). As a parameter for a character, he takes $(\alpha, \beta)$ with $\alpha = (\alpha_i)_{i\ge1}$, $\beta = (\beta_i)_{i\ge1}$, a pair of two decreasing sequences of non-negative real numbers such as

$$\begin{gathered} \alpha_1 \ge \alpha_2 \ge \alpha_3 \ge \cdots \ge 0, \quad \beta_1 \ge \beta_2 \ge \beta_3 \ge \cdots \ge 0; \\ \|\alpha\| + \|\beta\| \le 1 \quad \text{with} \ \ \|\alpha\| := \sum_{i\ge1} \alpha_i, \ \ \|\beta\| := \sum_{i\ge1} \beta_i. \end{gathered} \tag{5.24}$$

For a $\sigma \in \mathfrak{S}_\infty$, let $\sigma = \sigma_1\sigma_2\cdots\sigma_m$ be a cycle decomposition, and let $n_\ell(\sigma)$ denotes the number of $\sigma_j$ such that $\ell(\sigma_j) = \ell$ for $\ell \ge 2$. Then Thoma's formula for the character $f_{\alpha,\beta}$ is given as

$$f_{\alpha,\beta}(\sigma) = \prod_{\ell\ge2} \Big( \sum_{i\ge1} \alpha_i^{\ \ell} + (-1)^{\ell-1} \sum_{i\ge1} \beta_i^{\ \ell} \Big)^{n_\ell(\sigma)} \quad (\sigma \in \mathfrak{S}_\infty). \tag{5.25}$$

We rewrite this in the form of our formula for $\mathfrak{S}_\infty(T)$ in Theorem 3.3. Here we take $T$ as the trivial group $T = \{e_T\}$, and $\widehat{T} = \{\mathbf{1}_T\}$, where $\mathbf{1}_T$ is the trivial representation of the trivial group $T$, and put with $\zeta = \mathbf{1}_T$ and $\varepsilon = 0, 1$,

$$\alpha_{\zeta,0,i} := \alpha_i\,, \quad \alpha_{\zeta,1,i} := \beta_i\,, \quad \alpha_{\zeta,\varepsilon} := (\alpha_{\zeta,\varepsilon,i})_{i\in\boldsymbol{N}}\,;$$
$$\mu_\zeta := 1 - \sum_{\varepsilon=0,1} \|\alpha_{\zeta,\varepsilon}\|\,; \quad \chi_\varepsilon(\sigma) := \big(\mathrm{sgn}_{\mathfrak{S}}(\sigma)\big)^{\varepsilon} \ \ (\sigma \in \mathfrak{S}_\infty).$$

For a cycle $\sigma_j$ in the cycle decomposition of $\sigma$, we have $\mathrm{sgn}_{\mathfrak{S}}(\sigma_j) = (-1)^{\ell(\sigma_j)-1}$, and the formula above is rewritten as

$$f_{\alpha,\beta}(\sigma) = \prod_{1\le j\le m} \left( \sum_{\varepsilon=0,1} \sum_{i\in\boldsymbol{N}} (\alpha_{\zeta,\varepsilon,i})^{\ell(\sigma_j)} \cdot \chi_\varepsilon(\sigma_j) \right). \tag{5.26}$$

*Remark* 5.3. In [Hir2]–[Hir3], it is shown that all these characters $f_{\alpha,\beta}$ are obtained as various limits of centralizations of one matrix element $F = \mathrm{Ind}_H^G f_\pi$ of a unitary representation $\rho = \mathrm{Ind}_H^G \pi$, induced from one-dimensional character $\pi$ of a certain subgroup $H$ of wreath product type (cf. [Hir3, §15] in particular).

This fact means also that $f_{\alpha,\beta}$ can be obtained as a limit of a sequence of characters of certain standard induced representations (not irreducible in general) of $\mathfrak{S}_n$ as $n \to \infty$.

### 5.4. Limits of monomial terms of irreducible characters of $\mathfrak{S}_n(T)$

Let us continue the discussions in **5.1**.

STEP 5. We apply Theorem 5.4 to the $\zeta$-factor in (5.15), or more exactly to the increasing sequence of IURs $\pi(\Phi(\lambda^{n,\zeta}))$ of $\mathfrak{S}_{N_n} \nearrow \mathfrak{S}_\infty$, where $N_n = |I_{n,\zeta}| = |\lambda^{n,\zeta}|$ is the size of $\lambda^{n,\zeta}$. Then, coming back to $\pi(\lambda^{n,\zeta})$ of $\mathfrak{S}_{I_{n,\zeta}} \nearrow \mathfrak{S}_{I_{\infty,\zeta}} \cong \mathfrak{S}_\infty$, we get the following.

LEMMA 5.5. *The normalized character $\widetilde{\chi}(\lambda^{n,\zeta})$ of IURs $\pi(\lambda^{n,\zeta})$ of $\mathfrak{S}_{I_{n,\zeta}}$ converge pointwise if and only if there exist the following limits for any $k \geq 1$:*

$$\lim_{n\to\infty} \frac{r_k(\lambda^{n,\zeta})}{|I_{n,\zeta}|} = \alpha'_{\zeta,k}\,, \quad \lim_{n\to\infty} \frac{c_k(\lambda^{n,\zeta})}{|I_{n,\zeta}|} = \beta'_{\zeta,k}\,. \tag{5.27}$$

*In that case, put* $\alpha'_\zeta = (\alpha'_{\zeta,i})_{i\ge1}$, $\beta'_\zeta = (\beta'_{\zeta,i})_{i\ge1}$. *Then, the limit is given through a bijection* $\Phi : I_{\infty,\zeta} \to \boldsymbol{N}$ *in* (5.16) *and* $\varphi : \mathfrak{S}_{I_{n,\zeta}} \to \mathfrak{S}_\infty$ *in* (5.17) *as*

$$\lim_{n\to\infty} \widetilde{\chi}(\lambda^{n,\zeta};\tau) = f_{\alpha'_\zeta,\beta'_\zeta}\big(\varphi(\tau)\big) \quad (\tau \in \mathfrak{S}_{I_{n,\zeta}}), \tag{5.28}$$

*where* $\varphi(\tau) = \Phi\circ\tau\circ\Phi^{-1}$.

Step 6. We assume Condition I holds, that is, that for each $\zeta \in \widehat{T}$ there exists $\lim_{n\to\infty} |I_{n,\zeta}|/n = B_\zeta \ge 0$. Put $\widehat{T}^+ = \{\zeta \in \widehat{T}\ ;\ B_\zeta > 0\}$. For $\zeta \in \widehat{T}^+$, we assume that the limits (5.27) exist for $1 \le k < \infty$. Introduce a parameter $\varepsilon = 0, 1$ corresponding to a one-dimensional character $(\mathrm{sgn}_{\mathfrak{S}})^\varepsilon$, and put

$$\begin{aligned} &\alpha_{\zeta,0,k} := \lim_{n\to\infty} \frac{r_k(\lambda^{n,\zeta})}{n}, \quad \alpha_{\zeta,1,k} := \lim_{n\to\infty} \frac{c_k(\lambda^{n,\zeta})}{n}, \\ &\mu_\zeta := B_\zeta - \sum_{\varepsilon=0,1} \|\alpha_{\zeta,\varepsilon}\| \quad \text{with} \;\; \alpha_{\zeta,\varepsilon} = (\alpha_{\zeta,\varepsilon,i})_{i\ge1}. \end{aligned} \tag{5.29}$$

Then, $\alpha_{\zeta,0,k} = B_\zeta\,\alpha'_{\zeta,k}$, $\alpha_{\zeta,1,k} = B_\zeta\,\beta'_{\zeta,k}$ for $\zeta \in \widehat{T}^+$, and $\alpha_{\zeta,0,k} = 0$, $\alpha_{\zeta,1,k} = 0$, $\mu_\zeta = 0$ for $\zeta \notin \widehat{T}^+$. Since $B = \sum_{\zeta\in\widehat{T}} B_\zeta \le 1$, we have

$$\begin{aligned} &\mu_\zeta \ge 0, \quad \|\alpha_{\zeta,0}\| + \|\alpha_{\zeta,1}\| + \mu_\zeta = B_\zeta\,, \\ &\sum_{\zeta\in\widehat{T}} \sum_{\varepsilon=0,1} \|\alpha_{\zeta,\varepsilon}\| + \|\mu\| \le 1 \\ &\text{with} \;\; \mu = (\mu_\zeta)_{\zeta\in\widehat{T}}\,, \quad \|\mu\| = \sum_{\zeta\in\widehat{T}} \mu_\zeta\,. \end{aligned}$$

Let $\tau = \prod_{j\in J_\zeta} \sigma_j$ be a cycle decomposition of $\tau$, then $\varphi(\tau) = \prod_{j\in J_\zeta} \varphi(\sigma_j)$ is a cycle decomposition of $\varphi(\tau)$, and $\ell(\varphi(\sigma_j)) = \ell(\sigma_j)$. Hence we see from (5.26) that, with $\alpha'_{\zeta,0,i} = \alpha'_{\zeta,i}$, $\alpha'_{\zeta,1,i} = \beta'_{\zeta,i}$,

$$f_{\alpha'_\zeta,\beta'_\zeta}\big(\varphi(\tau)\big) = \prod_{j\in J_\zeta} \left( \sum_{\varepsilon=0,1} \sum_{i\in\boldsymbol{N}} \chi_\varepsilon(\sigma_j)(\alpha'_{\zeta,\varepsilon,i})^{\ell(\sigma_j)} \right). \tag{5.30}$$

Thus, taking into account (5.13) and (5.14), we get the following convergence lemma.

LEMMA 5.6. *Let $\mathcal{I}_n = (I_{n,\zeta})_{\zeta\in\widehat{T}}$ be a partition of $\boldsymbol{I}_n$ increasing along with $n \to \infty$ in such a way that $I_{n,\zeta} \subseteq I_{n+1,\zeta} \subseteq \cdots$. Assume* CONDITION I *holds, and put $\widehat{T}^+ = \{\zeta \in \widehat{T}\,;\, B_\zeta > 0\}$. Moreover assume that the limits in* (5.27) *exist for $\zeta \in \widehat{T}^+$.*

*Take an $h = (d,\sigma) \in H_{n_0} \subset G_{n_0} = \mathfrak{S}_{n_0}(T)$. Then $h \in H_n \subset G_n$ for $n \geq n_0$. Let $h = \xi_{q1}\xi_{q2}\cdots\xi_{q_r}g_1g_2\cdots g_m$ with $\xi_q = \big(t_q,(q)\big)$, $g_j = (d_j,\sigma_j)$ be a standard decomposition of $h$ as an element of $G = \mathfrak{S}_\infty(T)$. Then $\sigma = \sigma_1\sigma_2\cdots\sigma_m$ is a cycle decomposition of $\sigma$. Put $Q = \{q_1,q_2,\ldots,q_r\} =$ $\mathrm{supp}(d)\setminus\mathrm{supp}(\sigma)$ and $J = \{1,2,\ldots,m\}$. Take a pair $(\mathcal{Q},\mathcal{J})$ of partitions of $Q$ and $J$. Then, for the term corresponding to $(\mathcal{Q},\mathcal{J})$ in* (4.24) *of irreducible characters of $G_n$, the limit is given as follows.*

(i) *The limit of coefficients is given as*

$$\lim_{n\to\infty} c(\mathcal{I}_n;\mathcal{Q},\mathcal{J}) = c(\mathcal{Q},\mathcal{J}) := \prod_{\zeta\in\widehat{T}}\Big(B_\zeta^{|Q_\zeta|}\cdot\prod_{j\in J_\zeta} B_\zeta^{\ell(\sigma_j)}\Big),$$

(ii) *If $Q_\zeta \neq \emptyset$ or $J_\zeta \neq \emptyset$ for some $\zeta \notin \widehat{T}^+$, then*

$$\lim_{n\to\infty} c(\mathcal{I}_n;\mathcal{Q},\mathcal{J})\, X(\Lambda^n;\mathcal{Q},\mathcal{J};h) = 0,$$

*otherwise, $\bigsqcup_{\zeta\in\widehat{T}^+} Q_\zeta = Q$, $\bigsqcup_{\zeta\in\widehat{T}^+} J_\zeta = J$, and*

$$\lim_{n\to\infty} X(\Lambda^n;\mathcal{Q},\mathcal{J};h) = \prod_{\zeta\in\widehat{T}^+}\left(\prod_{q\in Q_\zeta}\frac{\chi_\zeta(t_q)}{\dim\zeta}\times\prod_{j\in J_\zeta}\frac{\chi_\zeta\big(P_{\sigma_j}(d_j)\big)}{(\dim\zeta)^{\ell(\sigma_j)}}\sum_{\varepsilon=0,1}\chi_\varepsilon(\sigma_j)\sum_{i\in\boldsymbol{N}}(\alpha'_{\zeta,\varepsilon,i})^{\ell(\sigma_j)}\right);$$

$$\lim_{n\to\infty} c(\mathcal{I}_n;\mathcal{Q},\mathcal{J})\, X(\Lambda^n;\mathcal{Q},\mathcal{J};h) = \prod_{\zeta\in\widehat{T}^+}\left\{\prod_{q\in Q_\zeta}\left(\big(\|\alpha_{\zeta,0}\|+\|\alpha_{\zeta,1}\|+\mu_\zeta\big)\frac{\chi_\zeta(t_q)}{\dim\zeta}\right)\times\prod_{j\in J_\zeta}\left(\sum_{\varepsilon=0,1}\sum_{i\in\boldsymbol{N}}(\alpha_{\zeta,\varepsilon,i})^{\ell(\sigma_j)}\,\chi_\varepsilon(\sigma_j)\cdot\frac{\chi_\zeta\big(P_{\sigma_j}(d_j)\big)}{(\dim\zeta)^{\ell(\sigma_j)}}\right)\right\}.$$

## §6. Limits of irreducible characters of $\mathfrak{S}_n(T)$

By Lemma 5.6 above, we get the limit of each monomial term $c(\mathcal{I}_n;\mathcal{Q},\mathcal{J})$ $X(\Lambda^n;\mathcal{Q},\mathcal{J};h)$ of $\widetilde{\chi}_{\Pi_n}(h)$ of $G_n = \mathfrak{S}_n(T)$ in (4.24) as $n \to \infty$, and then summing up these limits, we obtain the limit function $\lim_{n\to\infty} \widetilde{\chi}_{\Pi_n}(h)$ on $G = \mathfrak{S}_\infty(T)$ as follows.

Let $\mathcal{I}_n = (I_{n,\zeta})_{\zeta\in\widehat{T}}$ be a partition of $\boldsymbol{I}_n$ and $\Lambda^n = (\lambda^{n,\zeta})_{\zeta\in\widehat{T}}$ a set of Young diagrams such that $\lambda^{n,\zeta}$ has the size $|I_{n,\zeta}|$ and determines IURs $\pi(\lambda^{n,\zeta})$ of the subgroup $\mathfrak{S}_{I_{n,\zeta}}$ of $\mathfrak{S}_n$. Take a subgroup $H_n$ of $G_n$ and its IUR $\pi_n$ given as

$$(6.1)\qquad H_n = D_n \rtimes S_{[\eta_n]}, \quad D_n = D_n(T) = D_{\boldsymbol{I}_n}(T), \quad S_{[\eta_n]} = \prod_{\zeta\in\widehat{T}'} \mathfrak{S}_{I_{n,\zeta}},$$

$$(6.2)\qquad \pi_n = \eta_n \boxdot \xi_n, \quad \eta_n = \boxtimes_{\zeta\in\widehat{T}'} \zeta^{I_{n,\zeta}}, \quad \xi_n = \boxtimes_{\zeta\in\widehat{T}'} \pi(\lambda^{n,\zeta}),$$

and consider a sequence of IURs $\Pi_n = \mathrm{Ind}_{H_n}^{G_n} \pi_n$ of $G_n$ $(n = 3, 4, 5, \dots)$.

Assume that the pair $(\mathcal{I}_n, \Lambda^n)$ increases along with $n \to \infty$ in such a way that $I_{n,\zeta} \subseteq I_{n+1,\zeta} \subseteq \cdots$ and $\lambda^{n,\zeta} \subseteq \lambda^{n+1,\zeta} \subseteq \cdots$. Moreover assume that the sequence $(\mathcal{I}_n, \Lambda^n)$, $n \to \infty$, satisfies Condition I on $\mathcal{I}_n$, and Condition I$\Lambda$ on $(\mathcal{I}_n, \Lambda^n)$ given below:

(Condition I) *the following limits exist*: $B_\zeta := \lim_{n\to\infty} \frac{|I_{n,\zeta}|}{n}$ $(\zeta \in \widehat{T})$.

(Condition I$\Lambda$) *For each* $\zeta \in \widehat{T}^+ := \{\zeta \in \widehat{T}\ ;\ B_\zeta > 0\}$, *there exist limits*

$$(6.3)\qquad \lim_{n\to\infty} \frac{r_k(\lambda^{n,\zeta})}{|I_{n,\zeta}|} = \alpha'_{\zeta,k}\,, \quad \lim_{n\to\infty} \frac{c_k(\lambda^{n,\zeta})}{|I_{n,\zeta}|} = \beta'_{\zeta,k} \quad (1 \le k < \infty).$$

Put, for $\zeta \in \widehat{T}$,

$$(6.4)\qquad \begin{cases} \alpha_{\zeta,0,k} := \lim_{n\to\infty} \frac{r_k(\lambda^{n,\zeta})}{n}, \quad \alpha_{\zeta,1,k} := \lim_{n\to\infty} \frac{c_k(\lambda^{n,\zeta})}{n}, \\ \alpha_{\zeta,\varepsilon} = (\alpha_{\zeta,\varepsilon,k})_{k\ge 1} \quad \text{for} \quad \varepsilon = 0, 1; \\ \mu_\zeta := B_\zeta - \sum_{\varepsilon=0,1} \|\alpha_{\zeta,\varepsilon}\|, \quad \mu := (\mu_\zeta)_{\zeta\in\widehat{T}}. \end{cases}$$

Then, for $\zeta \in \widehat{T}^+$, $\alpha_{\zeta,0,k} = B_\zeta\, \alpha'_{\zeta,k}$, $\alpha_{\zeta,1,k} = B_\zeta\, \beta'_{\zeta,k}$, and for $\zeta \notin \widehat{T}^+$, $\alpha_{\zeta,0,k} = 0$, $\alpha_{\zeta,1,k} = 0$, and $\mu_\zeta = 0$, with $B_\zeta = 0$.

With these data, we define a parameter $A$ as $A = \big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\ ;\ \mu\big)$, then,

$$(6.5)\qquad \sum_{\zeta\in\widehat{T}} \sum_{\varepsilon\in\{0,1\}} \|\alpha_{\zeta,\varepsilon}\| + \|\mu\| = \sum_{\zeta\in\widehat{T}} B_\zeta \le 1.$$

Corresponding to this parameter $A$, we define a function $F^A$ on $G$. Take an element $g = (d, \sigma) \in G$. Let $g = \xi_{q1}\xi_{q2}\cdots\xi_{q_r}g_1g_2\cdots g_m$ with $\xi_q = \big(t_q, (q)\big)$, $g_j = (d_j, \sigma_j)$, be a standard decomposition of $g$. Put $Q = \mathrm{supp}(d) \setminus \mathrm{supp}(\sigma) = \{q_1, q_2, \ldots, q_r\}$ and $J = \{1, 2, \ldots, m\}$, then

$$\text{(6.6)}\quad F^A(g) := \prod_{q\in Q}\left\{\sum_{\zeta\in\widehat{T}}\left(\sum_{\varepsilon\in\{0,1\}}\|\alpha_{\zeta,\varepsilon}\| + \mu_\zeta\right)\frac{\chi_\zeta(t_q)}{\dim\zeta}\right\}$$
$$\times\prod_{j\in J}\left\{\sum_{\zeta\in\widehat{T}}\left(\sum_{\varepsilon\in\{0,1\}}\sum_{i\in\boldsymbol{N}}(\alpha_{\zeta,\varepsilon,i})^{\ell(\sigma_j)}\chi_\varepsilon(\sigma_j)\cdot\frac{\chi_\zeta\big(P_{\sigma_j}(d_j)\big)}{(\dim\zeta)^{\ell(\sigma_j)}}\right)\right\},$$

where $\chi_\varepsilon(\sigma_j) = \mathrm{sgn}_{\mathfrak{S}}(\sigma_j)^\varepsilon = (-1)^{\varepsilon(\ell(\sigma_j)-1)}$, and for $\sigma_j = (i_1\ i_2\ \cdots\ i_{\ell_j})$ with $\ell_j = \ell(\sigma_j)$ and $d_j = (t_i)_{i\in K_j}$ with $K_j := \mathrm{supp}(\sigma_j)$, $P_{\sigma_j}(d_j) := \big[t'_{\ell_j}t'_{\ell_j-1}\cdots t'_2t'_1\big] \in T/\!\sim$ with $t'_k = t_{i_k}$.

THEOREM 6.1. *Assume that the sequence $(\mathcal{I}_n, \Lambda^n)$, $n \to \infty$, satisfies* CONDITION I *on $\mathcal{I}_n$ and* CONDITION I$\Lambda$ *on $\Lambda^n$.*

(i) *In the case where $T$ is finite, we have*

$$\text{(6.7)}\qquad \lim_{n\to\infty}\widetilde{\chi}_{\Pi_n}(g) = F^A(g)\quad (g \in G),$$

*and the limit function $F^A$ is equal to the character $f_A$ in Theorem* 2.2 *with a parameter $A = \big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\ ;\ \mu\big)$ given above, which satisfies the equality condition* (2.9)*, that is, $\sum_{\zeta\in\widehat{T}}\sum_{\varepsilon\in\{0,1\}}\|\alpha_{\zeta,\varepsilon}\| + \|\mu\| = 1$.*

*The set of limits $\lim_{n\to\infty}\widetilde{\chi}_{\Pi_n}$ of normalized characters of IURs of $G_n$ is exactly equal to the set of all characters of factor representations of finite type of $G$.*

(ii) *In the case where $T$ is infinite, the function $F^A$ is continuous if and only if the equality condition* (2.9) *holds for $A$, or if and only if the following* CONDITION E *holds:*

$$\text{(CONDITION E)}\qquad \sum_{\zeta\in\widehat{T}}B_\zeta = 1.$$

*In that case, $\lim_{n\to\infty}\widetilde{\chi}_{\Pi_n}(g)$ is given by $F^A(g)$ as in* (6.7)*, and the limit function $F^A$ equals the character $f_A$ in Theorem* 2.2*, and the convergence is uniform on every compact subset of $G$.*

*The set of limit functions $\lim_{n\to\infty}\widetilde{\chi}_{\Pi_n}$ of normalized characters of IURs of $G_n$ which are continuous on $G$ is exactly equal to the set of all characters of factor representations of finite type of $G$.*

*Proof.* Let us prove the convergence $\widetilde{\chi}_{\Pi_n}(g) \to F^A(g)$ as $n \to \infty$. Fix $n_0$ and take $g = (d, \sigma) \in G_{n_0} \subset G$, then $g \in G_n$ for $n \geq n_0$. Let its standard decomposition be $g = \xi_{q_1}\xi_{q_2} \cdots \xi_{q_r} g_1 g_2 \cdots g_m$ as in (4.11), and correspondingly put $Q = \{q_1, q_2, \ldots, q_r\}$ and $J = \{1, 2, \ldots, m\}$ as in (4.12). By Theorem 4.5,

$$\widetilde{\chi}_{\Pi_n}(g) = \sum_{(\mathcal{Q},\mathcal{J})} c(\mathcal{I}_n; \mathcal{Q}, \mathcal{J})\, X\big(\Lambda^n; \mathcal{Q}, \mathcal{J}; g\big), \tag{6.8}$$

where, for a pair $(\mathcal{Q}, \mathcal{J})$ of partitions $\mathcal{Q} = (Q_\zeta)_{\zeta \in \widehat{T}}$ of $Q$ and $\mathcal{J} = (J_\zeta)_{\zeta \in \widehat{T}}$ of $J$,

$$c(\mathcal{I}_n; \mathcal{Q}, \mathcal{J}) = \prod_{\zeta \in \widehat{T}} \frac{|I_{n,\zeta}|\big(|I_{n,\zeta}| - 1\big) \cdots \big(|I_{n,\zeta}| - |Q_\zeta| - \sum_{j \in J_\zeta} |K_j| + 1\big)}{n(n-1)(n-2) \cdots \big(n - |Q| - |\operatorname{supp}(\sigma)| + 1\big)},$$

$$X\big(\Lambda^n; \mathcal{Q}, \mathcal{J}; g\big) = \prod_{\zeta \in \widehat{T}} \left( \prod_{q \in Q_\zeta} \frac{\chi_\zeta(t_q)}{\dim \zeta} \times \prod_{j \in J_\zeta} \frac{\chi_\zeta\big(P_{\sigma_j}(d_j)\big)}{(\dim \zeta)^{\ell(\sigma_j)}} \times \widetilde{\chi}\big(\lambda^{n,\zeta}; (\ell(\sigma_j))_{j \in J_\zeta}\big) \right).$$

The target function $F^A(g)$ is expressed in a similar way as

$$F^A(g) = \sum_{(\mathcal{Q},\mathcal{J})} c(\mathcal{Q}, \mathcal{J})\, X(\mathcal{Q}, \mathcal{J}; g), \tag{6.9}$$

where $(\mathcal{Q}, \mathcal{J})$ runs over pairs of partitions $\mathcal{Q} = (Q_\zeta)_{\zeta \in \widehat{T}^+}$ of $Q$ and $\mathcal{J} = (J_\zeta)_{\zeta \in \widehat{T}^+}$ of $J$, and

$$c(\mathcal{Q}, \mathcal{J}) = \lim_{n \to \infty} c(\mathcal{I}_n; \mathcal{Q}, \mathcal{J}) = \prod_{\zeta \in \widehat{T}^+} \Big( B_\zeta^{|Q_\zeta|} \cdot \prod_{j \in J_\zeta} B_\zeta^{\ell(\sigma_j)} \Big), \tag{6.10}$$

$$X(\mathcal{Q}, \mathcal{J}; g) := \lim_{n \to \infty} X\big(\Lambda^n; \mathcal{Q}, \mathcal{J}; g\big)$$

$$= \prod_{\zeta \in \widehat{T}^+} \left( \prod_{q \in Q_\zeta} \frac{\chi_\zeta(t_q)}{\dim \zeta} \times \prod_{j \in J_\zeta} \frac{\chi_\zeta\big(P_{\sigma_j}(d_j)\big)}{(\dim \zeta)^{\ell(\sigma_j)}} \times \left( \sum_{\varepsilon = 0,1} \chi_\varepsilon(\sigma_j) \sum_{i \in \boldsymbol{N}} (\alpha'_{\zeta,\varepsilon,i})^{\ell(\sigma_j)} \right) \right).$$

Since $B_\zeta = 0$ for $\zeta \in \widehat{T} \setminus \widehat{T}^+$, we can extend the summation in (6.9) over pairs $(\mathcal{Q}, \mathcal{J})$ of partitions $\mathcal{Q} = (Q_\zeta)_{\zeta\in\widehat{T}}$ of $Q$ and $\mathcal{J} = (J_\zeta)_{\zeta\in\widehat{T}}$ of $J$, just as in (6.8), by extending the definitions of $c(\mathcal{Q}, \mathcal{J})$ and $X(\mathcal{Q}, \mathcal{J}; g)$ naturally, where for the latter the parameters $(\alpha'_{\zeta,\varepsilon,i})$ can be taken arbitrary since the term $c(\mathcal{Q}, \mathcal{J})\, X(\mathcal{Q}, \mathcal{J}; g)$ are understood naturally as equals 0.

So doing, we evaluate the difference of two corresponding terms of the same kind by separating into two factors as

$$\begin{aligned}&\left|c(\mathcal{I}_n; \mathcal{Q}, \mathcal{J})\, X\big(\Lambda^n; \mathcal{Q}, \mathcal{J}; g\big) - c(\mathcal{Q}, \mathcal{J})\, X(\mathcal{Q}, \mathcal{J}; g)\right| \\ &\quad \le I(n; \mathcal{Q}, \mathcal{J}) + II(n; \mathcal{Q}, \mathcal{J}),\end{aligned}$$

where

$$\begin{aligned}I(n; \mathcal{Q}, \mathcal{J}) &:= \left|c(\mathcal{I}_n; \mathcal{Q}, \mathcal{J}) - c(\mathcal{Q}, \mathcal{J})\right|, \\ II(n; \mathcal{Q}, \mathcal{J}) &:= c(\mathcal{Q}, \mathcal{J}) \\ \times &\left| \prod_{\zeta\in\widehat{T}^+} \prod_{j\in J_\zeta} \widetilde{\chi}\big(\lambda^{n,\zeta}; (\ell(\sigma_j))_{j\in J_\zeta}\big) - \prod_{\zeta\in\widehat{T}^+} \prod_{j\in J_\zeta} \sum_{\varepsilon=0,1} \chi_\varepsilon(\sigma_j) \sum_{i\in\boldsymbol{N}} (\alpha'_{\zeta,\varepsilon,i})^{\ell(\sigma_j)} \right|.\end{aligned}$$

Here we put $II(n; \mathcal{Q}, \mathcal{J}) = 0$ in accordance with $c(\mathcal{Q}, \mathcal{J}) = 0$, when some of $Q_\zeta$ and $J_\zeta$ are not empty for a $\zeta \in \widehat{T} \setminus \widehat{T}^+$, and we have used the evaluation of monomial terms given as

$$\left|X\big(\Lambda^n; \mathcal{Q}, \mathcal{J}; g\big)\right| \le 1, \quad \left| \prod_{\zeta\in\widehat{T}} \left( \prod_{q\in Q_\zeta} \frac{\chi_\zeta(t_q)}{\dim\zeta} \times \prod_{j\in J_\zeta} \frac{\chi_\zeta\big(P_{\sigma_j}(d_j)\big)}{(\dim\zeta)^{\ell(\sigma_j)}} \right) \right| \le 1.$$

The difference between $\widetilde{\chi}_{\Pi_n}(g)$ and $F^A(g)$ is evaluated as follows. Let $\mathcal{F} = \mathcal{F}^Q \times \mathcal{F}^J$ be a finite subset of pairs $(\mathcal{Q}, \mathcal{J})$ of partitions of $Q$ and $J$, with finite subsets $\mathcal{F}^Q$ of $\mathcal{Q}$ and $\mathcal{F}^J$ of $\mathcal{J}$, then

$$\left|\widetilde{\chi}_{\Pi_n}(g) - F^A(g)\right| \le \sum_{(\mathcal{Q},\mathcal{J})} I(n; \mathcal{Q}, \mathcal{J}) + \sum_{(\mathcal{Q},\mathcal{J})} II(n; \mathcal{Q}, \mathcal{J});$$

where

$$\begin{aligned}\sum_{(\mathcal{Q},\mathcal{J})} I(n; \mathcal{Q}, \mathcal{J}) &\le J_1^0(n; \mathcal{F}) + J_2^0(n; \mathcal{F}) + J_3^0(\mathcal{F}), \\ J_1^0(n; \mathcal{F}) &:= \sum_{(\mathcal{Q},\mathcal{J})\in\mathcal{F}} \left|c(\mathcal{I}_n; \mathcal{Q}, \mathcal{J}) - c(\mathcal{Q}, \mathcal{J})\right|,\end{aligned}$$

$$J_2^0(n;\mathcal{F}) := \sum_{(\mathcal{Q},\mathcal{J})\notin\mathcal{F}} c(\mathcal{I}_n;\mathcal{Q},\mathcal{J})$$

$$J_3^0(\mathcal{F}) := \sum_{(\mathcal{Q},\mathcal{J})\notin\mathcal{F}} c(\mathcal{Q},\mathcal{J}) = \sum_{(\mathcal{Q},\mathcal{J})\notin\mathcal{F}} \prod_{\zeta\in\widehat{T}} \Big( B_\zeta^{|Q_\zeta|} \prod_{j\in J_\zeta} B_\zeta^{\ell(\sigma_j)} \Big),$$

and, since

$$\sum_{\mathcal{Q}} c(\mathcal{Q},\mathcal{J}) = c(\mathcal{J}) \quad \text{with} \;\; c(\mathcal{J}) := \Big(\sum_{\zeta\in\widehat{T}} B_\zeta\Big)^{|Q|} \cdot \prod_{\zeta\in\widehat{T}} \prod_{j\in J_\zeta} B_\zeta^{\ell(\sigma_j)},$$

$$\sum_{(\mathcal{Q},\mathcal{J})} II(n;\mathcal{Q},\mathcal{J}) \le J_4^0(n;\mathcal{F}^J) + J_5^0(\mathcal{F}^J),$$

$$J_4^0(n;\mathcal{F}^J) := \sum_{\mathcal{J}\in\mathcal{F}^J} \Bigg\{ c(\mathcal{J}) \cdot \Bigg| \prod_{\zeta\in\widehat{T}+} \prod_{j\in J_\zeta} \widetilde{\chi}\big(\lambda^{n,\zeta}; (\ell(\sigma_j))_{j\in J_\zeta}\big) - \prod_{\zeta\in\widehat{T}+} \prod_{j\in J_\zeta} \sum_{\varepsilon=0,1} \chi_\varepsilon(\sigma_j) \sum_{i\in\boldsymbol{N}} (\alpha'_{\zeta,\varepsilon,i})^{\ell(\sigma_j)} \Bigg| \Bigg\},$$

$$J_5^0(\mathcal{F}^J) := \sum_{\mathcal{J}\notin\mathcal{F}^J} 2\cdot c(\mathcal{J}) = 2\cdot \Big(\sum_{\zeta\in\widehat{T}} B_\zeta\Big)^{|Q|} \cdot \sum_{\mathcal{J}\notin\mathcal{F}^J} \prod_{\zeta\in\widehat{T}} \prod_{j\in J_\zeta} B_\zeta^{\ell(\sigma_j)}.$$

Note that

$$\sum_{(\mathcal{Q},\mathcal{J})} c(\mathcal{I}_n;\mathcal{Q},\mathcal{J}) = 1 \quad \text{(from (4.22))},$$

$$\sum_{(\mathcal{Q},\mathcal{J})} c(\mathcal{Q},\mathcal{J}) = \sum_{(\mathcal{Q},\mathcal{J})} \prod_{\zeta\in\widehat{T}} B_\zeta^{|Q_\zeta| + \sum_{j\in J_\zeta} \ell(\sigma_j)} = \Big(\sum_{\zeta\in\widehat{T}} B_\zeta\Big)^{|Q|+|\mathrm{supp}(\sigma)|}.$$

Now assume that Condition E holds, or $\sum_{\zeta\in\widehat{T}} B_\zeta = 1$. Then, in the last equality, we have $\sum_{(\mathcal{Q},\mathcal{J})} c(\mathcal{Q},\mathcal{J}) = 1$, and so the evaluation similar to (5.10) is possible and gives us $J_2^0(n;\mathcal{F}) \le J_1^0(n;\mathcal{F}) + J_3^0(\mathcal{F})$. Hence

$$\Big| \widetilde{\chi}_{\Pi_n}(g)) - F^A(g) \Big| \le 2J_1^0(n;\mathcal{F}) + 2J_3^0(\mathcal{F}) + J_4^0(n;\mathcal{F}^J) + J_5^0(\mathcal{F}^J).$$

Now, for a given small $\varepsilon > 0$, take a finite set $\mathcal{F} = \mathcal{F}^Q \times \mathcal{F}^J$ sufficiently large so that $J_3^0(\mathcal{F}) < \varepsilon$ and $J_5^0(\mathcal{F}^J) < \varepsilon$. Then, choosing $n$ sufficiently large, we have $J_1^0(n,\mathcal{F}) < \varepsilon$ by (6.10), and $J_4^0(n;\mathcal{F}^J) < \varepsilon$ by Lemma 5.6. So, we obtain

$$\big| \widetilde{\chi}_{\Pi_n}(g) - F^A(g) \big| \le 6\varepsilon.$$

This proves that $\widetilde{\chi}_{\Pi_n}(g)$ converges to $F^A(g)$ uniformly on $G_{K^0} = D_{K^0} \rtimes \mathfrak{S}_{K^0}$ with a fixed finite set $K^0 \supset \mathrm{supp}(g) = Q \sqcup K$, $K = \mathrm{supp}(\sigma) = \bigsqcup_{j \in J} \mathrm{supp}(\sigma_j)$.

Thus the convergence $\widetilde{\chi}_{\Pi_n}(g) \to F^A(g)$ is uniform on each $G_{n_0}$. Since any compact subset of $G$ in the inductive limit topology $\tau_{ind}$ is contained in some $G_n$, the convergence is uniform on every compact subsets of $G$.

The rests of the theorem are easy to prove. □

*Remark* 6.1. In the case where CONDITION I holds but CONDITION E does not, the function $F^A$ is nowhere continuous. However it is continuous on every $D'_Q D_K \times \{\sigma\}$ with $K = \mathrm{supp}(\sigma)$. Therefore $F^A$ is Borel measurable on $G$ in the topology $\tau_{ind}$.

PROBLEM 6.1. We suspect that $F^A$ is a weak limit in a certain sense, and also a pointwise limit "almost everywhere", of normalized irreducible characters $\widetilde{\chi}_{\Pi_n}$ of $\mathfrak{S}_n$. At least, the restriction $F_A|_{G_k}$ is the weak limit of $\widetilde{\chi}_{\Pi_n}|_{G_k}$ in each $L^2(G_k)$.

EXAMPLE 6.1. To give examples of discontinuous limits $\lim_{n\to\infty} \widetilde{\chi}_{\Pi_n}(g)$ on $G$, we take $G = \mathfrak{S}_\infty(T)$ with $T = \mathbf{T}^1$ one-dimensional torus. The dual of $T$ is given as $\widehat{T} = \{\zeta_k \,;\, k \in \boldsymbol{Z}\}$, where $\zeta_k(t) = t^k$ $(t \in T)$. For a fixed $n_0$, we take $h = (d, \sigma)$ with $d = (t_1, t_2, \ldots, t_{n_0}, e_T, e_T, \ldots)$. For $n \geq n_0$, take a partition $\boldsymbol{I}_n = \bigsqcup_{\zeta \in \widehat{T}} I_{n,\zeta}$ given as

$$
\begin{aligned}
I_{n,\zeta_s} &= \emptyset \quad (1 \leq s < n_0), \\
I_{n,\zeta_{n_0}} &= \boldsymbol{I}_{n_0} \sqcup \{n_0 + 2p + 1 \,;\, 0 \leq p \leq [(n - n_0 - 1)/2]\}, \\
I_{n,\zeta_{n_0+u}} &= \begin{cases} \{n_0 + 2u\} & \text{if } n_0 + 2u \leq n \\ \emptyset & \text{if } n_0 + 2u > n. \end{cases}
\end{aligned}
$$

Then, $B_{\zeta_{n_0}} = 1/2$ and $B_\zeta = 0$ for $\zeta \neq \zeta_{n_0}$, and $\sum_{\zeta \in \widehat{T}} B_\zeta = 1/2$.

The normalized character $\widetilde{\chi}_{\Pi_n}$ of irreducible representation $\Pi = \mathrm{Ind}_{H_n}^{G_n} \pi$ is given for $h = (d, \mathbf{1})$ with $d \in D'_Q$ by (5.2). In particular, if $h = (t_1, e_T, e_T, \ldots)$, $t_1 \neq e_T$, we have $Q = \{1\}$, $J = \emptyset$, and since $\dim \zeta = 1$,

$$
\widetilde{\chi}_{\Pi_n}(h) = \frac{|I_{n,\zeta_{n_0}}|}{n} \cdot \chi_{\zeta_{n_0}}(t_1) + \sum_{1 \leq u \leq (n-n_0)/2} \frac{1}{n} \cdot \chi_{\zeta_{n_0+u}}(t_1). \tag{6.11}
$$

Hence we get a discontinuous pointwise limit function as

$$\lim_{n\to\infty} \widetilde{\chi}_{\Pi_n}(h) = \frac{1}{2}\cdot\chi_{\zeta_{n_0}}(t_1) \quad \text{for } t_1 \neq e_T\,;$$
$$\lim_{n\to\infty} \widetilde{\chi}_{\Pi_n}(e) = 1 \quad \text{for the identity element } e \in G.$$

Similarly we have the restriction $F^A|_{D'_Q}$ as pointwise limit of $\widetilde{\chi}_{\Pi_n}\big((d,\mathbf{1})\big)$, $d \in D'_Q$.

Another simpler example is given as follows. Fix an integer $p > 0$, and $I_{n,\zeta_k} = \{k\}$ if $k = sp$, $1 \le s \le n$, and $I_{n,\zeta_k} = \emptyset$ otherwise. Then, for $h = (d,\mathbf{1})$ with $t_1 = t \neq e_T$,

$$\widetilde{\chi}_{\Pi_n}(h) = \sum_{1\le s\le n} \frac{1}{n}\,\zeta_{sp}(t) = \frac{1}{n} \times \begin{cases} \dfrac{t^{np+p}-t^p}{t^p-1} & \text{if } \ t^p \neq 1, \\ n & \text{if } \ t^p = 1. \end{cases}$$

Hence

$$\lim_{n\to\infty} \widetilde{\chi}_{\Pi_n}(h) = \begin{cases} 0 & \text{if } \ t^p \neq 1, \\ 1 & \text{if } \ t^p = 1. \end{cases}$$

One more example: we put

$$I_{n,\zeta_k} = \{2k+1\} \quad \text{for } \ 0 \le k \le (n-1)/2\,;$$
$$I_{n,\zeta_{-k}} = \{2k\} \quad \text{for } \ 1 \le k \le n/2.$$

Then, for $h = (d,\mathbf{1})$ with $t_1 = t \neq e_T$,

$$\widetilde{\chi}_{\Pi_n}(h) = \sum_{-n/2\le k\le (n-1)/2} \frac{1}{n}\,\zeta_k(t) = \frac{1}{n} \times \begin{cases} \dfrac{t^p - t^{-p}}{t-1} & \text{if } \ n = 2p\,, \\[2ex] \dfrac{t^{p+1} - t^{-p}}{t-1} & \text{if } \ n = 2p+1\,. \end{cases}$$

Put $t = e^{i\theta}$ $(0 \le \theta \le 2\pi)$ and $f(t) = \widetilde{\chi}_{\Pi_n}(h)$, then for $n = 2p+\epsilon$ $(\epsilon = 0,1)$,

$$f(e^{i\theta}) = \frac{1}{n}\,e^{(\epsilon-1)i\theta/2}\,\frac{\sin(n\theta/2)}{\sin(\theta/2)} \longrightarrow 0 \ \ (\text{if } e^{i\theta} \neq 1)$$

and $nf(e^{i\theta}) \longrightarrow \delta_1$ ($\delta$-distribution on $\boldsymbol{T}^1$ supported by $1 \in \boldsymbol{T}^1$).

## §7. Necessary and sufficient condition for existence of a limit of irreducible characters of $\mathfrak{S}_n(T)$

To construct explicitly IURs $\Pi_n = \mathrm{Ind}_{H_n}^{G_n} \pi_n$ of $G_n = \mathfrak{S}_n(T)$, we have used in Section 3 a partition $(I_{n,\zeta})_{\zeta\in\widehat{T}}$ of $\boldsymbol{I}_n = \{1, 2, \ldots, n\}$. However, to parametrize irreducible characters $\widetilde{\chi}_{\Pi_n}$, the essential thing is not the partition $(I_{n,\zeta})_{\zeta\in\widehat{T}}$ itself but the partition $(n_\zeta)_{\zeta\in\widehat{T}}$ of $n$ given by $n_\zeta = |I_{n,\zeta}| = |\lambda^{n,\zeta}|$, as seen in Remark 4.2. Therefore we rewrite CONDITION I on $\mathcal{I}_n$'s and CONDITION I$\Lambda$ on $(\mathcal{I}_n, \Lambda^n)$'s together in one condition as follows: for increasing sequence $\Lambda^n = (\lambda^{n,\zeta})_{\zeta\in\widehat{T}}$ of sets of Young diagrams $\lambda^{n,\zeta}$ with $n = \sum_{\zeta\in\widehat{T}} |\lambda^{n,\zeta}|$,

(CONDITION $\Lambda$) *there exist limits*: $B_\zeta := \lim_{n\to\infty} \dfrac{|\lambda^{n,\zeta}|}{n} \quad (\zeta \in \widehat{T}),$

*and for each* $\zeta \in \widehat{T}^+ := \{\zeta \in \widehat{T}\,;\, B_\zeta > 0\}$, *there exist limits*

$$\alpha_{\zeta,0,k} := \lim_{n\to\infty} \frac{r_k(\lambda^{n,\zeta})}{n}, \quad \alpha_{\zeta,1,k} := \lim_{n\to\infty} \frac{c_k(\lambda^{n,\zeta})}{n} \quad (1 \le k < \infty). \tag{7.1}$$

For later use in the second part [HHH2] of our present work, we summarize the results in the preceding section as in the following form. We put also for $\zeta \in \widehat{T} \setminus \widehat{T}^+$

$$\alpha_{\zeta,0,k} := \lim_{n\to\infty} \frac{r_k(\lambda^{n,\zeta})}{n} = 0, \quad \alpha_{\zeta,1,k} := \lim_{n\to\infty} \frac{c_k(\lambda^{n,\zeta})}{n} = 0 \quad (1 \le k < \infty).$$

and put for $\varepsilon = 0, 1$, $\alpha_{\zeta,\varepsilon} := (\alpha_{\zeta,\varepsilon,k})_{k\ge 1}$, and

$$\mu_\zeta := B_\zeta - \sum_{\varepsilon\in\{0,1\}} \|\alpha_{\zeta,\varepsilon}\|, \quad \mu := (\mu_\zeta)_{\zeta\in\widehat{T}}. \tag{7.2}$$

Then we have

$$\sum_{\zeta\in\widehat{T}} \sum_{\varepsilon\in\{0,1\}} \|\alpha_{\zeta,\varepsilon}\| + \|\mu\| = \sum_{\zeta\in\widehat{T}} B_\zeta \le 1.$$

THEOREM 7.1. *Let $G = \mathfrak{S}_\infty(T)$, $G_n = \mathfrak{S}_n(T)$ with $T$ any compact group. Let the normalized characters of $\widetilde{\chi}_{\Pi_n}$ of IURs $\Pi_n$ of $G_n$ be parametrized by $\Lambda^n$ as in Theorem 6.1.*

(i) *The following is a necessary and sufficient condition for the existence of the pointwise limit of $\widetilde{\chi}_{\Pi_n}$ as $n \to \infty$, as a continuous function on $G$ in case $T$ is infinite.*

(i-1) *In the case where $T$ is finite, the limit* $\lim_{n\to\infty}\widetilde{\chi}_{\Pi_n}$ *exists if and only if* Condition Λ *holds. Here* Condition E *holds automatically, or* $\sum_{\zeta\in\widehat{T}} B_\zeta = 1$.

(i-2) *In the case where $T$ is infinite, the limit* $\lim_{n\to\infty}\widetilde{\chi}_{\Pi_n}$ *exists and is continuous on $G$, if and only if both* Condition Λ *and* Condition E *hold. Here the convergence is compact uniform on $G$.*

(ii) *In both cases* (i-1) *and* (i-2)*, the set of all the limits* $\lim_{n\to\infty}\widetilde{\chi}_{\Pi_n}$ *coincides with the set of all characters* (*of factor representations of finite type*) *of $G$. The parameter* $A = \big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\,;\,\mu\big)$ *for a character of $G$ thus obtained is given by* (7.1) *and* (7.2).

## §8. Case of canonical subgroups $G^S$ for $T$ abelian

Let $T$ be a compact abelian group and $S$ its subgroup, not necessarily closed. Define subgroups of $G = \mathfrak{S}_\infty(T) = D_\infty(T) \rtimes \mathfrak{S}_\infty$, and $G_n = \mathfrak{S}_n(T) = D_n(T) \rtimes \mathfrak{S}_n$, which are called *canonical* as follows:

$$G^S = \mathfrak{S}_\infty(T)^S = D_\infty(T)^S \rtimes \mathfrak{S}_\infty, \quad G_n^S = \mathfrak{S}_n(T)^S = D_n(T)^S \rtimes \mathfrak{S}_n,$$

$$\text{with } D_\infty(T)^S = D_\infty^S := \big\{d = (t_i)_{i\in\boldsymbol{N}} \in D_\infty(T)\,;\, P(d) := \textstyle\prod_{i\in\boldsymbol{N}} t_i \in S\big\},$$

$$D_n(T)^S = D_n^S := \{d = (t_i)_{i\in\boldsymbol{I}_n} \in D_n(T)\,;\, P(d) \in S\}.$$

Then, by Theorem 2.4, the restriction of a character of $G$ onto $G^S$ gives a character of $G^S$, and in case $S$ is open in $T$ any character of $G^S$ is obtained by this restriction. Note that in the case of $G^S$ the factorizability of positive definite functions in Definition 2.1 used to characterize characters is no longer well-fitted to the situation because the canonical decomposition in (2.6) of $g \in G^S$ should be considered in the bigger group $G$.

In this section we study, for $G^S = \lim_{n\to\infty} G_n^S$, limits of irreducible characters of $G_n^S$, and in particular ask if all characters of $G^S$ can be obtained as limits of irreducible characters of $G_n^S$.

### 8.1. Restriction of IURs of $G_n$ on its subgroup $G_n^S$

Let us first study the structure of the restriction $\Pi_n|_{G_n^S}$ of an IUR $\Pi_n$ of $G_n$ on its subgroup $G_n^S$, and in particular, study if $\Pi_n|_{G_n^S}$ remains still irreducible or not. As is proved in Section 3, every IUR $\Pi_n$ is realized as an induced representation as $\Pi_n = \mathrm{Ind}_{H_n}^{G_n} \pi_n$. We keep here notations in Section 3:

$\mathcal{I}_n = (I_{n,\zeta})_{\zeta\in\widehat{T}}$ : a partition of $\boldsymbol{I}_n$,

$\Lambda^n = (\lambda^{n,\zeta})_{\zeta\in\widehat{T}}$ : a set of Young diagrams such that $|\lambda^{n,\zeta}| = |I_{n,\zeta}|$,

$\pi(\lambda^{n,\zeta})$ : an IUR of $\mathfrak{S}_{I_{n,\zeta}} \subset \mathfrak{S}_n$ determined by $\lambda^{n,\zeta}$,

a subgroup $H_n \subset G_n$ and its IUR $\pi_n$ : with $\widehat{T}'_n := \{\zeta \in \widehat{T}\,;\, I_{n,\zeta} \neq \emptyset\}$,

$$H_n = D_n \rtimes S_{[\eta_n]}, \quad D_n = D_n(T) = D_{\boldsymbol{I}_n}(T), \quad S_{[\eta_n]} = \prod_{\zeta\in\widehat{T}'_n} \mathfrak{S}_{I_{n,\zeta}},$$

$$\pi_n = \eta_n \boxdot \xi_n, \quad \eta_n = \boxtimes_{\zeta\in\widehat{T}'_n} \zeta^{I_{n,\zeta}}, \quad \xi_n = \boxtimes_{\zeta\in\widehat{T}'_n} \pi(\lambda^{n,\zeta}).$$

Moreover the space $V(\Pi_n)$ of $\Pi_n = \mathrm{Ind}_{H_n}^{G_n} \pi_n$ consists of $V(\pi_n)$-valued continuous functions on $G_n$ satisfying $\varphi(hg) = \pi_n(h)\big(\varphi(g)\big)$ $(h \in H_n,\ g \in G_n)$, and the representation operator is given by $\big(\Pi_n(g^0)\varphi\big)(g) = \varphi(gg^0)$ $(g, g^0 \in G_n)$.

Let us examine the algebra of intertwining operators $\mathcal{I}(\Pi_n|_{G_n^S})$ of the representation $\Pi_n|_{G_n^S}$ of $G_n^S$ with itself.

STEP 1. Since $\dim V(\Pi_n) < \infty$, any linear transformation $L$ of $V(\Pi_n)$ is given as follows by an $\mathcal{L}\big(V(\pi_n)\big)$-valued continuous kernel $K'(g,g')$ satisfying

$$K'(hg, h'g') = \pi_n(h)\, K'(g,g')\, \pi_n(h')^{-1} \quad (h, h' \in H_n,\ g, g' \in G_n),$$

$$L\varphi(g) = \int_{H_n\backslash G_n} K'(g,g')\,\varphi(g')\, d\mu_{H_n\backslash G_n}(\widehat{g'}) = \int_{G_n} K'(g,g')\,\varphi(g')\, d\mu_{G_n}(g'),$$

where $\widehat{g'} = H_n g'$. Suppose $J$ be an intertwining operator of $G_n^S$-module $\Pi_n|_{G_n^S}$. Then it is expressed as an integral operator with a kernel $K'(g,g')$ satisfying

$$K'(gg^S, g'g^S) = K'(g,g') \quad (g, g' \in G_n,\ g^S \in G_n^S).$$

Note that $H_n G_n^S = G_n$, then we see that $K'(g,g')$ is uniquely determined by $K(g) := K'(g,e)$ as

$$K'(g, h'g^S) = K(g(g^S)^{-1})\pi_n(h') \quad (g \in G,\ h' \in H_n,\ g^S \in G^S). \tag{8.1}$$

The latter $K(g)$ is determined by the system $\{K(\tau)\}$, where $\tau$ runs over a complete system of representatives of double coset space

$$H_n\backslash G_n/(H_n \cap G^S) \cong S_{[\eta_n]}\backslash \mathfrak{S}_n / S_{[\eta_n]}\,, \tag{8.2}$$

with $S_{[\eta_n]} = H_n \cap \mathfrak{S}_n = \prod_{\zeta\in\widehat{T}'_n} \mathfrak{S}_{I_{n,\zeta}}$, and for a fixed $\tau$,

$$K(h\tau h') = \pi_n(h)K(\tau)\pi_n(h') \quad (h \in H_n, h' \in H_n \cap G^S).$$

Noting that $H_n \cap G^S = D_n^S \rtimes S_{[\eta_n]}$ and $H_n \cap \tau(H_n \cap G^S)\tau^{-1} = D_n^S \rtimes \big(S_{[\eta_n]} \cap \tau S_{[\eta_n]}\tau^{-1}\big)$, we obtain the following lemma.

Lemma 8.1. *For a representative $\tau$ of a double coset in $S_{[\eta_n]}\backslash\mathfrak{S}_n/S_{[\eta_n]}$, the operator $K(\tau) \in \mathcal{L}\big(V(\pi_n)\big)$ intertwines two representations of $H_n \cap \tau(H_n \cap G^S)\tau^{-1} = D_n^S \rtimes \big(S_{[\eta_n]} \cap \tau S_{[\eta_n]}\tau^{-1}\big)$ as*

$$\pi_n(h)\, K(\tau) = K(\tau)\, \pi_n(\tau^{-1}h\tau) \quad \big(h \in D_n^S \rtimes \big(S_{[\eta_n]} \cap \tau S_{[\eta_n]}\tau^{-1}\big)\big).$$

Step 2. Let us first study $D_n^S$-module structure. For $h = (d, \mathbf{1})$ with $d = (t_i)_{i\in\boldsymbol{I}_n} \in D_n^S$, we have $P(d) = \prod_{i\in\boldsymbol{I}_n} t_i \in S$, and

$$\pi_n(h) = \eta_n(d) \boxdot \xi_n(\mathbf{1}), \quad \eta_n(d) = \prod_{\zeta\in\widehat{T}'_n} \prod_{i\in I_{n,\zeta}} \zeta_i(t_i)$$

with $\zeta_i = \zeta$ for $i \in I_{n,\zeta}$. Furthermore $\tau h\tau^{-1} = (\tau(d), \mathbf{1})$ with $\tau(d) = (t_{\tau^{-1}(i)})_{i\in\boldsymbol{I}_n}$,

$$\pi_n(\tau h\tau^{-1}) = \eta_n\big(\tau(d)\big) \boxdot \xi_n(\mathbf{1}), \quad \eta_n\big(\tau(d)\big) = \prod_{\zeta\in\widehat{T}'_n} \prod_{i\in I_{n,\zeta}} \zeta_i(t_{\tau^{-1}(i)}).$$

Hence, supposing $K(\tau) \neq 0$, we have $\eta_n(d) = \eta_n\big(\tau(d)\big)$ or

$$\prod_{i\in\boldsymbol{I}_n} \zeta_i(t_i) = \prod_{i\in\boldsymbol{I}_n} \zeta_{\tau(i)}(t_i) \quad \text{if} \quad \prod_{i\in\boldsymbol{I}_n} t_i \in S. \tag{8.3}$$

On the other hand, consider characters of $D_n = \prod_{i\in\boldsymbol{I}_n} T_i$, $T_i = T$, which are trivial on $D_n^S$. A character of $T$ is trivial on $S$ if and only if so is on the closure $\overline{S}$, and so it is naturally considered as a character of $T/\overline{S}$. Since $D_n/D_n^S \cong T/S$, characters of $D_n$ vanishing on $D_n^S$ are of the form

$$\mathop{\boxtimes}_{i\in\boldsymbol{I}_n} \zeta'_i \quad \text{with} \quad \zeta'_i = \zeta^S \ (\forall i) \ \text{ for some } \ \zeta^S \in \big(T/\overline{S}\big)^\wedge.$$

Therefore we see from (8.3) that $\zeta_{\tau(i)} = \zeta^S\zeta_i$ ($\forall i$) for some $\zeta^S \in \big(T/\overline{S}\big)^\wedge$, and so $(\zeta^S)^n = 1$. Denote by $\overline{I_{n,\zeta}}$ the underlying set of integers of $I_{n,\zeta}$, then we have $\tau\big(\overline{I_{n,\zeta}}\big) = \overline{I_{n,\zeta^S\zeta}}$. By adjusting the representative $\tau$ of a double coset

$S_{[\eta_n]}\backslash\mathfrak{S}_n/S_{[\eta_n]}$, we have $\tau(I_{n,\zeta}) = I_{n,\zeta^S\zeta}$ as ordered sets, that is, $\tau(i) > \tau(i')$ in $I_{n,\zeta^S\zeta}$ if $i > i'$ in $I_{n,\zeta}$. Hence the map

$$\mathfrak{S}_{I_{n,\zeta}} \ni \sigma \longmapsto \tau\sigma\tau^{-1} \in \mathfrak{S}_{I_{n,\zeta^S\zeta}} \tag{8.4}$$

gives a natural isomorphism between components of $S_{[\eta_n]}$, and those of $S_{[\eta_n]} \cap \tau S_{[\eta_n]}\tau^{-1} = S_{[\eta_n]}$.

Now turn to study the module structure for $S_{[\eta_n]}$. Then we get the following lemma.

Lemma 8.2. (i) *Suppose* $K(\tau) \neq 0$. *There exists a unique* $\zeta^S \in \big(T/\overline{S}\big)\widehat{}$ *such that* $\tau\big(\overline{I_{n,\zeta}}\big) = \overline{I_{n,\zeta^S\zeta}}$ $(\zeta \in \widehat{T}'_n)$, *and*

$$\lambda^{n,\zeta} = \lambda^{n,\zeta^S\zeta} \quad (\zeta \in \widehat{T}'_n) \qquad (\textit{as abstract Young diagrams}). \tag{8.5}$$

(ii) *Normalize a representative* $\tau$ *of double coset in* $S_{[\eta_n]}\backslash\mathfrak{S}_n/S_{[\eta_n]}$ *such that*

$$\tau(I_{n,\zeta}) = I_{n,\zeta^S\zeta} \quad (\zeta \in \widehat{T}'_n) \qquad (\textit{as ordered sets}). \tag{8.6}$$

*Then the intertwining operator* $K(\tau)$ *is a scalar multiple of the unitary operator* $U(\tau)$ *of exchange of the order of components in the exterior tensor product* $\xi_n = \boxtimes_{\zeta\in\widehat{T}'_n}\pi(\lambda^{n,\zeta})$ *for* $S_{[\eta_n]} = \prod_{\zeta\in\widehat{T}'_n}\mathfrak{S}_{I_{n,\zeta}}$ *by replacing* $\pi(\lambda^{n,\zeta})$ *with* $\pi(\lambda^{n,\zeta^S\zeta})$ *together with the same exchange in* $\eta_n = \boxtimes_{\zeta\in\widehat{T}'_n}\zeta^{I_{n,\zeta}}$ *for* $D_n$.

Let $p = p(\zeta^S)$ be the order of $\zeta^S$. Take a partition of $\widehat{T}'_n$ into subsets of $p$ elements of the form $Z(\zeta) := \{\zeta, (\zeta^S)\zeta, \ldots, (\zeta^S)^{p-1}\zeta\}$, and a complete set of representatives $\Delta_n = \Delta_{n,\zeta^S} = \{\zeta\}$, $|\Delta_n| = |\widehat{T}'_n|/p$, then

$$\widehat{T}'_n = \bigsqcup_{\zeta\in\Delta_n} Z(\zeta), \quad n = pN \ \text{ with } \ N := \sum_{\zeta\in\Delta_n}|I_{n,\zeta}|, \tag{8.7}$$

$$\lambda^{n,\zeta'} = \lambda^{n,\zeta''} \ \ (\zeta',\zeta'' \in Z(\zeta)) \quad (\text{as abstract Young diagrams}). \tag{8.8}$$

Accordingly the operator $U(\tau)$ for the normalized representative $\tau$, associated to $\zeta^S$, is of order $p$, and if we determine well the positive multiplicative scalar, the operator $L(\zeta^S)$ with the integral kernel $K'(g,g')$ given by (8.1) through $K(g)$, which corresponds to $K(\tau)$ and zero outside $H_n\tau H_n$, is unitary and of order $p$ and intertwines $\Pi_n|_{G_n^S}$ with itself. Here $K'(h\tau g^S, h'g^S) =$

$\pi_n(h)K(\tau)\pi_n(h'^{-1})$ $(h, h' \in H_n, g^S \in G_n^S)$ and $K'(g, g') = 0$ if $g(g')^{-1} \notin H_n \tau H_n$, and $L(\zeta^S)$ is given as follows: for $g = (d, \sigma) = d\sigma \in G_n$,

$$L(\zeta^S)\varphi(g) := \int_{G_n} K'(g, g')\,\varphi(g')\,d\mu_{G_n}(g') = \frac{1}{|\mathfrak{S}_n|}\sum_{\sigma' \in \mathfrak{S}_n} K(d\sigma\sigma'^{-1})\,\varphi(\sigma')$$
$$= \eta_n(d)\cdot\frac{1}{|\mathfrak{S}_n|}\sum_{\sigma' \in \mathfrak{S}_n} K(\sigma\sigma'^{-1})\,\varphi(\sigma').$$

LEMMA 8.3. *For $K(\tau) = c\,U(\tau)$, $c > 0$, the associated operator $L(\zeta^S)$ is unitary if $c = \mu_{G_n}(H_n)^{-1} = |\mathfrak{S}_n|/|S_{[\eta_n]}|$. Put $K^U(g) := c^{-1}K(g)$, then $K^U(g)$ is associated to $U(\tau)$ and*

$$L(\zeta^S)\varphi(g) = \eta_n(d)\cdot\sum_{\sigma' \in S_{[\eta_n]}\backslash\mathfrak{S}_n} K^U(\sigma\sigma'^{-1})\,\varphi(\sigma'). \tag{8.9}$$

*Proof.* Define $\varphi^{\sigma,\boldsymbol{v}} \in V(\Pi_n)$ for $\sigma \in \mathfrak{S}_n$ and $\boldsymbol{v} \in V(\pi_n)$ as $\varphi^{\sigma,\boldsymbol{v}}(h\sigma) := \pi_n(h)\boldsymbol{v}$ $(h \in H_n)$, and $\varphi^{\sigma,\boldsymbol{v}}(h\sigma) := 0$ outside $H_n\sigma$. Then $\|\varphi^{\sigma,\boldsymbol{v}}\|^2 = \mu_{G_n}(H_n)\,\|\boldsymbol{v}\|^2$, and

$$L(\zeta^S)\varphi^{\sigma,\boldsymbol{v}} = c\cdot\mu_{G_n}(H_n)\,\varphi^{\tau\sigma,U(\tau)\boldsymbol{v}}.$$

Comparing lengths of vectors in both sides, we have $c\cdot\mu_{G_n}(H_n) = 1$. □

Let $\mathcal{Z}(\pi_n)$ be the set of all $\zeta^S \in \big(T/\overline{S}\big)^\wedge$ which satisfy (8.5):

$$\mathcal{Z}(\pi_n) := \{\zeta^S \in \big(T/\overline{S}\big)^\wedge;\ \lambda^{n,\zeta} = \lambda^{n,\zeta^S\zeta}\ \ (\zeta \in \widehat{T}'_n)\}. \tag{8.10}$$

Then it is a subgroup of order $\le n$ consisting of elements of orders dividing $n$. From the structure theory for abelian groups of finite orders, we see that $\mathcal{Z}(\pi_n)$ is a direct product of cyclic groups as $\mathcal{Z}(\pi_n) \cong \prod_{1\le j\le b}\boldsymbol{Z}_{p_j}$.

PROPOSITION 8.4. *The algebra $\mathcal{I}(\Pi_n|_{G_n^S})$ of intertwining operators for the restriction $\Pi_n|_{G_n^S}$ of IUR $\Pi_n = \mathrm{Ind}_{H_n}^{G_n}\pi_n$ on $G_n^S$ is isomorphic to the group algebra of $\mathcal{Z}(\pi_n)$, and there corresponds to a group element $\zeta^S$ the operator $L(\zeta^S) \in \mathcal{L}\big(V(\Pi_n)\big)$.*

*Since $\mathcal{I}(\Pi_n|_{G_n^S})$ is abelian, the irreducible decomposition of $\Pi_n|_{G_n^S}$ is of multiplicity free, and the number of irreducible components equals the order $|\mathcal{Z}(\pi_n)|$.*

*In particular, the restriction of IUR $\Pi_n$ of $G_n$ on its subgroup $G_n^S$ remains irreducible if there exists no non-trivial character $\zeta^S \in \big(T/\overline{S}\big)^\wedge$ for which (8.5) holds.*

Conversely, now consider an IUR $\rho_n$ of $G_n^S$ and ask if it appears in a restriction of an IUR $\Pi_n$ of $G_n$. Note that the map $G^S \ni g \mapsto \rho_n(g)$ is uniformly continuous and can be uniquely extended by continuity to an IUR $\overline{\rho_n}$ of the closure $\overline{G_n^S}$ which is equal to $G_n^{\overline{S}}$. From the general theory of representations of compact groups, we extract Frobenius reciprocity given in the following lemma, and from it we know that every irreducible component of the induced representation $\mathrm{Ind}_{G_n^{\overline{S}}}^{G_n} \overline{\rho_n}$ contains a multiple of $\overline{\rho_n}$ when it is restricted on $G_n^{\overline{S}}$ and naturally a multiple of $\rho_n$.

LEMMA 8.5. $\left[\mathrm{Ind}_{G_n^{\overline{S}}}^{G_n} \overline{\rho_n} : \Pi_n\right] = \left[\Pi_n|_{G_n^{\overline{S}}} : \overline{\rho_n}\right]$.

## 8.2. Limits of irreducible characters of $G_n^S$ as $n \to \infty$

DEFINITION 8.1. For IURs $\Pi_n^S$ of $G_n^S$ and $\Pi_{n+1}^S$ of $G_{n+1}^S$, suppose that they are restrictions of IURs $\Pi_n$ of $G_n$ and $\Pi_{n+1}$ of $G_{n+1}$ respectively. Then we say $\Pi_n^S$ increases to $\Pi_{n+1}^S$ (notation: $\Pi_n^S \nearrow \Pi_{n+1}^S$) if and only if $\Pi_n$ increases to $\Pi_{n+1}$.

Taking increasing sequences of IURs $\Pi_n$ of $G_n$, $n \to \infty$, we restrict their each terms to subgroups $G_n^S$, and apply the results in **8.1**. Then, with the help of Theorem 2.4, we obtain from Proposition 8.4 the following results on limits of irreducible characters of $G_n^S$ and on characters of $G^S$.

THEOREM 8.6. (i) *Every character of $G^S$ is a limit of characters of sequence of increasing IURs of $G_n^S$ as $n \to \infty$. In this case, the convergence is compact uniform.*

(ii) *Take an increasing sequence $\Pi_n$ of IURs of $G_n$ for which the normalized characters $\widetilde{\chi}_{\Pi_n}$ converge to a character $f_A$ of $G$. Then, their restrictions $\Pi_n^S = \Pi_n|_{G_n^S}$ remain irreducible for an infinite number of $n$, and thus obtained sequence of normalized characters $\widetilde{\chi}_{\Pi_n^S}$ of IURs of $G_n^S$ (even though it may have an infinite number of $n$ of lacks or defects) converges compact uniformly to the character $f_A^S = f_A|_{G^S}$ of $G^S$.*

(iii) *In the case of* (ii), *suppose an infinite number of $\Pi_n^S = \Pi_n|_{G_n^S}$ are reducible. Then, for the sequence consisting of reducible $\Pi_n^S$, their normalized characters $\widetilde{\chi}_{\Pi_n^S}$ also converge compact uniformly to the character $f_A^S = f_A|_{G^S}$ of $G^S$ as $n \to \infty$.*

*In this case, the character $f_A$ of $G$ is characterized by the property that, for some non-trivial $\zeta^S \in (T/\overline{S})\widehat{\ }$,*

$$\alpha_{\zeta,\varepsilon} = \alpha_{\zeta\zeta^S,\varepsilon}, \quad \mu_\zeta = \mu_{\zeta\zeta^S} \quad (\zeta \in \widehat{T},\ \varepsilon = 0,1) \tag{8.11}$$

*Proof.* If the restriction $\Pi_n^S = \Pi_n|_{G_n^S}$ is reducible, there hold (8.6) and (8.8) or $|I_{n,\zeta}| = |I_{n,\zeta^S\zeta}|$ and $\lambda^{n,\zeta} = \lambda^{n,\zeta^S\zeta}$. These conditions can not occur consecutively for $n$. Moreover, for any character $f_A$ of $G$, we can find a sequence of $\Pi_n$ for which none of $\Pi_n$ satisfies these conditions and $f_A = \lim_{n\to\infty} \widetilde{\chi}_{\Pi_n}$. This gives us a sequence of IURs $\Pi_n^S$ for which $f_A^S = \lim_{n\to\infty} \widetilde{\chi}_{\Pi_n^S}$, and so the assertion (i).

Taking into account the above conditions (8.6) and (8.8), we can deduce other assertions easily from Proposition 8.4. □

*Remark* 8.1. For another type of subgroups $\mathfrak{A}_n(T) := D_n(T) \rtimes \mathfrak{A}_n$ of $\mathfrak{S}_n(T)$ and $\mathfrak{A}_\infty(T) := D_\infty(T) \rtimes \mathfrak{A}_\infty$ of $\mathfrak{S}_\infty(T)$, we can study the analogous problem. By Theorem 16.1 [HH6], all the characters of $\mathfrak{A}_\infty(T)$ are obtained as restrictions of those of $\mathfrak{S}_\infty(T)$, and conversely the restriction of any character of the latter is also a character of the former.

### 8.3. Characters of irreducible components of the restriction $\Pi_n|_{G_n^S}$

Analysis of reducible restrictions $\Pi_n|_{G_n^S}$ of IURs $\Pi_n = \mathrm{Ind}_{H_n}^{G_n} \pi_n$ of $G_n$ is interesting, and necessary to study further the situations of $G_n^S$ and $G^S$ independently from $G_n$ and $G$, for instance, to extend Definition refdefn8.1 to general IURs of $G_n^S$, and to clarify the situation in Theorem 8.6 (iii).

The algebra $\mathcal{I}(\Pi_n|_{G_n^S})$ of intertwining operators for the restriction $\Pi_n|_{G_n^S}$ of IUR $\Pi_n = \mathrm{Ind}_{H_n}^{G_n} \pi_n$ is abelian and isomorphic to the group algebra of

$$(8.12) \qquad \mathcal{Z}(\pi_n) = \left\{\zeta^S \in \left(T/\overline{S}\right)^\wedge ; \lambda^{n,\zeta} = \lambda^{n,\zeta^S\zeta} \;\; (\zeta \in \widehat{T}_n')\right\} \cong \prod_{1\le k\le b} \boldsymbol{Z}_{p_k}.$$

The set of operators $L(\zeta^S)$ on $V(\Pi_n)$ is isomorphic to the above group in such a way that $L(zz') = L(z)L(z')$ $(z, z' \in \mathcal{Z}(\pi_n))$, and we can construct from them projections onto each irreducible subspaces. Then we can also determine characters of irreducible constituents of $\Pi_n|_{G_n^S}$.

Take an $L_k = L(\zeta^S)$ corresponding to the generator $\zeta^S$ of $k$-th cyclic component $\boldsymbol{Z}_{p_k}$ in the right hand side of the above isomorphism. Put $L = L_k$, $p = p_k$ for a moment, and for cyclic group $\boldsymbol{Z}_p$, we solve $L^p = I$ (the identity operator) to get $p$ orthogonal projections corresponding to different minimal idempotents in the group algebra of $\boldsymbol{Z}_p$. Put $\alpha_p = \exp(2\pi i/p)$ a $p$-th elementary root of 1, then $\prod_{0\le s<p}(L - \alpha_p^s I) = 0$. Differentiate the

identity $X^p - 1 = \prod_{0\le s<p}(X - \alpha_p^s I)$ in an indeterminate $X$, and then multiply by $X$, we have

$$\sum_{0\le u<p} Q_u(X) = X^p \quad \text{with } Q_u(X) = \frac{1}{p}X \cdot \prod_{0\le s<p,\, s\ne u}(X - \alpha_p^s). \tag{8.13}$$

Put $Q_u := Q_u(L)$, then $Q_u \cdot Q_v = 0$ $(u \ne v)$. Take firstly the square of the first identity in (8.13) with $X = L$, and secondly multiply $Q_v$ to it, then we obtain

$$\sum_{0\le u<p} Q_u^2 = (L^p)^2 = I,$$

$$\sum_{0\le u<p} Q_u \cdot Q_v = L^p \cdot Q_v \quad \therefore\ Q_v^2 = Q_v.$$

Therefore $Q_u$, $0 \le u < p$, are projections (not necessarily self-adjoint) mutually orthogonal. Note that, in the expression $Q_u = a_{0,u}I + a_{1,u}L + a_{2,u}L^2 + \cdots + a_{p-1,u}L^{p-1}$, the "constant terms" are the same: $a_{0,u}I = \frac{1}{p}I$.

Come back to the notations with indices $k$. Let $Q_u^{(k)}$, $0 \le u \le p_k$, be the projection constructed from $L_k$. Then the set of possible products

$$Q_{u_1,u_2,\ldots,u_s} := \prod_{1\le k\le s} Q_{u_k}^{(k)} \quad (1 \le u_k \le p_k) \tag{8.14}$$

are all of minimal projections onto different irreducible components of $\Pi_n|_{G_n^S}$, and so the total number of irreducible components equals $p_1p_2\cdots p_b$ $= |\mathcal{Z}(\pi_n)|$. Note that the common "constant term" of $Q_{u_1,u_2,\ldots,u_s}$ is equal to $\frac{1}{|\mathcal{Z}(\pi_n)|}I$.

Now let us calculate the trace of $L(\zeta^S)\Pi_n^S(g^S)$, $g^S \in G_n^S$, which is called *virtual character* of $L(\zeta^S)\Pi_n^S$, and is denoted by $\chi_{L(\zeta^S)\Pi_n^S}$. On the space $V(\Pi_n)$, the operator $L(\zeta^S)\Pi_n^S(g^S) = L(\zeta^S)\Pi_n(g^S)$ is given with an integral kernel as

$$\begin{aligned}
\big(L(\zeta^S)\Pi_n^S(g^S)\big)\varphi(g') &= \int_{H_n\backslash G_n} K'(g', g'')\varphi(g''g^S)\, d\mu_{H_n\backslash G_n}(g'') \\
&= \int_{H_n\backslash G_n} K'(g', g''(g^S)^{-1})\varphi(g'')\, d\mu_{H_n\backslash G_n}(g'') \\
&= \int_{H_n\backslash G_n} K'(g'g^S, g'')\varphi(g'')\, d\mu_{H_n\backslash G_n}(g'').
\end{aligned}$$

Therefore its trace is given by an integral, and then by a sum as

$$\begin{aligned}\operatorname{tr}\big(L(\zeta^S)\Pi_n^S(g^S)\big) &= \int_{H_n\backslash G_n} \operatorname{tr}\big(K'(g'g^S,g')\big)\,d\mu_{H_n\backslash G_n}(g') \\ &= \frac{1}{|\mathfrak{S}_n|}\sum_{\sigma'\in\mathfrak{S}_n} \operatorname{tr}\big(K'(\sigma'g^S\sigma'^{-1})\big) = \sum_{\sigma'\in S_{[\eta_n]}\backslash\mathfrak{S}_n} \operatorname{tr}\big(K^U(\sigma'g^S\sigma'^{-1})\big),\end{aligned}$$

because, for $h\in H_n = D_n \rtimes S_{[\eta_n]}$, we have

$$\begin{aligned}K'(h\sigma'g^S, h\sigma') &= \pi_n(h)K'(\sigma'g^S,\sigma')\pi_n(h)^{-1} \\ &= \pi_n(h)K(\sigma'g^S\sigma'^{-1})\pi_n(h)^{-1}.\end{aligned}$$

Let $g^S = (d^S,\sigma)\in G_n^S = D_n^S \rtimes \mathfrak{S}_n$, and we identify $d^S$ and $\sigma$ with their images in $G_n$. Then $\sigma'g^S\sigma'^{-1} = \big(\sigma'(d^S), \sigma'\sigma\sigma'^{-1}\big)$, and so

$$K^U(\sigma'g^S\sigma'^{-1}) = \pi_n\big(\sigma'(d^S)\big)K^U(\sigma'\sigma\sigma'^{-1}),$$

$$\operatorname{tr}\big(L(\zeta^S)\Pi_n^S(g^S)\big) = \sum_{\sigma'\in S_{[\eta_n]}\backslash\mathfrak{S}_n} \operatorname{tr}\Big(\pi_n\big(\sigma'(d^S)\big)K^U\big(\sigma'\sigma\sigma'^{-1}\big)\Big). \tag{8.15}$$

Here, with an element $\tau = \tau_{\zeta^S}\in\mathfrak{S}_n$ determined by $\zeta^S$ such that $\tau S_{[\eta_n]}\tau^{-1} = S_{[\eta_n]}$ and $\tau\neq\mathbf{1}$, the kernel is given as

$$K^U(\sigma') = \begin{cases} \pi_n(\kappa)K^U(\tau)\pi_n(\kappa') & \text{for } \sigma' = \kappa\tau\kappa',\ \kappa,\kappa'\in S_{[\eta_n]}, \\ 0 & \text{outside of } S_{[\eta_n]}\tau S_{[\eta_n]} = S_{[\eta_n]}\tau. \end{cases}$$

**Case I.** Case of $g = (d,\sigma)$, where $\sigma$ is not conjugate to any $\kappa\tau$ with $\kappa\in S_{[\eta_n]}$ (in particular $\sigma = \mathbf{1}$):

$$\operatorname{tr}\big(L(\zeta^S)\Pi_n^S\big)\big((d,\sigma)\big) = 0.$$

LEMMA 8.7. *For each minimal projection $Q = Q_{u_1,u_2,\ldots,u_s}$ in (8.14), let $\Pi_n^{(Q)} = Q\cdot\Pi_n^S\cdot Q$ be the corresponding irreducible component of $\Pi_n$. Then their dimensions are all equal and*

$$\dim\Pi_n^{(Q)} = \frac{1}{|\mathcal{Z}(\pi_n)|}\cdot\dim\Pi_n.$$

*Proof.* Express $Q = Q_{u_1,u_2,\dots,u_s}$ in (8.14) as a linear combination of the identity operator $I$ and $L(\zeta^S)$ for non-trivial $\zeta^S \in \mathcal{Z}(\pi_n)$ as $Q = a_0 I + \sum_{\zeta^S} a_{\zeta^S} L(\zeta^S)$. Then, as remarked before, $a_0 = 1/|\mathcal{Z}(\pi_n)|$.

On the other hand, $\mathrm{tr}\big(L(\zeta^S)\Pi_n(e)\big) = 0$ as above for non-trivial $\zeta^S$, and so

$$\dim \Pi_n^{(Q)} = \mathrm{tr}\big(\Pi_n^{(Q)}(e)\big) = a_0 \cdot \mathrm{tr}\big(\Pi_n(e)\big) = a_0 \cdot \dim \Pi_n.$$

□

**Case II.** Case of $g = (d, \sigma)$, where $\sigma$ is conjugate to $\kappa\tau$, $\kappa \in S_{[\eta_n]}$: From $\sigma'\sigma\sigma'^{-1} = \kappa\tau$, we have

$$\begin{aligned}\pi_n\big(\sigma'(d^S)\big)K^U(\sigma'\sigma\sigma'^{-1}) &= \pi_n\big(\sigma'(d^S)\big)\pi_n(\kappa)U(\tau)\\ &= \Big(\eta_n\big(\sigma'(d^S)\big) \boxtimes \xi_n(\kappa)\Big)U(\tau).\end{aligned}$$

Here $\eta_n(\,\cdot\,)$ is a one-dimensional character given as

$$\eta_n\big(\sigma'(d^S)\big) = \prod_{i\in \boldsymbol{I}_n} \zeta_i(t_{\sigma'^{-1}(i)}) = \prod_{i\in \boldsymbol{I}_n} \zeta_{\sigma'(i)}(t_i) \quad \text{for } d^S = (t_i). \tag{8.16}$$

Decompose $\kappa \in S_{[\eta_n]}$ as $\kappa = \prod_{\zeta\in\widehat{T}'_n} \kappa_\zeta$, $\kappa_\zeta \in \mathfrak{S}_{I_{n,\zeta}}$, then

$$\xi_n(\kappa) = \mathop{\boxtimes}_{\zeta\in\widehat{T}'_n} \pi(\lambda^{n,\zeta})(\kappa_\zeta).$$

Moreover $K(\tau)$ sends each component $\pi(\lambda^{n,\zeta})$ of $\xi_n$ to $\pi(\lambda^{n,\zeta^S\zeta})$.

To compute the trace, we apply the following lemma.

LEMMA 8.8. *Let $V$ be a Hilbert space of finite dimension, and take its copies $V^{(k)}$, $0 \le k \le p-1$. Let $U$ be a cycle of linear isomorphisms of $V^{(k)}$ onto $V^{(k+1)}$ coming from the identity map on $V$ for $0 \le k \le p-1$ such that*

$$V^{(0)} \longrightarrow V^{(1)} \longrightarrow \cdots \longrightarrow V^{(p-1)} \longrightarrow V^{(0)},$$

*where superfices $k$ are considered modulo $p$, that is, superfix $p$ is understood as $0$. In another expression, $U$ is a linear map on $X := \bigoplus_{0\le k\le p-1}$ sending $V^{(k)} = V$ onto $V^{(k+1)} = V$ through the identity map on $V$. Let $L$ be a linear transformation of tensor product space $W = \bigotimes_{0\le k\le p-1} V^{(k)}$ given by permuting components by $U$ as*

$$L:\ \bigotimes_{0\le k\le p-1} v^{(k)} \longmapsto \bigotimes_{0\le k\le p-1} Uv^{(k-1)} \quad (v^{(k)} \in V^{(k)}).$$

*Let $A^{(k)}$ be a linear transformation on $V^{(k)}$ for $0 \le k \le p-1$. Then the trace of linear transformation $\big(\bigotimes_{0\le k\le p-1} A^{(k)}\big)\cdot L$ on $W$ is given by*

$$\text{(8.17)}\qquad \begin{aligned} &\mathrm{tr}_W\Big(\Big(\bigotimes_{0\le k\le p-1} A^{(k)}\Big)\cdot L\Big)\\ &\quad= \mathrm{tr}_{V^{(0)}}\big(UA^{(p-1)}UA^{(p-2)}\cdots UA^{(1)}UA^{(0)}\big)\\ &\quad= \mathrm{tr}_V\big(A^{(p-1)}A^{(p-2)}\cdots A^{(1)}A^{(0)}\big), \end{aligned}$$

*where, on the last right hand side,* $\mathrm{tr}_V$ *denotes the trace on $V$, and each $A^{(k)}$ on $V^{(k)}$ is pulled back as a linear transformation on $V \cong V^{(k)}$.*

*Let $A^{(k)}$ and $B^{(k)}$ be two linear transformation on $V^{(k)}$. Then the trace of linear transformation $\big(\bigotimes_{0\le k\le p-1} A^{(k)}\big)\cdot L\cdot\big(\bigotimes_{0\le k\le p-1} B^{(k)}\big)$ on $W$ is given by*

$$\text{(8.18)}\qquad \begin{aligned} &\mathrm{tr}_W\Bigg(\Big(\bigotimes_{0\le k\le p-1} A^{(k)}\Big)\cdot L\cdot\Big(\bigotimes_{0\le k\le p-1} B^{(k)}\Big)\Bigg)\\ &\quad= \mathrm{tr}_{V^{(0)}}\big(A^{(0)}UB^{(p-1)}A^{(p-1)}UB^{(p-2)}A^{(p-2)}UB^{(p-3)}\cdots A^{(1)}UB^{(0)}\big)\\ &\quad= \mathrm{tr}_V\big(A^{(0)}B^{(p-1)}A^{(p-1)}B^{(p-2)}A^{(p-2)}B^{(p-3)}\cdots A^{(1)}B^{(0)}\big). \end{aligned}$$

*Proof.* Let $N = \dim V$, and take an orthonormal basis $\{e_m,\ 1 \le m \le N\}$ of $V$, and take its copy in $V^{(k)}$ as an orthonormal basis $\{e^{(k)}_m,\ 1 \le m \le N\}$, for $0 \le k \le p-1$. Take an orthonormal basis of $W$ given by $\bigotimes_{0\le k\le p-1} e^{(k)}_{m_k}$, $1 \le m_k \le N$. Then, by definition, $L$ maps $\bigotimes_{0\le k\le p-1} e^{(k)}_{m_k}$ to $\bigotimes_{0\le k\le p-1} e^{(k)}_{m_{k-1}}$. Put matrix elements of $A^{(k)}$ as $a^{(k)}(m,m') = \langle A^{(k)}e^{(k)}_{m'},\ e^{(k)}_m\rangle$. Then

$$\begin{aligned} &\Big\langle\Big(\Big(\bigotimes_{0\le k\le p-1} A^{(k)}\Big)\cdot L\Big)\bigotimes_{0\le k\le p-1} e^{(k)}_{m_k},\ \bigotimes_{0\le k\le p-1} e^{(k)}_{m_k}\Big\rangle\\ &\quad= \Big\langle\Big(\bigotimes_{0\le k\le p-1} A^{(k)}\Big)\bigotimes_{0\le k\le p-1} e^{(k)}_{m_{k-1}},\ \bigotimes_{0\le k\le p-1} e^{(k)}_{m_k}\Big\rangle\\ &\quad= a^{(0)}(m_0,m_{p-1})\,a^{(1)}(m_1,m_0)\ \cdots\ a^{(p-1)}(m_{p-1},m_{p-2}). \end{aligned}$$

Summing up over $1 \le m_0, m_1, \ldots, m_{p-1} \le N$, we get $\mathrm{tr}_V\big(A^{(p-1)}A^{(p-2)}\cdots A^{(1)}A^{(0)}\big)$.

Put matrix elements of $B^{(k)}$ as $b^{(k)}(m,m') = \langle B^{(k)} e^{(k)}_{m'}, e^{(k)}_m \rangle$. Then

$$\begin{aligned}
&\Big\langle \Big( \bigotimes_{0\le k\le p-1} A^{(k)} \Big) \cdot L \cdot \Big( \bigotimes_{0\le k\le p-1} B^{(k)} \Big) \bigotimes_{0\le k\le p-1} e^{(k)}_{m_k}, \bigotimes_{0\le k\le p-1} e^{(k)}_{m_k} \Big\rangle \\
&= \Big\langle \Big( \bigotimes_k A^{(k)} \Big) \cdot L \sum_{1\le n_0,n_1,\dots,n_{p-1}\le N} \prod_k b^{(k)}(n_k, m_k) \bigotimes_k e^{(k)}_{n_k}, \bigotimes_k e^{(k)}_{m_k} \Big\rangle \\
&= \Big\langle \Big( \bigotimes_k A^{(k)} \Big) \sum_{n_0,n_1,\dots,n_{p-1}} \prod_k b^{(k)}(n_k, m_k) \bigotimes_k e^{(k)}_{n_{k-1}}, \bigotimes_k e^{(k)}_{m_k} \Big\rangle \\
&= \sum_{n_0,n_1,\dots,n_{p-1}} \prod_k b^{(k)}(n_k, m_k) \prod_k a^{(k)}(m_k, n_{k-1}) \\
&= \sum_{n_0,n_1,\dots,n_{p-1}} a^{(0)}(m_0, n_{p-1})\, b^{(p-1)}(n_{p-1}, m_{p-1})\, a^{(1)}(m_1, n_0)\, b^{(0)}(n_0, m_0) \\
&\qquad\qquad \cdots a^{(p-1)}(m_{p-1}, n_{p-2})\, b^{(p-2)}(n_{p-2}, m_{p-2}).
\end{aligned}$$

Summing up over $1 \le m_0, m_1, \dots, m_{p-1} \le N$, we get the second formula. □

Now we apply the above lemma by taking $V = V\big(\pi(\lambda^{n,\zeta})\big)$ for a fixed $\zeta \in \widehat{T}'_n$, and $V^{(k)} = V\big(\pi(\lambda^{n,(\zeta^S)^k\zeta})\big)$, $0 \le k \le p-1$, and $U$ the identification of $V\big(\pi(\lambda^{n,(\zeta^S)^k\zeta})\big)$ with $V\big(\pi(\lambda^{n,(\zeta^S)^{k+1}\zeta})\big)$ through the order-preserving correspondence $\tau(I_{n,(\zeta^S)^k\zeta}) = I_{n,(\zeta^S)^{k+1}\zeta}$ for $0 \le k \le p-1$, and $L = K(\tau)$. Then

$$W = \bigotimes_{0\le k\le p-1} V^{(k)} = V(\eta_n \boxdot \xi_n) = V(\pi_n).$$

Put $A^{(k)} = \pi(\lambda^{n,(\zeta^S)^k\zeta})(\kappa_{(\zeta^S)^k\zeta})$, then it is pulled back to a linear transformation on $V^{(0)}$ as $U^{-k}A^{(k)}U^k = \pi(\lambda^{n,\zeta})(\tau^{-k}\kappa_{(\zeta^S)^k\zeta}\,\tau^k)$. Since $\tau^p = \mathbf{1}$, we have

$$\begin{aligned}
&UA^{(p-1)}UA^{(p-2)} \cdots UA^{(1)}UA^{(0)} \\
&\quad = \pi(\lambda^{n,\zeta})\big(\tau\kappa_{(\zeta^S)^{p-1}\zeta}\,\tau\kappa_{(\zeta^S)^{p-2}\zeta} \cdots \tau\kappa_{(\zeta^S)\zeta}\,\tau\kappa_\zeta\big).
\end{aligned}$$

Take a partition of $\widehat{T}'_n$ in (8.7) into subsets of $p$ elements of the form $Z(\zeta) := \{\zeta, (\zeta^S)\zeta, \dots, (\zeta^S)^{p-1}\zeta\}$ with a complete set of representatives $\Delta_n = \Delta_{n,\zeta^S}$, $|\Delta_n| = |\widehat{T}'_n|/p$. The results in **Case I** and **Case II** together give us the following. For simplicity, a representative $\sigma'$ of a coset in $S_{[\eta_n]}\backslash\mathfrak{S}_n$ is denoted as $\sigma' \in S_{[\eta_n]}\backslash\mathfrak{S}_n$.

THEOREM 8.9. *Suppose that, for IUR* $\Pi_n = \mathrm{Ind}_{H_n}^{G_n}\pi_n$*, its restriction* $\Pi_n^S = \Pi_n|_{G_n^S}$ *onto the subgroup* $G_n^S$ *is reducible. Take a non-trivial* $\zeta^S \in \mathcal{Z}(\pi_n) \subset \bigl(T/\overline{S}\bigr)^\frown$ *in* (8.12)*, and* $\tau = \tau_{n,\zeta^S} \in \mathfrak{S}_n$ *such that* $\tau(I_{n,\zeta}) = I_{n,\zeta^S\zeta}$ *is an order-preserving map for every* $\zeta \in \widehat{T}'_n$*. Let* $L(\zeta^S)$ *be the corresponding intertwining operator for* $\Pi_n^S$*. Then the virtual character* $\chi_{L(\zeta^S)\Pi_n^S}(g^S) = \mathrm{tr}\bigl(L(\zeta^S)\Pi_n^S(g^S)\bigr)$ *is given as follows.*

*Take a* $g^S = (d^S, \sigma) \in G_n^S$*. A* $\kappa \in S_{[\eta_n]}$ *is decomposed as* $\kappa = \prod_{\zeta\in\widehat{T}'_n}\kappa_\zeta$ *according to* $S_{[\eta_n]} = \prod_{\zeta\in\widehat{T}'_n}\mathfrak{S}_{I_{n,\zeta}}$*. Then,*

$$\chi_{L(\zeta^S)\Pi_n^S}(g^S) = 0 \ \ \textit{if } g^S \textit{ is not conjugate under } G \textit{ to an element in } H_n\tau,$$

*and*

$$\begin{aligned}
\chi_{L(\zeta^S)\Pi_n^S}(g^S) &= \sum_{\sigma'\in S_{[\eta_n]}\backslash\mathfrak{S}_n} \eta_n\bigl(\sigma'(d^S)\bigr)\cdot \mathrm{tr}\bigl(K^U(\sigma'\sigma\sigma'^{-1})\bigr)\\
&= \frac{1}{|S_{[\eta_n]}|}\sum_{\kappa\in S_{[\eta_n]}}\ \sum_{\substack{\sigma'\in\mathfrak{S}_n:\\ \sigma'\sigma\sigma'^{-1}=\kappa\tau}} \eta_n\bigl(\sigma'(d^S)\bigr)\cdot\mathrm{tr}\bigl(K^U(\kappa\tau)\bigr)\\
&= \frac{1}{|S_{[\eta_n]}|}\sum_{\kappa\in S_{[\eta_n]}}\ \sum_{\substack{\sigma'\in\mathfrak{S}_n:\\ \sigma'\sigma\sigma'^{-1}=\kappa\tau}} \eta_n\bigl(\sigma'(d^S)\bigr)\\
&\qquad\times\prod_{\zeta\in\Delta_n}\chi_{\pi(\lambda^{n,\zeta})}\bigl(\tau\kappa_{(\zeta^S)^{p-1}\zeta}\,\tau\kappa_{(\zeta^S)^{p-2}\zeta}\cdots\tau\kappa_{(\zeta^S)\zeta}\,\tau\kappa_\zeta\bigr).
\end{aligned}$$

We define the *normalized virtual character* as $\widetilde{\chi}_{L(\zeta^S)\Pi_n^S} = \chi_{L(\zeta^S)\Pi_n^S}/\dim\Pi_n$. Then, since $\dim\Pi_n = |\mathfrak{S}_n|/|S_{[\eta_n]}|\cdot\dim\pi_n$, we get the following corollary.

COROLLARY 8.10. *The normalized virtual character* $\widetilde{\chi}_{L(\zeta^S)\Pi_n^S} = \chi_{L(\zeta^S)\Pi_n^S}/\dim\Pi_n$ *is given as follows: for* $g^S = (d^S,\sigma) \in G^S = D_n^S \rtimes \mathfrak{S}_n$*,*

$$\begin{aligned}
\widetilde{\chi}_{L(\zeta^S)\Pi_n^S}(g^S) &= \frac{1}{|\mathfrak{S}_n|}\sum_{\kappa\in S_{[\eta_n]}}\ \sum_{\substack{\sigma'\in\mathfrak{S}_n:\\ \sigma'\sigma\sigma'^{-1}=\kappa\tau}} \eta_n\bigl(\sigma'(d^S)\bigr)\\
&\times\prod_{\zeta\in\Delta_n}\frac{1}{\bigl(\dim\pi(\lambda^{n,\zeta})\bigr)^{p-1}}\cdot\widetilde{\chi}_{\pi(\lambda^{n,\zeta})}\bigl(\tau\kappa_{(\zeta^S)^{p-1}\zeta}\,\tau\kappa_{(\zeta^S)^{p-2}\zeta}\cdots\tau\kappa_{(\zeta^S)\zeta}\,\tau\kappa_\zeta\bigr).
\end{aligned}$$

Now consider the situation in Theorem 8.6 (iii), where picking up $\Pi_n$ with reducible $\Pi_n^S = \Pi_n|_{G_n^S}$, we have still a subsequence such that $\lim_{n\to\infty}\widetilde{\chi}_{\Pi_n} = f_A$.

THEOREM 8.11. *Suppose that for these reducible* $\Pi_n = \mathrm{Ind}_{H_n}^{G_n} \pi_n$, *the groups* $\mathcal{Z}(\pi_n) \subset (T/\overline{S})^\frown$, *which generate intertwining operators of* $\Pi_n^S$, *become stable along with a subsequence* $n = n_q$, $q \to \infty : n_1 < n_2 < \cdots \nearrow \infty$, *or* $\mathcal{Z}(\pi_{n_q}) = \mathcal{Z}(\pi_{n_{q+1}})$ *for* $q \gg 1$. *In particular, this is realized if* $T$ *is finite or* $\overline{S}$ *is open in* $T$.

*Then, the number of irreducible components of* $\Pi_n^S$ *becomes stable as equal to the order of the group* $\mathcal{Z} = \lim_{n\to\infty} \mathcal{Z}(\pi_n)$ *along this subsequence* $n = n_q$. *Corresponding to each minimal projections* $Q = Q_{u_1,u_2,\ldots,u_s}$ *in* (8.14) *onto irreducible components, there exist a sequence* $\Pi_n^{(Q)} = Q \cdot \Pi_n^S \cdot Q$ *of IURs of* $G_n^S$, $n = n_q \nearrow \infty$, *with dimension* $\dim \Pi_n^{(Q)} = \dim \Pi_n / |\mathcal{Z}|$. *The limits of their normalized characters* $\widetilde{\chi}_{\Pi_n^{(Q)}} = \chi_{\Pi_n^{(Q)}} / \dim \Pi_n^{(Q)}$ *are all the same and equal to* $f_A^S = f_A|_{G^S}$.

Taking into account Lemma 8.7 and its proof, we see that, to prove this theorem, it is enough to prove that, for any non-trivial $\zeta^S \in \mathcal{Z}$, the normalized virtual character $\widetilde{\chi}_{L(\zeta^S)\Pi_n^S} = \chi_{L(\zeta^S)\Pi_n^S}(g^S)/\dim \Pi_n$ tends to 0 as $n \to \infty$. More than this, we prove that, for a fixed $G_k^S = \mathfrak{S}_k(T)^S$, $\widetilde{\chi}_{L(\zeta^S)\Pi_n^S} = 0$ on $G_k^S$ if $n > k$.

We give below an explicit evaluation of the absolute value $|\widetilde{\chi}_{L(\zeta^S)\Pi_n^S}(g^S)|$ on $G_n^S$. From this evaluation and a similar one for $\big|\widetilde{\chi}_{\Pi_n}(g^S) - f_A^S(g^S)\big|$, we can get an asymptotic evaluation of $\widetilde{\chi}_{\Pi_n^{(Q)}}$ around its limit $f_A^S = \lim_{n\to\infty} \widetilde{\chi}_{\Pi_n^{(Q)}}$, or an evaluation of

$$\sup_{G_n^S} \Big| \widetilde{\chi}_{\Pi_n^{(Q)}} - \lim_{n\to\infty} \widetilde{\chi}_{\Pi_n^{(Q)}} \Big| . \tag{8.19}$$

PROPOSITION 8.12. *Let* $\zeta^S \in \mathcal{Z}(\pi_n)$ *be non-trivial and let* $p$ *be its order, then* $n = pN$. *For* $g^S = (d^S, \sigma) \in G^S = D_n^S \rtimes \mathfrak{S}_n$, *let* $\sigma = \sigma_1\sigma_2\cdots\sigma_m$ *be a cycle decomposition of* $\sigma$, *and put* $\ell_j = \ell(\sigma_j)$ *the length of* $\sigma_j$. *Then*

$$\big|\widetilde{\chi}_{L(\zeta^S)\Pi_n^S}(g^S)\big| \le \mathcal{E}_n(\sigma), \tag{8.20}$$

*where* $\mathcal{E}_n(\sigma) = 0$ *if* $|\mathrm{supp}(\sigma)| = \sum_{1\le j\le m} \ell_j < n$ *or one of* $\ell_j$ *is not a multiple of* $p$, *otherwise*

$$\begin{aligned} \mathcal{E}_n(\sigma) &= \frac{M_n(\sigma) \cdot p^m \cdot \prod_{\zeta' \in \widehat{T}'_n} |I_{n,\zeta'}|!}{n!} \\ &= \frac{M_n(\sigma) \cdot p^m \cdot \prod_{\zeta \in \Delta_n} (|I_{n,\zeta}|!)^p}{n!}, \end{aligned} \tag{8.21}$$

*where* $M_n(\sigma)$ *denotes the number of partitions* $\mathcal{J} = \{J_\zeta\}_{\zeta\in\Delta_n}$ *of* $J = \{1, 2, \ldots, m\}$ *satisfying*

$$\sum_{j\in J_\zeta} \ell_j = |I_{Z(\zeta)}| = p\,|I_{n,\zeta}| \quad (\forall\zeta \in \Delta_n). \tag{8.22}$$

*Proof.* We use the explicit formula in Corollary 8.10 to evaluate the absolute value of $\widetilde{\chi}_{L(\zeta^S)\Pi_n^S}(g^S)$ as

(8.23)
$$\bigl|\widetilde{\chi}_{L(\zeta^S)\Pi_n^S}(g^S)\bigr| \le \frac{\mathcal{E}'_n(\sigma)}{n!}, \quad \mathcal{E}'_n(\sigma) := \bigl|\bigl\{\sigma' \in \mathfrak{S}_n\,;\, \sigma'\sigma\sigma'^{-1} \in S_{[\eta_n]}\tau\bigr\}\bigr|.$$

Recall that $\tau$ corresponds to $\zeta^S$ as $\tau(I_{n,\zeta'}) = I_{n,\zeta^S\zeta'}$ $(\zeta' \in \widehat{T}'_n)$ with order $p$. Put $I_{Z(\zeta)} := \bigsqcup_{\zeta'\in Z(\zeta)} I_{n,\zeta'}$ with $Z(\zeta) = \{\zeta, \zeta^S\zeta, \ldots, (\zeta^S)^{p-1}\zeta\}$, then

$$\boldsymbol{I}_n = \bigsqcup_{\zeta\in\Delta_n} I_{Z(\zeta)}, \quad \bigl|I_{Z(\zeta)}\bigr| = p\,|I_{n,\zeta}|, \quad \sum_{\zeta\in\Delta_n} |I_{n,\zeta}| = N.$$

We see that, for $\sigma' \in \mathfrak{S}_n$, the condition $\sigma'\sigma\sigma'^{-1} \in S_{[\eta_n]}\tau$ is equivalent to

$$\sigma'\sigma\sigma'^{-1}(I_{n,\zeta'}) = I_{n,\zeta^S\zeta'} \quad (\zeta' \in \widehat{T}'_n). \tag{8.24}$$

The above condition is closed inside of each $I_{Z(\zeta)}$ for $\zeta \in \Delta_n$. Take a cycle $\sigma_j = (i_1\ \ i_2\ \ \cdots\ \ i_{\ell_j})$ of $\sigma$ with $i_1 = \min\{i_1, i_2, \ldots, i_{\ell_j}\}$, then $\sigma'\sigma_j\sigma'^{-1} = \bigl(\sigma'(i_1)\ \ \sigma'(i_2)\ \ \cdots\ \ \sigma'(i_{\ell_j})\bigr)$, and the above condition says that

$(\star)$ *if* $\sigma'(i_1) \in I_{n,\zeta'}$*, then* $\sigma'(i_2) \in I_{n,\zeta^S\zeta'}, \ldots, \sigma'(i_p) \in I_{n,(\zeta^S)^{p-1}\zeta'}$*, and so on, that is,* $\sigma'(i_{k'p+k})$*,* $k'p + k \le \ell_j$ $(0 \le k \le p-1)$*, belongs to* $I_{n,(\zeta^S)^k\zeta'}$ *cyclically modulo* $p$*.*

Now suppose the condition (8.24) holds. Put $K_j := \mathrm{supp}(\sigma_j)$, then $|K_j| = \ell_j$, $\bigsqcup_{1\le j\le m} K_j = \boldsymbol{I}_n$, and $\sigma'(K_j) \subset I_{Z(\zeta)}$ so that $K_j$'s are grouped up into $|\Delta_n|$ number of subsets as

$$J = \{1, 2, \ldots, m\} = \bigsqcup_{\zeta\in\Delta_n} J_\zeta, \quad \bigsqcup_{j\in J_\zeta} \sigma'(K_j) = I_{Z(\zeta)} \ \ (\zeta \in \Delta_n). \tag{8.25}$$

Denote by $\mathcal{J} = \{J_\zeta\}_{\zeta\in\Delta_n}$ a partition of $J$ satisfying

$$\sum_{j\in J_\zeta} |K_j| = |I_{Z(\zeta)}| = p\,|I_{n,\zeta}| \quad (\forall\zeta \in \Delta_n), \tag{8.26}$$

and pick up one $\sigma' \in \mathfrak{S}_n$ satisfying (8.25) and also ($\star$), and denote it by $\sigma_{\mathcal{J}}$.

Fix such a $\mathcal{J}$. Then the deviation of general $\sigma'$ satisfying (8.25) and $\sigma_{\mathcal{J}}$ comes from $\sigma'' \in \prod_{\zeta \in \Delta_n} \mathfrak{S}_{I_{Z(\zeta)}}$ such that $\sigma' = \sigma'' \sigma_{\mathcal{J}}$. To choose $\sigma'(i_1)$, we first fix $\zeta' \in Z(\zeta)$ such that $\sigma'(i_1) \in I_{n,\zeta'}$ and the number of choices are $p = |Z(\zeta)|$. Thus the total number of $\sigma' \in \mathfrak{S}_n$ satisfying (8.25) for $\mathcal{J}$ is

$$\prod_{\zeta \in \Delta_n} \Big( p^{|J_\zeta|} \prod_{\zeta' \in Z(\zeta)} |I_{n,\zeta'}|! \Big) = p^m \cdot \prod_{\zeta \in \Delta_n} (|I_{n,\zeta}|!)^p.$$

The sum over $\mathcal{J}$ gives the number $\mathcal{E}_n'(\sigma)$. ∎

COROLLARY 8.13. *Let $\zeta^S \in \mathcal{Z}(\pi_n)$ be non-trivial and let $p$ be its order, then $n = pN$, and*

$$\sup_{g^S \in G_n^S} \big| \widetilde{\chi}_{L(\zeta^S)\Pi_n^S}(g^S) \big| \le \mathcal{E}_n, \quad \mathcal{E}_n = \frac{N! \cdot p^N \cdot \prod_{\zeta \in \Delta_n} (|I_{n,\zeta}|!)^{p-1}}{n!}.$$

*Proof.* In (8.21), the maximum $\mathcal{E}_n$ of $\mathcal{E}_n(\sigma)$ is achieved in the case where $m = N$ and all $\ell_j$ are equal to $p$. In that case, $M_n(\sigma) = N!/\big(\prod_{\zeta \in \Delta_n} |I_{n,\zeta}|!\big)$. ∎

## §9. Cases of infinite Weyl groups of type BC and of type D

Take $T = \boldsymbol{Z}_2$ and put $G_n = \mathfrak{S}_n(\boldsymbol{Z}_2) = D_n(\boldsymbol{Z}_2) \rtimes \mathfrak{S}_n$ and $G = \mathfrak{S}_n(\boldsymbol{Z}_2)$. Then $G_n$ is isomorphic to the Weyl group $W_{\mathrm{BC}_n}$ of type $\mathrm{BC}_n$, and $G$ is called the infinite Weyl group $W_{\mathrm{BC}_\infty}$ of type BC. Take $S = \{e_T\} = \{1\}$ the trivial subgroup of $T$, the subgroups $G_n^S$ and $G^S$ is defined as

(9.1) $$G_n^S := \{(d, \sigma) \in G_n \,;\, P(d) \in S\}, \quad G^S := \{(d, \sigma) \in G \,;\, P(d) \in S\},$$

where $P(d) := \prod_{i \in \boldsymbol{I}_n} t_i$ for $d = (t_i)_{i \in \boldsymbol{I}_n}$. Then $G_n^S$ is isomorphic to the Weyl group $W_{\mathrm{D}_n}$ of type $\mathrm{D}_n$, and $G^S$ is called the infinite Weyl group $W_{\mathrm{D}_\infty}$ of type D.

As representative systems of cosets $G_n/G_n^S$ and $G/G^S$, we can take $\{e,\, h^0 = (d^0, \mathbf{1})\}$ with $d^0 = (-1, 1, 1, \ldots)$. When $n$ is odd, $h^0$ can be replaced by a central element $h^{(1)} = (d^{(1)}, \mathbf{1})$ with $d^{(1)} = (-1, -1, \ldots, -1)$.

The characters of infinite Weyl groups of type BC and of type D are studied in detail in [HH1], and summarized in Section 6 of [HH4]. For the stochastic discussions using the space of paths in Dynkin diagrams, see Part II of the present work [HHH2].

### 9.1. Case of Weyl groups $W_{\mathrm{BC}_n}$ and $W_{\mathrm{BC}_\infty}$

For the infinite Weyl group $G = \mathfrak{S}_\infty(\boldsymbol{Z}_2) \cong W_{\mathrm{BC}_\infty}$, we have $\widehat{T} = \{\zeta^0, \zeta^1\}$ with $T = \boldsymbol{Z}_2 = \{1, -1\}$, where $\zeta_s(\epsilon) = \epsilon^s$ $(s = 0, 1)$ for $\epsilon \in \boldsymbol{Z}_2$. All characters on $G$ are given as $f_A$ with parameters

$$(9.2) \qquad \begin{cases} A = \Big( (\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\,;\mu \Big), \quad \mu = (\mu_\zeta)_{\zeta\in\widehat{T}}, \ \ \mu_\zeta \geq 0, \\ \alpha_{\zeta,\varepsilon} = (\alpha_{\zeta,\varepsilon,i})_{i\in\boldsymbol{N}}: \ \ \alpha_{\zeta,\varepsilon,1} \geq \alpha_{\zeta,\varepsilon,2} \geq \alpha_{\zeta,\varepsilon,3} \geq \cdots \geq 0, \\ \sum_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}} \|\alpha_{\zeta,\varepsilon}\| + \|\mu\| = 1. \end{cases}$$

We put as in [HH1] and [HH6, §6],

$$(9.3) \qquad \alpha := \alpha_{\zeta^0,0}\,, \ \ \beta := \alpha_{\zeta^0,1}\,, \ \ \gamma := \alpha_{\zeta^1,0}\,, \ \ \delta := \alpha_{\zeta^1,1}\,; \ \ \kappa := \mu_{\zeta^0} - \mu_{\zeta^1}.$$

Then $\qquad \|\alpha\| + \|\beta\| + \|\gamma\| + \|\delta\| + |\kappa| \leq 1.$

A unique non-trivial $\zeta^S \in (T/S)\widehat{\ } = \widehat{T}$ is $\zeta^S = \zeta^1$, and its action on the parameter $A$ defined by $\alpha_{\zeta,\varepsilon} \to \alpha_{\zeta^S\zeta,\varepsilon}$, $\mu_\zeta \to \mu_{\zeta^S\zeta}$ is realized in the new parameter as an exchange of $(\alpha, \beta)$ and $(\gamma, \delta)$, and $\kappa \to -\kappa$, that is,

$$(9.4) \qquad (\alpha, \beta\,;\, \gamma, \delta\,;\, \kappa) \longrightarrow (\gamma, \delta\,;\, \alpha, \beta\,;\, -\kappa).$$

The character $f_A$ of $G$ is factorizable. For $g \in G$, take its standard decomposition

$$(9.5) \qquad g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r} g_1 g_2 \cdots g_m,$$
$$\xi_q = (t_q, (q)), \ t_q = -1 \in \boldsymbol{Z}_2, \ \ g_j = (d_j, \sigma_j), \ \mathrm{supp}(d_j) \subset \mathrm{supp}(\sigma_j),$$

and $\sigma_j$ is a cycle with length $\ell_j = \ell(\sigma_j)$. Then, $f_A(g)$ is expressed as

$$(9.6) \qquad f_A(g) = \prod_{1\leq k\leq r} \Phi_1^{\alpha,\beta;\gamma,\delta;\kappa}(\xi_{q_k}) \cdot \prod_{1\leq j\leq m} \Phi_{\ell_j}^{\alpha,\beta;\gamma,\delta;\kappa}\big((d_j, \sigma_j)\big),$$

with factors $\Phi_\ell^{\alpha,\beta;\gamma,\delta;\kappa}$, $\ell \geq 1$, given as

$$\begin{aligned} \Phi_1^{\alpha,\beta;\gamma,\delta;\kappa}\big((t_q, (q))\big) &= \sum_{i\in\boldsymbol{N}} \alpha_i + \sum_{i\in\boldsymbol{N}} \beta_i + \mu_{\zeta^0} - \sum_{i\in\boldsymbol{N}} \gamma_i - \sum_{i\in\boldsymbol{N}} \delta_i - \mu_{\zeta^1} \\ &= \|\alpha\| + \|\beta\| - \|\gamma\| - \|\delta\| + \kappa, \\ \Phi_{\ell_j}^{\alpha,\beta;\gamma,\delta;\kappa}\big((d_j, \sigma_j)\big) &= \sum_{i\in\boldsymbol{N}} (\alpha_i)^{\ell_j} + (-1)^{\ell_j-1} \sum_{i\in\boldsymbol{N}} (\beta_i)^{\ell_j} \\ &\quad + \zeta^1\big(P(d_j)\big) \sum_{i\in\boldsymbol{N}} (\gamma_i)^{\ell_j} + \zeta^1\big(P(d_j)\big)(-1)^{\ell_j-1} \sum_{i\in\boldsymbol{N}} (\delta_i)^{\ell_j}. \end{aligned}$$

An IUR $\Pi_n = \mathrm{Ind}_{H_n}^{G_n} \pi_n$ of $G_n \cong W_{\mathrm{BC}_n}$ are parametrized by a pair of Young diagrams $\Lambda^n = (\lambda^{n,\lambda_0}, \lambda^{n,\lambda_1})$ since $\widehat{T} = \{\zeta^0, \zeta^1\}$. Its normalized character $\widetilde{\chi}_{\Pi_n}$ is given explicitly by Theorem 4.5 as follows. Take $g = (d, \sigma) \in G_n$ and let its standard decomposition be as in (9.5), and put $Q = \{q_1, q_2, \ldots, q_r\}$ and $J = \{1, 2, \ldots, m\}$, then

$$\widetilde{\chi}_{\Pi_n}(g) = \sum_{\mathcal{Q},\mathcal{J}} c(\Lambda^n; \mathcal{Q}, \mathcal{J})\, X\big(\Lambda^n; \mathcal{Q}, \mathcal{J}; g\big), \tag{9.7}$$

$$c(\Lambda^n; \mathcal{Q}, \mathcal{J}) = \frac{\prod_{\zeta \in \widehat{T}} |\lambda^{n,\zeta}|(|\lambda^{n,\zeta}| - 1) \cdots \big(|\lambda^{n,\zeta}| - |Q_\zeta| - \sum_{j \in J_\zeta} |K_j| + 1\big)}{n(n-1)(n-2) \cdots \big(n - |Q| - |\operatorname{supp}(\sigma)| + 1\big)},$$

$$X\big(\Lambda^n; \mathcal{Q}, \mathcal{J}; g\big) = \widetilde{\chi}\big(\lambda^{n,\zeta^0}; (\ell(\sigma_j))_{j \in J_{\zeta^0}}\big) \times (-1)^{|Q_{\zeta^1}|} \cdot \prod_{j \in J_{\zeta^1}} \zeta^1\big(P(d_j)\big) \times \widetilde{\chi}\big(\lambda^{n,\zeta^1}; (\ell(\sigma_j))_{j \in J_{\zeta^1}}\big),$$

where $\mathcal{Q} = (Q_\zeta)_{\zeta \in \widehat{T}}$ and $\mathcal{J} = (J_\zeta)_{\zeta \in \widehat{T}}$ run over partitions of $Q$ and $J$ respectively, and $\widetilde{\chi}\big(\lambda^{n,\zeta}; *\big)$ denotes the normalized character of IUR $\pi(\lambda^{n,\zeta})$.

PROPOSITION 9.1. *The parameters for the limit $f_A = \lim_{n\to\infty} \widetilde{\chi}_{\Pi_n}$ are given as*

$$\alpha_i = \lim_{n\to\infty} \frac{r_i(\lambda^{n,\lambda_0})}{n}, \quad \beta_i = \lim_{n\to\infty} \frac{c_i(\lambda^{n,\lambda_0})}{n};$$
$$\gamma_i = \lim_{n\to\infty} \frac{r_i(\lambda^{n,\lambda_1})}{n}, \quad \delta_i = \lim_{n\to\infty} \frac{c_i(\lambda^{n,\lambda_1})}{n},$$
$$\mu_{\zeta^0} = \lim_{n\to\infty} \frac{|\lambda^{n,\lambda_0}|}{n}, \quad \mu_{\zeta^1} = \lim_{n\to\infty} \frac{|\lambda^{n,\lambda_1}|}{n}.$$

### 9.2. Case of Weyl groups $W_{\mathrm{D}_n}$ and $W_{\mathrm{D}_\infty}$

Characters of $G^S \cong W_{\mathrm{D}_\infty}$ are obtained by the restriction $f_A^S = f_A|_{G^S}$ of $f_A$. If $g$ belongs to $G^S$, its standard decomposition (9.5) in $G$ satisfies

$$(-1)^r \prod_{1 \le j \le m} \zeta^1\big(P(d_j)\big) = 1.$$

We proved in [HH4, §17] and [HH6, §15] that $f_A^S = f_{A'}^S$ with $(\alpha', \beta'; \gamma', \delta'; \kappa')$ for $A'$ if and only if

$$(\alpha', \beta'; \gamma', \delta'; \kappa') = (\alpha, \beta; \gamma, \delta; \kappa) \;\text{ or }\; (\gamma, \delta; \alpha, \beta; -\kappa),$$

and in these cases, we have $f_{A'} = f_A$ or $f_{A'}(g) = \zeta^1\big(P(d)\big)\cdot f_A(g)$ for $g=(d,\sigma)$.

The inverse image of the restriction $f_A \to f_A^S$ is unique if and only if $(\alpha,\beta;\gamma,\delta;\kappa)=(\gamma,\delta;\alpha,\beta;-\kappa)$, or

$$\alpha_{\zeta^0,\varepsilon}=\alpha_{\zeta^1,\varepsilon}\ \ (\varepsilon=0,1), \quad \mu_{\zeta^0}=\mu_{\zeta^1}. \tag{9.8}$$

Proposition 9.2. *The case* (9.8) *is exactly equal to the case where the character $f_A$ is totally zero outside of $G_n^S$.*

*Proof.* The condition $f_A=0$ on $G_n\setminus G_n^S$ is equivalent to

$$\Phi_1^{\alpha,\beta;\gamma,\delta;\kappa}=0 \quad \text{and} \quad \Phi_{\ell_j}^{\alpha,\beta;\gamma,\delta;\kappa}\big((d_j,\sigma_j)\big)=0 \quad \text{if } P(d_j)=-1.$$

In turn, these conditions are equivalent to

$$\|\alpha\|+\|\beta\|-\|\gamma\|-\|\delta\|+\kappa=0,$$
$$\sum_{i\in\boldsymbol{N}}(\alpha_i)^{\ell_j}+(-1)^{\ell_j-1}\sum_{i\in\boldsymbol{N}}(\beta_i)^{\ell_j}=\sum_{i\in\boldsymbol{N}}(\gamma_i)^{\ell_j}+(-1)^{\ell_j-1}\sum_{i\in\boldsymbol{N}}(\delta_i)^{\ell_j},$$

for $\ell_j\ge 2$. Multiply by $z^{\ell_j-2}$ both sides of the last equation and sum up over $\ell_j\ge 2$. Comparing poles of obtained functions in $z$, we see $\alpha_i=\gamma_i$, $\beta_i=\delta_i$ $(i\ge 1)$. Then we get from the first equation that $\kappa=\mu_{\zeta^0}-\mu_{\zeta^1}=0$. □

By Lemma 8.4, the restriction $\Pi_n^S=\Pi_n|_{G_n^S}$ of $\Pi_n=\mathrm{Ind}_{H_n}^{G_n}\pi_n$ is reducible if and only if the group $\mathcal{Z}(\pi_n)$ in (8.10) is not trivial. Here $\mathcal{Z}(\pi_n)=\{\zeta^S\in\widehat{\boldsymbol{Z}_2}\,;\,\lambda^{n,\zeta^1}=\lambda^{n,\zeta^S\zeta^0}\}$.

In the reducible case, we have $\mathcal{Z}(\pi_n)=\{\zeta^0,\zeta^1\}$, and for $\zeta^S=\zeta^1$, take a unique $\tau=\tau_n\in\mathfrak{S}_n$ such that $\tau^2=\mathbf{1}$ and $\tau(I_{n,\zeta^0})=I_{n,\zeta^1}$ in an order-preserving manner. Then the intertwining operator $L(\zeta^S)=L(\zeta^1)\in\mathcal{L}\big(V(\Pi_n)\big)$ is defined by $K(\tau)=c\,U(\tau)\in\mathcal{L}\big(V(\pi_n)\big)$, where $c=\mu_{G_n}(H_n)^{-1}=|\mathfrak{S}_n|/|S_{[\eta_n]}|$, and $U(\tau)$ is a simultaneous exchange through $\tau$ of factors $\zeta^0$, $\zeta^1$ in $\eta_n=\prod_{i\in\boldsymbol{I}_n}\zeta_i$ with $\zeta_i=\zeta^k$ $(i\in I_{n,\zeta^k},\ k=0,1)$ for $D_n=\boldsymbol{Z}_2^n$ and of factors $V\big(\pi(\lambda^{n,\zeta^0})\big)$, $V\big(\pi(\lambda^{n,\zeta^1})\big)$ in the tensor product space $V(\xi_n)=V\big(\pi(\lambda^{n,\zeta^0})\big)\otimes V\big(\pi(\lambda^{n,\zeta^1})\big)$ for $S_{[\eta_n]}=\mathfrak{S}_{I_{n,\zeta^0}}\times\mathfrak{S}_{I_{n,\zeta^1}}$.

From Theorem 8.6 (iii) together with the above facts, we have the following.

PROPOSITION 9.3. *The case* (9.8) *is exactly equal to the case where a sequence of reducible* $\Pi_n^S = \Pi_n|_{G_n^S}$ *can attain* $f_A^S$ *as its limit.*

In the case where the restriction $\Pi_n^S = \Pi_n|_{G_n^S}$ is reducible, we have two projections onto two irreducible components as

$$Q_+ = \frac{I + L(\zeta^1)}{2}, \quad Q_- = \frac{I - L(\zeta^1)}{2}, \quad L(\zeta^1) = Q_+ - Q_-.$$

From Lemma 8.7, the dimensions of two irreducible constituents $\Pi_n^{(\pm)} := Q_\pm \Pi_n Q_\pm$ are equal and $\dim \Pi_n^{(\pm)} = \frac{1}{2} \dim \Pi_n$.

Put $N = n/2 = |I_{n,\zeta^0}| = |I_{n,\zeta^1}|$, then $S_{[\eta_n]} = \mathfrak{S}_{I_{n,\zeta^0}} \times \mathfrak{S}_{I_{n,\zeta^1}} \cong \mathfrak{S}_N \times \mathfrak{S}_N$. According to this decomposition, $\kappa \in \mathfrak{S}_{[\eta_n]}$ is decomposed as $\kappa = \kappa_{\zeta^0} \kappa_{\zeta^1}$ with $\kappa_\zeta \in \mathfrak{S}_{I_{n,\zeta}}$. By applying Corollary 8.10, we get the following character formula.

PROPOSITION 9.4. *For an even integer* $n = 2N > 0$, *normalized characters of two irreducible components* $\Pi_n^{(\pm)} := Q_\pm \Pi_n Q_\pm$ *of* $\Pi_n$ *are given as follows. For a* $g^S = (d^S, \sigma) \in G_n^S = D_n^S \rtimes \mathfrak{S}_n$,

$$\widetilde{\chi}_{\Pi_n^{(\pm)}}(g^S) := \frac{1}{\dim \Pi_n^{(\pm)}} \chi_{\Pi_n^{(\pm)}}(g^S) = \widetilde{\chi}_{\Pi_n}(g^S) \pm \widetilde{\chi}_{L(\zeta^1)\Pi_n}(g^S),$$

*and* $\widetilde{\chi}_{\Pi_n^{(+)}}(g^S) = \widetilde{\chi}_{\Pi_n^{(-)}}(g^S)$ *if* $g^S$ *is not conjugate under* $G_n$ *to an element in* $H_n\tau$, *and*

$$\begin{aligned}
&\widetilde{\chi}_{\Pi_n^{(+)}}(g^S) - \widetilde{\chi}_{\Pi_n^{(-)}}(g^S) = 2\,\widetilde{\chi}_{L(\zeta^S)\Pi_n^S}(g^S) \\
&= \frac{2}{n!} \sum_{\kappa \in S_{[\eta_n]}} \sum_{\substack{\sigma' \in \mathfrak{S}_n: \\ \sigma'\sigma\sigma'^{-1} = \kappa\tau}} \eta_n\big(\sigma'(d^S)\big) \cdot \frac{1}{\dim \pi(\lambda^{n,\zeta^0})} \cdot \widetilde{\chi}_{\pi(\lambda^{n,\zeta^0})}\big(\tau\kappa_{\zeta^1}\,\tau\kappa_{\zeta^0}\big).
\end{aligned}$$

By this result and the character formula for $\widetilde{\chi}_{\Pi_n}$, the normalized irreducible characters $\widetilde{\chi}_{\Pi_n^{(\pm)}}$ are explicitly given, and so we can evaluate separately asymptotic behaviors of $\widetilde{\chi}_{\Pi_n^{(+)}}(g^S)$ and $\widetilde{\chi}_{\Pi_n^{(-)}}(g^S)$ as $n \to \infty$.

Here we evaluate their difference:

$$\begin{aligned}
\big|\widetilde{\chi}_{\Pi_n^{(+)}}(g^S) - \widetilde{\chi}_{\Pi_n^{(-)}}(g^S)\big| &\leq \frac{2}{n!} \cdot \big|\{\sigma' \in \mathfrak{S}_n\,;\, \sigma'\sigma\sigma'^{-1} \in S_{[\eta_n]}\tau\}\big| \\
&=: \mathcal{E}_n^0(\sigma) \ \ \text{(put)}.
\end{aligned}$$

If $\sigma$ is not conjugate under $\mathfrak{S}_n$ to an element in $S_{[\eta_n]}\tau$, $\tau = \tau_n$, then $\mathcal{E}_n^0(\sigma) = 0$.

Otherwise, $\sigma \neq \mathbf{1}$ and for $\sigma' \in \mathfrak{S}_n$,

$$\sigma'\sigma\sigma'^{-1} \in S_{[\eta_n]}\tau \Longleftrightarrow \sigma'\sigma\sigma'^{-1}(I_{n,\zeta^0}) = I_{n,\zeta^1} \text{ and } \sigma'\sigma\sigma'^{-1}(I_{n,\zeta^1}) = I_{n,\zeta^0},$$

because $\boldsymbol{I}_n = I_{n,\zeta^0} \sqcup I_{n,\zeta^1}$. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_m$ be a cycle decomposition of $\sigma$, then $\sigma'\sigma\sigma'^{-1} = \sigma_1'\sigma_2'\cdots\sigma_m'$ with $\sigma_j' := \sigma'\sigma_j\sigma'^{-1}$ is that of $\sigma'\sigma\sigma'^{-1}$. Take a $\sigma_j = (i_1 \ \ i_2 \ \ \cdots \ \ i_{\ell_j})$, then $\sigma_j' = \big(\sigma'(i_1) \ \ \sigma'(i_2) \ \ \cdots \ \ \sigma'(i_{\ell_j})\big)$. The above property of $\sigma'\sigma\sigma'^{-1}$ is equivalent to that, in this expression of $\sigma_j'$, elements of $I_{n,\zeta^0}$ and $I_{n,\zeta^1}$ appear alternately, for $1 \leq \forall j \leq m$.

Therefore, in case where some length $\ell_j = \ell(\sigma_j)$ is odd for $\sigma$, then $\mathcal{E}_n^0(\sigma) = 0$. This is also true if $\sigma$ contains a "cycle of length 1", or $\sum_{1\leq j\leq m} \ell_j = |\operatorname{supp}(\sigma)| < n$.

Suppose all $\ell_j$ are even and $|\operatorname{supp}(\sigma)| = n$. Let $i_1$ in $\sigma_j$ be the minimum of $i_1, i_2, \ldots, i_{\ell_j}$, then we have two possibility: $\sigma'(i_1) \in I_{n,\zeta^0}$ or $\sigma'(i_1) \in I_{n,\zeta^1}$. Counting this for all $\sigma_j$, we get the number of $\sigma'$ satisfying the above condition so that, with $n = 2N$,

$$\mathcal{E}_n^0(\sigma) = \frac{2 \cdot 2^m \, (N!)^2}{(2N)!}. \tag{9.9}$$

Thus we get an evaluation for the difference of normalized characters as follows.

PROPOSITION 9.5. (i) *For $g^S = (d^S, \sigma) \in G_n^S = D_n^S \rtimes \mathfrak{S}_n$ with $n = 2N$,*

$$\big|\widetilde{\chi}_{\Pi_n^{(+)}}(g^S) - \widetilde{\chi}_{\Pi_n^{(-)}}(g^S)\big| \leq \mathcal{E}_n^0(\sigma),$$

*where $\mathcal{E}_n^0(\sigma) = 0$ if $\sigma$ contains a cycle of odd length or $|\operatorname{supp}(\sigma)| < n$, otherwise $\mathcal{E}_n^0(\sigma)$ is given by (9.9) when $\sigma$ is decomposed into $m$ cycles.*

(ii) *For a fixed $G_k^S = \mathfrak{S}_k(T)^S$, we have $\widetilde{\chi}_{\Pi_n^{(+)}}(g^S) = \widetilde{\chi}_{\Pi_n^{(-)}}(g^S)$ $(g^S \in G_k^S)$ if $n > k$. Moreover*

$$\sup_{G_n^S} \big|\widetilde{\chi}_{\Pi_n^{(+)}} - \widetilde{\chi}_{\Pi_n^{(-)}}\big| \leq \mathcal{E}_n^0, \quad \mathcal{E}_n^0 = \frac{2 \cdot 2^N (N!)^2}{(2N)!}. \tag{9.10}$$

# Chapter III. Analysis of limiting process of induced characters of $\mathfrak{S}_n(T)$ as $n \to \infty$

## §10. Problem setting for limiting process of induced characters of $\mathfrak{S}_n(T)$

### 10.1. Centralization of positive definite functions and their limits

In the series of works [Hir2]–[Hir3], [HH1]–[HH2] and [HH4]–[HH5], to get characters of $G = \mathfrak{S}_\infty$, those of $G = \mathfrak{S}_\infty(T)$ with $T$ finite abelian group, and of $G = \mathfrak{S}_\infty(T)$ with $T$ finite group and then with $T$ compact group in general, we have applied the method of taking limits of centralizations of the trivial inducing up $F = \mathrm{Ind}_H^G f_\pi$ of a diagonal matrix element $f_\pi$ of a unitary representation (= UR) $\pi$ of a subgroup $H$ of $G$. The limits thus obtained turn out to be characters, and moreover all the characters of $G$ are obtained in this manner.

Let us compare this method with the present method of taking limits of irreducible characters. To do so, first summarize our method in these previous papers. It goes principally along the following steps. Here let $G = \mathfrak{S}_\infty(T)$ with a compact group $T$.

Step I. We fix a subgroup $H$ of $G$, and an irreducible unitary representation (= IUR) $\pi$ of $H$ as follows. Take a partition of $\boldsymbol{N}$ as

$$\boldsymbol{N} = \Big( \bigsqcup_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}} \Big( \bigsqcup_{p\in P_{\zeta,\varepsilon}} I_p \Big)\Big) \sqcup \Big( \bigsqcup_{\zeta\in\widehat{T}} I_\zeta \Big), \tag{10.1}$$

where each $P_{\zeta,\varepsilon}$ is a countably infinite index set if not empty, and the subsets $I_p$ are infinite and so are $I_\zeta$ if not empty. Corresponding to this partition, we define a subgroup

$$H = \Big( \prod'_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}} \Big( \prod'_{p\in P_{\zeta,\varepsilon}} H_p \Big)\Big) \times \Big( \prod'_{\zeta\in\widehat{T}} H_\zeta \Big), \tag{10.2}$$

$$\text{with} \quad H_p = \mathfrak{S}_{I_p}(T), \quad H_\zeta = D_{I_\zeta}(T) \subset \mathfrak{S}_{I_\zeta}(T),$$

where $\prod'$ denotes the restricted direct product. As an IUR $\pi$ of $H$, we take

$$\pi = \Big( \bigotimes^b_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}} \Big( \bigotimes^{b_{\zeta,\varepsilon}}_{p\in P_{\zeta,\varepsilon}} \pi_p \Big)\Big) \otimes \Big( \bigotimes^{b'}_{\zeta\in\widehat{T}} \pi_\zeta \Big), \tag{10.3}$$

omitting appropriately the factors for empty $P_{\zeta,\varepsilon}$ and $I_\zeta$, where $b$, $b_{\zeta,\varepsilon}$, $b'$ are reference vectors. Here $b_{\zeta,\varepsilon} = (b_p)_{p\in P_{\zeta,\varepsilon}}$ with $b_p \in V(\pi_p)$, $\|b_p\| = 1$ $(p \in P_{\zeta,\varepsilon})$, and an IUR $\pi_p$ of $H_p = \mathfrak{S}_{I_p}(T)$ is given as

(10.4)

$$\pi_p\big((d,\sigma)\big) = \Big(\bigotimes\nolimits_{i\in I_p}^{a_p} \zeta_i(t_i)\Big)\, I(\sigma)\, \mathrm{sgn}_{\mathfrak{S}}(\sigma)^\varepsilon \quad \text{for } \ d = (t_i)_{i\in I_p},\ \sigma \in \mathfrak{S}_{I_p},$$

where $a_p = (a_i)_{i\in I_p}$ is a reference vector with $a_i \in V(\zeta_i)$, $\|a_i\| = 1$, and $\zeta_i = \zeta$ as an IUR of $T_i = T$ $(i \in I_p)$, and $I(\sigma)$ is defined as

(10.5)

$$I(\sigma) : v = \bigotimes\nolimits_{i\in I_p} v_i \longmapsto \bigotimes\nolimits_{i\in I_p} v_i',\ \ v_i' = v_{\sigma^{-1}(i)} \ \ (v_i \in V(\zeta_i),\ i \in I_p)\,.$$

Moreover $b' = (b_\zeta)_{\zeta\in\widehat{T}}$ with $b_\zeta \in V(\zeta)$, $\|b_\zeta\| = 1$, and for $\zeta \in \widehat{T}$, $\pi_\zeta$ of $H_\zeta$ is given as

$$\pi_\zeta(d) = \bigotimes\nolimits_{i\in I_\zeta}^{a_\zeta} \zeta_i(t_i) \quad \text{for } \ d = (t_i)_{i\in I_\zeta} \in H_\zeta = D_{I_\zeta}(T), \tag{10.6}$$

where $a_\zeta = (a_i)_{i\in I_\zeta}$ with $a_i \in V(\zeta_i)$, $\|a_i\| = 1$, and $\zeta_i = \zeta$ for $T_i = T$ $(i \in I_\zeta)$.

Step II. Put $\widehat{b} := \bigotimes_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}} b_{\zeta,\varepsilon}$, $b_{\zeta,\varepsilon} := \bigotimes_{p\in P_{\zeta,\varepsilon}} b_p$, and $\widehat{b'} := \bigotimes_{\zeta\in\widehat{T}} b_\zeta$, $b_\zeta = \bigotimes_{i\in I_\zeta} a_i$, then we have a unit vector $w_0 := \widehat{b}\otimes\widehat{b'} \in V(\pi)$. Take a diagonal matrix element for $w_0$ as

$$f_\pi(h) := \langle \pi(h)w_0, w_0\rangle, \quad f_\pi(e) = 1.$$

Denote by $\mathrm{Ind}_H^G f_\pi$ a trivial extension of $f_\pi$ to $G$, which is, by definition, equal to $f_\pi$ on $H$ and to zero outside $H$. Then $F := \mathrm{Ind}_H^G f_\pi$ is a positive definite function on $G$ normalized as $F(e) = 1$, and is continuous because $H$ is open in $G$.

Step III. For a continuous function $F$ on $G$ and a compact subgroup $G'$ of $G$, we define a *centralization* $F^{G'}$ of $F$ with respect to $G'$ as

$$F^{G'}(g) := \int_{G'} F(g'g(g')^{-1})\, d\mu_{G'}(g'), \tag{10.7}$$

where $\mu_{G'}$ denotes the normalized Haar measure on $G'$. Here we take $F = \mathrm{Ind}_H^G f_\pi$ and $G' = \mathfrak{S}_J(T) = D_J(T) \rtimes \mathfrak{S}_J$ for a finite subset $J \subset \boldsymbol{N}$, then $F^{G'}$ is a continuous positive definite function on $G$ invariant under $G'$. We calculate the centralization $F^{G'}$ explicitly.

Step IV. We choose an increasing sequence $J_N$, $N = 1, 2, \ldots$, of finite subsets of $\boldsymbol{N}$ such that $J_N \nearrow \boldsymbol{N}$, and the corresponding sequence of canonical subgroups $G'_N := \mathfrak{S}_{J_N}(T)$, demanding an asymptotic condition as

$$(10.8) \qquad \frac{|I_p \cap J_N|}{|J_N|} \longrightarrow \lambda_p \ \ (p \in P), \qquad \frac{|I_\zeta \cap J_N|}{|J_N|} \longrightarrow \mu_\zeta \ \ (\zeta \in \widehat{T}),$$

where $P := \bigsqcup_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}} P_{\zeta,\varepsilon}$ is the union of index sets. For each $(\zeta,\varepsilon) \in \widehat{T} \times \{0,1\}$, let reorder the numbers $\{\lambda_p \,;\, p \in P_{\zeta,\varepsilon}\}$ in the decreasing order and put it as $\alpha_{\zeta,\varepsilon} := (\alpha_{\zeta,\varepsilon,i})_{i\in\boldsymbol{N}}$: $\alpha_{\zeta,\varepsilon,1} \geq \alpha_{\zeta,\varepsilon,2} \geq \cdots \geq 0$, and also put $\mu := (\mu_\zeta)_{\zeta\in\widehat{T}}$. Then,

$$\sum_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}} \|\alpha_{\zeta,\varepsilon}\| + \|\mu\| \leq 1\,.$$

Pick up the case where the equality holds here, then by direct calculations we have a compact-uniform limit of the sequence of centralizations $F^{G'_N}$ as $N \to \infty$, which gives the character $f_A$ with a parameter $A = \big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}} \,;\, \mu\big)$ together with the general character formula in Theorem 2.3.

## 10.2. Recapitulation of centralization of positive definite functions

Note that, when $T$ is not finite, the group $G = \mathfrak{S}_\infty(T)$ with the inductive limit topology is no longer locally compact. However, since the quotient space $H\backslash G \cong \mathfrak{S}_H\backslash\mathfrak{S}_\infty$ with $\mathfrak{S}_H := H \cap \mathfrak{S}_\infty$ is countable, we can define an induced representation $\Pi := \mathrm{Ind}_H^G \pi$ from $H$ to $G$ by a standard method, on a vector-valued $\ell^2$-space on $H\backslash G \cong \mathfrak{S}_H\backslash\mathfrak{S}_\infty$. Then we see that the trivial extension $F = \mathrm{Ind}_H^G f_\pi$ is a diagonal matrix element of $\Pi$. Therefore the above method of centralizing $F = \mathrm{Ind}_H^G f_\pi$ and then taking limits is a special case of the following method:

(I) *take a diagonal matrix element* $F_0$ *of a UR* $\Pi_0$ (*not necessarily irreducible*), *or simply a continuous positive definite function* $F_0$ *on* $G$ (*as in* §§1–5 *of* [Hir3]),

(II) *take an increasing sequence of compact subgroups* $G'_N \nearrow G$, *and centralize* $F_0$ *with respect to* $G'_N$. *Then take* $\lim_{N\to\infty} F_0^{G'_N}$ *if exists.*

Then the pointwise limit $\lim_{N\to\infty} F_0^{G'_N}$ gives us an invariant positive definite function on $G$, which may be continuous.

Now, from the stand point of asymptotic approximation by normalized (trace) characters of compact subgroups of $G$, we can reformulate the method in **10.1** as follows.

As an increasing sequence $G'_N \nearrow G = \mathfrak{S}_\infty(T)$ of compact subgroups, we chose $G'_N = \mathfrak{S}_{J_N}(T) = D_{J_N}(T) \rtimes \mathfrak{S}_{J_N}$ with $J_N \nearrow \boldsymbol{N}$. For centralization $F^{G'_N}$ of a positive definite function $F = \mathrm{Ind}_H^G f_\pi$, we have

$$F^{G'_N}(h) = \int_{G'_N} f_\pi(g'hg'^{-1})\, d\mu_{G'_N}(g') \quad (h \in H), \tag{10.9}$$

where the integrand $f_\pi$ is extended outside $H$ trivially. Put

$$H'_N := G'_N \cap H,\ \ \pi'_N := \pi|_{H'_N},\ \ \Pi'_N := \mathrm{Ind}_{H'_N}^{G'_N} \pi'_N,\ \ F'_N := F|_{G'_N}.$$

The restriction $f_\pi|_{H'_N}$ is a diagonal matrix element $f_{\pi'_N}$ of an IUR $\pi'_N$ of $H'_N$, and $F'_N = \mathrm{Ind}_{H'_N}^{G'_N} f_{\pi'_N}$ is a diagonal matrix element of $\Pi'_N$, normalized as $F'_N(e) = 1$ since $f_\pi(e) = 1$. From the formula (10.9), we have the following.

PROPOSITION 10.1. *The restriction $F^{G'_N}$ onto $G'_N$ is the centralization of a diagonal matrix element $F'_N$ of the induced representation $\Pi'_N = \mathrm{Ind}_{H'_N}^{G'_N} \pi'_N$ of $G'_N$ with respect to $G'_N$, and is equal to the normalized character $\widetilde{\chi}_{\Pi'_N} := \chi_{\Pi'_N} / \dim \Pi'_N$.*

*Remark* 10.1. The induced representation $\Pi = \mathrm{Ind}_H^G \pi$ of the full group $G$ is irreducible if and only if all $I_\zeta$'s are empty. This is proved as for the infinite symmetric group $\mathfrak{S}_\infty$ in [Hir1].

However, at the stage of $G'_N$, almost all the induced representations $\Pi'_N = \mathrm{Ind}_{H'_N}^{G'_N} \pi'_N$ of $G'_N$ are not irreducible as is seen from the discussions in Section 3.

### 10.3. Problem setting

The subgroup $G'_N = \mathfrak{S}_{J_N}(T)$ is isomorphic to $G_n = \mathfrak{S}_n(T)$ with $n = m_N := |J_N|$ through a bijective correspondence $J_N \leftrightarrow \boldsymbol{I}_n$. By this isomorphism, the induced representation $\Pi'_N$ in Proposition 10.1 is interpreted as an induced representation $\Pi''_n$ of $G_n$ with $n = m_N$. Therefore the result explained in **10.1** is that *any character $f_A$ of $G$ is obtained as a limit of normalized characters of induced representations $\widetilde{\chi}_{\Pi''_n}$ of $G_n$ as $n = m_N \to \infty$.*

Usually $\Pi'_N$ and accordingly $\Pi''_n$, $n = m_N$, are not irreducible. Let

$$\Pi''_n = \sum_{1\le r\le R_n}^{\oplus} m_n(r)\cdot \Pi_n^{(r)} \tag{10.10}$$

be an irreducible decomposition of $\Pi''_n$, where $\Pi_n^{(r)}$ is an IUR of $G_n$ and $m_n(r)$ denotes its multiplicity in $\Pi''_n$. Then the normalized character is expressed as

$$\widetilde{\chi}_{\Pi''_n} = \sum_{1\le r\le R_n} \frac{m_n(r)\dim \Pi_n^{(r)}}{\dim \Pi''_n}\cdot \widetilde{\chi}_{\Pi_n^{(r)}}. \tag{10.11}$$

Taking into account our result Theorem 7.1 in the present paper and the result in [HH6] explained in **10.1** and **10.2**, we ask naturally *which part of irreducible components in* (10.10) *or in* (10.11) *is responsible for having limit* $\lim_{n=m_N\to\infty}\widetilde{\chi}_{\Pi''_n} = f_A$. Thus we can formulate the following problems.

PROBLEM 2007-1. *In what case, almost all induced representations* $\Pi'_N = \mathrm{Ind}_{H'_N}^{G'_N}\pi'_N$ *of* $G'_N$ *are irreducible as* $N\to\infty$*?*

In this case, a character $f_A$ is obtained as pointwise limit of normalized irreducible characters $\lim_{N\to\infty}\widetilde{\chi}_{\Pi'_N}$, and this is proved by direct calculations without appealing to the evaluation (5.23) of Murnaghan or to the result Theorem 5.1 quoted from [VK1]–[VK2].

Suppose now we are in an opposite situation of Problem 10.3. For each $n = m_N$, pick up an irreducible component $\Pi_n^{(r_n)}$ $(1\le \exists r_n\le R_n)$ of $\Pi''_n$ in (10.10), and we call such a sequence $\Pi_n^{(r_n)}$, $n = m_N$ $(N\to\infty)$ a *generalized path of IURs of* $G_n$ *to infinity.*

PROBLEM 2007-2. *For what kind of generalized paths* $\Pi_n^{(r_n)}$, $n = m_N$ $(N\to\infty)$, *we have the same limit as* $\lim_{n=m_N\to\infty}\widetilde{\chi}_{\Pi_n^{(r_n)}} = \lim_{N\to\infty}\widetilde{\chi}_{\Pi'_N} = f_A$*? In addition, in what cases, the ratio* $\frac{m_n(r)\dim\Pi_n^{(r)}}{\dim\Pi''_n}$ *do not vanish as* $n = m_N\to\infty$*?*

Roughly speaking, this asks which irreducible components are responsible to the limit $\lim_{N\to\infty}\widetilde{\chi}_{\Pi'_N} = f_A$. Does there exist 'principal' irreducible constituents which take the whole responsibility?

*Remark* 10.2. In some cases, it is also disputable if Problem 2007-2 is well-posed. For example, consider the case where we have, at the limit, $\mu_{\zeta^0} = 1$ for a $\zeta^0 \in \widehat{T}$ and other factors in the parameter $A$ of a character $f_A$ in Theorem 2.3 are all zero and so

$$(10.12) \quad Y_1^A(t) = \frac{1}{\dim \zeta^0}\, \chi_{\zeta^0}(t) = \widetilde{\chi}_{\zeta^0}(t), \quad Y_\ell^A(t) \equiv 0 \quad \text{for } \ell \geq 2 \ \ (t \in T).$$

We get this character by the method explained in **10.1** by considering a sequence $J_N \nearrow \boldsymbol{N}$ such that $|J_N \cap I_{\zeta^0}|/|J_N| \to 1$. Simplify this situation, then it is essentially very near to the case where we take $I_{\zeta^0} = \boldsymbol{N}$ and $J_N = \boldsymbol{I}_n$, and so $H = D_\infty \subset G = D_\infty \rtimes \mathfrak{S}_\infty$. In this simplified case, we take for $H'_N = G'_N \cap H$ and $\Pi'_N = \mathrm{Ind}_{H'_N}^{G'_N} \pi'_N$ respectively

$$H_n = D_n \subset G_n \quad \text{and} \quad \Pi_n = \mathrm{Ind}_{D_n}^{G_n} \eta_n, \ \ \eta_n = \mathop{\boxtimes}_{i \in \boldsymbol{I}_n} \zeta_i \ \text{ with } \ \zeta_i = \zeta^0.$$

Then $\Pi_n = \eta_n \boxtimes \mathcal{R}_n$ for $G_n = D_n \rtimes \mathfrak{S}_n$, where $\mathcal{R}_n$ denotes the regular representation of $\mathfrak{S}_n$. Then, for $(d, \sigma) \in G$, $d = (t_q)_{q \in \boldsymbol{N}}$, and $n$ sufficiently large,

$$\widetilde{\chi}_{\Pi_n}(d, \sigma) = \prod_{q \in \boldsymbol{I}_n} \widetilde{\chi}_{\zeta^0}(t_q) \times \delta_{\mathbf{1}}^{\mathfrak{S}_n} \longrightarrow \prod_{q \in \boldsymbol{N}} Y_1^A(t_q) \times \delta_{\mathbf{1}}^{\mathfrak{S}_\infty} = f_A\big((d, \sigma)\big),$$

where $\delta_{\mathbf{1}}^{\mathfrak{S}_n}$ and $\delta_{\mathbf{1}}^{\mathfrak{S}_\infty}$ are delta-functions on $\mathfrak{S}_n$ and $\mathfrak{S}_\infty$ respectively, supported on the identity element $\mathbf{1}$. At the stage $G_n$, every IUR $\pi \in \widehat{\mathfrak{S}_n}$ contributes to $\widehat{\chi}_{\mathcal{R}_n} = \delta_{\mathbf{1}}^{\mathfrak{S}_n}$ as its own right with coefficient $(\dim \pi)^2/|\mathfrak{S}_n|$ for $\widetilde{\chi}_\pi$, and we cannot say which parts of $\widehat{\mathfrak{S}_n}$ $(n \geq 2)$ are responsible to the limit $\delta_{\mathbf{1}}^{\mathfrak{S}_\infty}$, without introducing some other criterion.

*Remark* 10.3. In contrast with the above method, the above character $f_A$ in (10.12) is also obtained as limits of characters of IURs $\Pi_n$ of $G_n = D_n \rtimes \mathfrak{S}_n$ as in Theorem 6.1. We can characterize some of such sequences of IURs $\Pi_n$. Take

$$(10.13) \qquad (\mathcal{I}_n, \Lambda^n), \quad \mathcal{I}_n = (I_{n,\zeta})_{\zeta \in \widehat{T}}, \quad \Lambda^n = (\lambda^{n,\zeta})_{\zeta \in \widehat{T}},$$

for IUR $\Pi_n = \mathrm{Ind}_{H_n}^{G_n} \pi_n$ in the beginning of Section 6 as $I_{n,\zeta^0} = \boldsymbol{I}_n$, $I_{n,\zeta} = \emptyset$ $(\zeta \neq \zeta^0)$ and $\lambda^{n,\zeta} = \emptyset$ $(\zeta \neq \zeta^0)$. Then, in (6.1) and (6.2), $\eta_n = \boxtimes_{i \in \boldsymbol{I}_n} \zeta_i$, $\zeta_i = \zeta^0$, and so $S_{[\eta_n]} = \mathfrak{S}_n$, $H_n = D_n \rtimes S_{[\eta_n]} = G_n$. Accordingly $\Pi_n = \pi_n =$

$\eta_n \boxdot \xi_n = \eta_n \boxdot \pi(\lambda^{n,\zeta^0})$, where $\pi(\lambda^{n,\zeta^0})$ is an IUR of $\mathfrak{S}_n$ corresponding to a Young diagram $\lambda^{n,\zeta^0}$.

*Assume that Young diagrams $\lambda^{n,\zeta^0}$ increase along with $n$. Denote by $r_k(\lambda^{n,\zeta^0})$ (resp. $c_k(\lambda^{n,\zeta^0})$) the length of $k$-th row (resp. $k$-th column) of $\lambda^{n,\zeta^0}$. Then the sequence of normalized irreducible characters $\widetilde{\chi}_{\Pi_n}$ converges to the character $f_A$ if and only if*

$$r_1(\lambda^{n,\zeta^0})/n \longrightarrow 0, \quad c_1(\lambda^{n,\zeta^0})/n \longrightarrow 0. \tag{10.14}$$

Such sequences of Young diagrams are obtained by taking $\lambda^{n,\zeta^0}$ like isosceles triangles or like regular squares, where $r_1(\lambda^{n,\zeta^0})$ and $c_1(\lambda^{n,\zeta^0})$ are of the order $\sqrt{2n}$ or $\sqrt{n}$ respectively.

Comparing with Remark 10.2, we see that the above necessary and sufficient condition (10.14) has something to do with "limit" of Plancherel measures $\mu_n([\pi]) = (\dim \pi)^2/|\mathfrak{S}_n|$, $[\pi] \in \widehat{\mathfrak{S}_n}$, or a measure on the space of paths of Young diagrams of infinite lengths explained in **5.1**, which comes from the projective system of measures $(\mu_n)_{n\geq 1}$ (cf. §2 of Part II of this work [HHH2]).

## §11. Limits of induced characters in irreducible cases

For Problem 2007-1, we can give an answer in this section, and for Problem 2007-2, we can give only a partial answer by discussing examples. Let the notation be as in Section 10. The subgroup $H'_N$ and its IUR $\pi'_N$ are given as

$$H'_N = \left(\prod_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}} \left(\prod_{p\in P_{\zeta,\varepsilon}} H_{p,N}\right)\right) \times \left(\prod_{\zeta\in\widehat{T}} H_{\zeta,N}\right), \tag{11.1}$$

$$\text{with } H_{p,N} = \mathfrak{S}_{J_N\cap I_p}, \quad H_{\zeta,N} = D_{J_N\cap I_\zeta}(T) \subset \mathfrak{S}_{J_N\cap I_\zeta}(T),$$

$$\pi'_N = \left(\bigotimes_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}} \left(\bigotimes_{p\in P_{\zeta,\varepsilon}} \pi_{p,N}\right)\right) \otimes \left(\bigotimes_{\zeta\in\widehat{T}} \pi_{\zeta,N}\right), \tag{11.2}$$

where $\pi_{p,N}$ and $\pi_{\zeta,N}$ are IURs of $H_{p,N}$ and $H_{\zeta,N}$ respectively defined similarly as (10.4)–(10.6) replacing $I_p$ and $I_\zeta$ by $I_{p,N} := I_p \cap J_N$ and $I_{\zeta,N} := I_\zeta \cap J_N$. Here the products are actually finite since $J_N$ is finite.

PROPOSITION 11.1. (i) *If the induced representation* $\Pi'_N = \mathrm{Ind}_{H'_N}^{G'_N} \pi'_N$ *of* $G'_N = \mathfrak{S}_{J_N}(T)$ *is irreducible, then for each* $(\zeta, \varepsilon) \in \widehat{T} \times \{0,1\}$,

$$\tag{11.3} \big|\{p \in P_{\zeta,\varepsilon}\,;\, J_N \cap I_p \neq \emptyset\}\big| \le 1, \quad \big|J_N \cap I_\zeta\big| \le 1.$$

(ii) *The induced representation* $\Pi'_N = \mathrm{Ind}_{H'_N}^{G'_N} \pi'_N$ *of* $G'_N$ *is irreducible, if and only if for any* $\zeta \in \widehat{T}$,

$$\tag{11.4} \big|\{p \in P_{\zeta,0} \sqcup P_{\zeta,1}\,;\, J_N \cap I_p \neq \emptyset\}\big| + \big|J_N \cap I_\zeta\big| \le 1.$$

*Proof.* Proof for (i) is given by showing the existence of non-trivial intertwining operators for $\pi'_N$. The discussion is standard as in Section 3.

Proof for (ii) needs some more detailed explicit calculation of intertwining operators for the induced representation $\Pi'_N$, similarly as in the proof of Theorem 3.3 (i) (cf. discussions in **12.2**). □

Since we are interested in the asymptotic behavior as $J_N \nearrow \boldsymbol{N}$, it is enough to treat the case where (slightly modifying the setting) $P_{\zeta,\varepsilon}$'s are one point sets and $I_\zeta$'s are all empty. This case contains the case for Problem 2007-1, as seen from Proposition 11.1 (ii).

So, we take a partition $\mathcal{I}_{\boldsymbol{N}} = (I_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}$ of $\boldsymbol{N}$ given as $\boldsymbol{N} = \bigsqcup_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}} I_{\zeta,\varepsilon}$, and define newly a subgroup $H$ of $G = \mathfrak{S}_\infty(T)$ as

$$\tag{11.5} H = \prod_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}{}' G_{\zeta,\varepsilon} = D_\infty(T) \rtimes S_{\mathcal{I}_{\boldsymbol{N}}},$$

$$G_{\zeta,\varepsilon} := \mathfrak{S}_{I_{\zeta,\varepsilon}}(T) = D_{I_{\zeta,\varepsilon}}(T) \rtimes \mathfrak{S}_{I_{\zeta,\varepsilon}}, \quad S_{\mathcal{I}_{\boldsymbol{N}}} := \prod_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}{}' \mathfrak{S}_{I_{\zeta,\varepsilon}}.$$

First define an IUR $\pi_{\zeta,\varepsilon}$ of $G_{\zeta,\varepsilon}$. For $d = (t_i)_{i\in I_{\zeta,\varepsilon}} \in D_{I_{\zeta,\varepsilon}}(T)$, put

$$\bigotimes\nolimits_{i\in I_{\zeta,\varepsilon}}^{a_{\zeta,\varepsilon}} \zeta_i \ \text{ with } \ \zeta_i = \zeta \ \text{ for } \ T_i = T \ \ (i \in I_{\zeta,\varepsilon})$$

with respect to a reference vector $a_{\zeta,\varepsilon} = (a_i)_{i\in I_{\zeta,\varepsilon}}$, $a_i \in V(\zeta_i)$, $\|a_i\| = 1$, on the tensor product space $V_{\zeta,\varepsilon} = \bigotimes_{i\in I_{\zeta,\varepsilon}}^{a_{\zeta,\varepsilon}} V(\zeta_i)$. For $\sigma \in \mathfrak{S}_{I_{\zeta,\varepsilon}}$, put

$$\tag{11.6} I(\sigma)\big(\bigotimes\nolimits_{i\in I_{\zeta,\varepsilon}}^{a_{\zeta,\varepsilon}} v_i\big) := \bigotimes\nolimits_{i\in I_{\zeta,\varepsilon}}^{a_{\zeta,\varepsilon}} v_{\sigma^{-1}(i)} \quad \text{with } \ v_i \in V(\zeta_i), \ v_i = a_i \ (i \gg 1).$$

Then, for $g = (d, \sigma) \in G_{\zeta,\varepsilon}, d = (t_i)_{i\in I_{\zeta,\varepsilon}}$,

$$\tag{11.7} \pi_{\zeta,\varepsilon}(g) = \pi_{\zeta,\varepsilon}\big((d,\sigma)\big) := \big(\bigotimes\nolimits_{i\in I_{\zeta,\varepsilon}}^{a_{\zeta,\varepsilon}} \zeta_i(t_i)\big) I(\sigma)\,\mathrm{sgn}(\sigma)^{\varepsilon}.$$

For the subgroup $H = \prod'_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}} G_{\zeta,\varepsilon}$, we define an IUR $\pi = \bigotimes^b_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}} \pi_{\zeta,\varepsilon}$ as a tensor product acting on $V = \bigotimes^b_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}} V_{\zeta,\varepsilon}$ with a reference vector $b$.

For $\boldsymbol{I}_n = \{1,2,\ldots,n\}$, put $G_n = \mathfrak{S}_{\boldsymbol{I}_n}(T)$ and $H_n = H \cap G_n$ as before, and

$$\text{(11.8)} \qquad \begin{cases} I_{n,\zeta,\varepsilon} = \boldsymbol{I}_n \cap I_{\zeta,\varepsilon}, \quad I_{n,\zeta} = \bigsqcup_{\varepsilon=0,1} I_{n,\zeta,\varepsilon}, \\ G_{n,\zeta,\varepsilon} = H_n \cap G_{\zeta,\varepsilon}, \quad G_{n,\zeta} = \mathfrak{S}_{I_{n,\zeta}}(T), \\ \widehat{T}_n = \{\zeta \in \widehat{T}\,;\, I_{n,\zeta} \neq \emptyset\}. \end{cases}$$

Then $H_n = \prod_{\zeta\in\widehat{T}_n} H_{n,\zeta}$ with $H_{n,\zeta} = \prod_{\varepsilon=0,1} G_{n,\zeta,\varepsilon} \subset G_{n,\zeta}$, and we have an IUR $\pi_n$ of $H_n$ on a space $V_n$ as

$$\text{(11.9)} \qquad \begin{cases} \pi_n = \boxtimes_{\zeta\in\widehat{T}_n} \pi_{n,\zeta} \quad \text{with} \quad \pi_{n,\zeta} = \boxtimes_{\varepsilon=0,1} \pi_{n,\zeta,\varepsilon}, \\ V_n = \bigotimes_{\zeta\in\widehat{T}_n} V_{n,\zeta} \quad \text{with} \quad V_{n,\zeta} = \bigotimes_{\varepsilon=0,1} V_{n,\zeta,\varepsilon}, \end{cases}$$

where, for each $(\zeta,\varepsilon) \in \widehat{T}_n \times \{0,1\}$, $\pi_{n,\zeta,\varepsilon}$ is an IUR of $G_{n,\zeta,\varepsilon} = D_{I_{n,\zeta,\varepsilon}}(T) \rtimes \mathfrak{S}_{I_{n,\zeta,\varepsilon}}$ on a tensor product space $V_{n,\zeta,\varepsilon} = \bigotimes_{i\in I_{n,\zeta,\varepsilon}} V(\zeta_i)$ given similarly as $\pi_{\zeta,\varepsilon}$ on $V_{\zeta,\varepsilon}$ in (11.7).

The normalized character of a finite-dimensional UR $\Pi$ is defined as $\widetilde{\chi}_\Pi = \chi_\Pi / \dim \Pi$. Then $\widetilde{\chi}_{\Pi_n}$ for $\Pi_n = \mathrm{Ind}_{H_n}^{G_n} \pi_n$ is given by the *centralization* of the normalized character $\widetilde{\chi}_{\pi_n} := \chi_{\pi_n}/\dim \pi_n$ of $\pi_n$ with respect to the normalized Haar measure $d\mu_{G_n}$ on $G_n$. Though the induced representation $\Pi_n$ is not necessarily irreducible here, we get an explicit character formula for its normalized character $\widetilde{\chi}_{\Pi_n}$ quite similar to that in Theorem 4.5 for irreducible induced representations (cf. Sections 4–6, and cf. [HH5, §14] or [HH6, §12]).

For $g = (d,\sigma)$, let

$$\text{(11.10)} \qquad g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r} g_1 g_2 \cdots g_m, \quad \xi_q = \big(t_q, (q)\big), \quad g_j = (d_j, \sigma_j),$$

be a standard decomposition and put

$$\text{(11.11)} \qquad Q = \{q_1, q_2, \ldots, q_r\}, \quad J = \{1,2,\ldots,m\}.$$

PROPOSITION 11.2. *The normalized character of the induced representations $\Pi_n = \mathrm{Ind}_{H_n}^{G_n} \pi_n$ of $G_n$ is given as follows. Let $\mathcal{I}'_n := (I_{n,\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}$*

*be the partition of* $\boldsymbol{I}_n$ *corresponding to* $\Pi_n$. *For* $g=(d,\sigma)\in G_n=D_n(T)\rtimes\mathfrak{S}_n$ *above, let* $\mathcal{Q}'=(Q_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}$ *and* $\mathcal{J}'=(J_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}$ *be partitions of* $Q$ *and* $J$ *respectively. Then*

$$\widetilde{\chi}_{\Pi_n}(g)=\sum_{\mathcal{Q}',\mathcal{J}'}c'(\mathcal{I}'_n;\mathcal{Q}',\mathcal{J}')\times\prod_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\left(\prod_{q\in Q_{\zeta,\varepsilon}}\frac{\chi_\zeta(t_q)}{\dim\zeta}\cdot\prod_{j\in J_{\zeta,\varepsilon}}\frac{\chi_\zeta\big(P_{\sigma_j}(d_j)\big)}{(\dim\zeta)^{\ell(\sigma_j)}}\operatorname{sgn}(\sigma_j)^\varepsilon\right),$$

*with coefficients* $c'(\mathcal{I}'_n;\mathcal{Q}',\mathcal{J}')$ *given by*

$$\frac{\prod_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}|I_{n,\zeta,\varepsilon}|\big(|I_{n,\zeta,\varepsilon}|-1\big)\cdots\Big(|I_{n,\zeta,\varepsilon}|-|Q_{\zeta,\varepsilon}|-\sum_{j\in J_{\zeta,\varepsilon}}|K_j|+1\Big)}{n(n-1)\cdots\big(n-|Q|-|\operatorname{supp}(\sigma)|+1\big)},$$

*where* $\mathcal{Q}'$ *and* $\mathcal{J}'$ *run over all partitions of* $Q$ *and* $J$ *respectively.*

Take a $g\in G$. Then, starting from a certain $n=n_0$, $g$ is contained in $G_n$, and so we can consider the limit of the normalized character $\widetilde{\chi}_{\Pi_n}(g)$ for $g\in G_{n_0}\subset G_n$ as $n\geq n_0$ tends to $\infty$.

THEOREM 11.3. (i) *The sequence of unitary representations* $\Pi_n=\operatorname{Ind}_{H_n}^{G_n}\pi_n$ *of* $G_n=\mathfrak{S}_n(T)$ *given above is determined by a partition* $\mathcal{I}_{\boldsymbol{N}}=(I_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}$ *of* $\boldsymbol{N}$. *Put* $I_{n,\zeta,\varepsilon}=\boldsymbol{I}_n\cap I_{\zeta,\varepsilon}$ *and assume that the following limits exist*:

$$B_{\zeta,\varepsilon}=\lim_{n\to\infty}\frac{|I_{n,\zeta,\varepsilon}|}{n}\quad\big((\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}\big).\tag{11.12}$$

*Then, there exists a pointwise limit of normalized characters* $F_{\mathcal{I}_{\boldsymbol{N}}}:=\lim_{n\to\infty}\widetilde{\chi}_{\Pi_n}$ *on* $G=\mathfrak{S}_\infty(T)$. *For a* $g=(d,\sigma)\in G$ *with standard decomposition in* (11.10)–(11.11),

$$F_{\mathcal{I}_{\boldsymbol{N}}}(g)=\prod_{q\in Q}\left(\sum_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\left(\frac{B_{\zeta,\varepsilon}}{\dim\zeta}\right)\chi_\zeta(t_q)\right)\times\prod_{j\in J}\left(\sum_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\left(\frac{B_{\zeta,\varepsilon}}{\dim\zeta}\right)^{\ell(\sigma_j)}\chi_\zeta\big(P_{\sigma_j}(d_j)\big)\operatorname{sgn}(\sigma_j)^\varepsilon\right).$$

(ii) *The limit function* $F_{\mathcal{I}_{\boldsymbol{N}}}$ *is the character* $f_A$ *in Theorem* 2.3, *for which* $\alpha_{\zeta,\varepsilon}=(B_{\zeta,\epsilon},0,0,\dots)$, $\mu_\zeta=0$ *for* $(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}$ *in* $A=\big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}};\mu\big)$.

*Sketch of Proof.* The limits $\lim_{n\to\infty}\widetilde{\chi}_{\Pi_n}$ can be calculated directly from the explicit formula for $\widetilde{\chi}_{\Pi_n}$. The calculations go on similarly as in Section 5, but in the present case, we need not to appeal to Theorem 5.4 since irreducible representations of symmetric groups to be induced up are all one-dimensional here. □

This theorem contains the case of a sequence of IURs (of a degenerate form) of $G_n = \mathfrak{S}_n(T)$ which answers Problem 2007-1. It gives limits of irreducible characters of $G_n$ by direct calculation (cf. [HH4, §12] and [HH6, **13.3**]).

**Answer to Problem 2007-1:**

In the limiting process $F^{G'_N} \to f_A$ explained in **10.1**–**10.2**, the restriction $F^{G'_N}|_{G'_N}$ is the normalized character $\widetilde{\chi}_{\Pi'_N}$ of the induced representation $\Pi'_N = \mathrm{Ind}_{H'_N}^{G'_N}\pi'_N$ (Proposition 10.1). Suppose that there exists an infinite subsequence of $\Pi'_N$ which are irreducible. Then it is essentially the case in Theorem 11.3, where

$$I_{n,\zeta,\varepsilon} = \emptyset \ \ (\forall n) \quad \text{for } \varepsilon = 0 \text{ or } = 1 \text{ for any } \zeta \in \widehat{T}.$$

The characters $f_A$ obtained as $\lim_{N\to\infty} F^{G'_N}$ in this irreducible case are given in (ii) of Theorem 11.3.

*Remark* 11.1. In the special case where $I_{\zeta,\varepsilon} = \boldsymbol{N}$ for a fixed $(\zeta,\varepsilon)$, all $\Pi_n$ are irreducible and we have the following characters as limits of irreducible characters $\widetilde{\chi}_{\Pi_n}$. For $g = (d,\sigma) \in G$ as in Theorem 11.3,

$$F_{\zeta,\varepsilon}(g) = \prod_{q\in Q} \frac{\chi_\zeta(t_q)}{\dim\zeta} \times \prod_{j\in J} \frac{\chi_\zeta\big(P_{\sigma_j}(d_j)\big)}{(\dim\zeta)^{\ell(\sigma_j)}}\,\mathrm{sgn}(\sigma_j)^\varepsilon, \tag{11.13}$$

which is equal to $f_A$ with $\alpha_{\zeta,\varepsilon} = (1,0,0,\dots)$. Especially when $\zeta = \mathbf{1}_T \in \widehat{T}$, the trivial representation of $T$, we have $F_{\mathbf{1}_T,\varepsilon}(g) = \mathrm{sgn}(\sigma)^\varepsilon$ for $g = (d,\sigma) \in G$, one-dimensional character. So, for any $\zeta \in \widehat{T}$, $F_{\zeta,\varepsilon}(g) = F_{\zeta,0}(g)\,F_{\mathbf{1}_T,\varepsilon}(g)$ $(g \in G)$.

## §12. Irreducible decomposition of the induced representations $\boldsymbol{\Pi_n}$

### 12.1. Towards a partial answer to Problem 2007-2 by examples

For Problem 2007-2, we can only give a partial answer by giving some examples of sequences of reducible induced representations $\Pi_n$ of $G_n$ satisfying:

**(12-1)** each $\Pi_n$ splits into irreducible components $\Pi_n^{(r)}$, $1 \le r \le R$, where $R$ is a power of 2 (in the extreme case, $R = \infty$),

**(12-2)** one can form several natural sequences $\Pi_n^{(r_n)}$, $n \to \infty$, of irreducible components of $\Pi_n$ for which the sequences of normalized irreducible characters $\widetilde{\chi}_{\Pi_n^{(r_n)}}$ have the same limits as the original one $\lim_{n\to\infty} \widetilde{\chi}_{\Pi_n}$.

We take an open subgroup $H$ of $G$ again as in (11.5). For the subgroup $G_n = \mathfrak{S}_{\boldsymbol{I}_n}(T)$, the subgroup $H_n = H \cap G_n$ and the IUR $\pi_n$ in (11.9) are factored according to (11.8)–(11.9) as follows:

$$H_n = \prod_{\zeta \in \widehat{T}_n} H_{n,\zeta}, \quad H_{n,\zeta} = \prod_{\varepsilon=0,1} G_{n,\zeta,\varepsilon} \subset G_{n,\zeta};$$

$$(\pi_n, V_n) = \boxtimes_{\zeta \in \widehat{T}_n} (\pi_{n,\zeta}, V_{n,\zeta}).$$

To study the space $\mathcal{I}(\Pi_n)$ of intertwining operators between $\Pi_n = \mathrm{Ind}_{H_n}^{G_n} \pi_n$ and itself, we first remark that, as can be proved by direct calculations of intertwining operators for $\Pi_n = \mathrm{Ind}_{H_n}^{G_n} \pi_n$, a non-trivial intertwining operator originates only from the inside of each "$\zeta$-component" $\mathrm{Ind}_{H_{n,\zeta}}^{G_{n,\zeta}} \pi_{n,\zeta}$, where $\pi_{n,\zeta} = \boxtimes_{\varepsilon=0,1} \pi_{n,\zeta,\varepsilon}$.

Therefore, for our study, we can restrict ourselves to the case where only one fixed $\zeta$ appears, and we pursuit this fundamental case further on.

In this reduced case, each induced representations $\Pi_n$ of $G_n$ split into two irreducible components $\Pi_{n,0}$ and $\Pi_{n,1}$. The character $\chi_{\Pi_n} = \chi_{\Pi_{n,0}} + \chi_{\Pi_{n,1}}$ has been given in Proposition 11.2. For a certain intertwining operator $\widetilde{U} = \widetilde{U}_n$, the virtual character $\chi_{\widetilde{U}\Pi_n} := \mathrm{tr}(\widetilde{U}\Pi_n)$ is calculated in Theorem 13.7. Thus the explicit formula for characters $\chi_{\Pi_{n,\gamma}}$, $\gamma = 0, 1$, are obtained from Proposition 11.2 and Theorem 13.7 through the equality (12.12).

Both sequences of the normalized irreducible characters $\widetilde{\chi}_{\Pi_{n,\gamma}}$, $\gamma = 0, 1$, have the same limit as $n \to \infty$ as the original one $\lim_{n\to\infty} \widetilde{\chi}_{\Pi_n}$ (Theorem 14.2).

Using these results, we can analyse through Theorem 14.1 the asymptotic behavior of the difference $\widetilde{\chi}_{\Pi_{n,0}} - \widetilde{\chi}_{\Pi_{n,1}}$ as $n \to \infty$. This gives an evaluation of fluctuations of $\widetilde{\chi}_{\Pi_{n,0}}$ and $\widetilde{\chi}_{\Pi_{n,1}}$ around the common limit (Theorem 14.3).

### 12.2. Combinatorial lemmas and irreducible decomposition of $\Pi_n$

Now we put newly for $G_n = \mathfrak{S}_n(T) = D_n(T) \rtimes \mathfrak{S}_n$,

$$\begin{gathered} \boldsymbol{I}_n = I_{n,0} \sqcup I_{n,1}, \quad H_n = H_{n,0} \times H_{n,1}, \quad H_{n,\varepsilon} = \mathfrak{S}_{I_{n,\varepsilon}}(T) \ \ (\varepsilon = 0, 1), \\ \pi_n = \pi_{n,\zeta} = \boxtimes_{\varepsilon=0,1} \pi_{n,\zeta,\varepsilon}, \quad V_n = \bigotimes_{\varepsilon=0,1} V_{n,\zeta,\varepsilon}, \end{gathered} \tag{12.1}$$

where, for $g = (d, \sigma) \in H_{n,\varepsilon} = D_{I_{n,\varepsilon}}(T) \rtimes \mathfrak{S}_{I_{n,\varepsilon}}$, $d = (t_i)_{i \in I_{n,\varepsilon}}$, we put as in (11.7)

$$\pi_{n,\zeta,\varepsilon}(g) = \pi_{n,\zeta,\varepsilon}\big((d,\sigma)\big) := \big(\textstyle\bigotimes_{i \in I_{n,\varepsilon}} \zeta_i(t_i)\big) I(\sigma)\, \mathrm{sgn}(\sigma)^{\varepsilon}. \tag{12.2}$$

To fix the situation more exactly, we put $I_{n,0} = \{1, 2, \ldots, n_0\}$, $I_{n,1} = \{n_0 + 1, n_0 + 2, \ldots, n_0 + n_1 = n\}$ with $n_0, n_1 > 0$, and $\mathfrak{S}_{n_0,n_1} := \mathfrak{S}_{I_{n,0}} \times \mathfrak{S}_{I_{n,1}} \cong \mathfrak{S}_{n_0} \times \mathfrak{S}_{n_1}$. For $\sigma = \kappa_0\kappa_1 \in \mathfrak{S}_{n_0,n_1}$ with $\kappa_0 \in \mathfrak{S}_{I_{n,0}}$, $\kappa_1 \in \mathfrak{S}_{I_{n,1}}$, we put $\mathrm{sgn}_1(\sigma) := \mathrm{sgn}(\kappa_1)$.

We determine the space $\mathcal{I}(\Pi_n)$ of intertwining operators for $\Pi_n = \mathrm{Ind}_{H_n}^{G_n} \pi_n$ by a similar method as in the proof of Theorem 3.3, preparing several lemmas successively as follows. Recall the formulas (3.6)–(3.9).

LEMMA 12.1. (i) *Let $\tau_i = (n_0 - i + 1 \;\; n_0 + i)$ be the transposition of element $n_0 - i + 1 \in I_{n,0}$ and $n_0 + i \in I_{n,1}$. Then the space of double cosets $H_n \backslash G_n / H_n$ has a complete set of representatives given by*

$$\mathcal{T} := \big\{\mathbf{1},\, \tau_1,\, \tau_1\tau_2, \ldots,\, \tau_1\tau_2\cdots\tau_N\big\} \quad \text{for } \ N = \min\{n_0, n_1\}. \tag{12.3}$$

(ii) *Let $\tau = \tau_1\tau_2\cdots\tau_p \in \mathcal{T}$, $1 \le p \le N$. Then $H_n \cap \tau H_n \tau^{-1} = D_n(T) \rtimes S_\tau$, where $S_\tau = \mathfrak{S}_{n_0,n_1} \cap \tau\mathfrak{S}_{n_0,n_1}\tau^{-1}$ consists of permutations expressed by blockwise diagonal matrices $\mathrm{diag}(\sigma_1, \kappa_1; \kappa_2, \sigma_2)$, where $\sigma_1$, $\kappa_1$, $\kappa_2$ and $\sigma_2$ are respectively permutations on*

$$\begin{gathered} \{1, 2, \ldots, n_0 - p\}, \quad \{n_0 - p + 1, n_0 - p + 2, \ldots, n_0\}; \\ \{n_0 + 1, n_0 + 2, \ldots, n_0 + p\}, \quad \{n_0 + p + 1, n_0 + p + 2, \ldots, n_0 + n_1\}. \end{gathered}$$

LEMMA 12.2. *Let $\tau = \tau_1\tau_2\cdots\tau_p \in \mathcal{T}$, $1 \le p \le N$. Then a linear transformation $K(\tau)$ on $V_n$ satisfying*

$$\pi_n(h)K(\tau) = K(\tau)\pi_n(\tau^{-1}h\tau) \quad (h \in H_n \cap \tau H_n \tau^{-1}) \tag{12.4}$$

*is necessarily $0$ for $p \ge 2$.*

LEMMA 12.3. (i) *Let* $\tau_1 = (n_0\ \ n_0+1) \in \mathcal{T}$. *Then,* $H_n \cap \tau_1 H_n {\tau_1}^{-1} = D_n(T) \rtimes S_{\tau_1}$ *with* $S_{\tau_1}$ *the centralizer of* $\tau_1$ *in* $\mathfrak{S}_{n_0,n_1}$ *consisting of permutations expressed as* $\mathrm{diag}(\sigma_1, 1; 1, \sigma_2)$, *where* $\sigma_1$ *and* $\sigma_2$ *are respectively permutations on*

$$\{1, 2, \ldots, n_0 - 1\}\ (\subset I_{n,0}), \quad \{n_0+2, n_0+3, \ldots, n_0+n_1\}\ (\subset I_{n,1}).$$

*The transformation* $h \mapsto {\tau_1}^{-1} h \tau_1$ *for* $h = (d, \sigma) \in D_n(T) \rtimes S_{\tau_1}$ *is given as* ${\tau_1}^{-1} h \tau_1 = ({\tau_1}^{-1}(d), \sigma)$.

(ii) *The operator* $K(\tau_1)$ *which satisfies* (12.4) *is a constant multiple of the following operator on* $V_n = \bigotimes_{1\le i\le n_0+n_1} V(\zeta_i)$ *with* $\zeta_i = \zeta\ (\forall\, i)$

$$U = I(\tau_1) : \bigotimes\nolimits_{1\le i\le n_0+n_1} v_i \longmapsto \bigotimes\nolimits_{1\le i\le n_0+n_1} v_{{\tau_1}^{-1}(i)}, \tag{12.5}$$

*where* $v_i \in V(\zeta_i)$, *and* $\pi_n({\tau_1}^{-1} h \tau_1) = U^{-1}\pi_n(h)U$ $(h \in H_n \cap \tau_1 H_n {\tau_1}^{-1})$. *Here* $U = I(\tau_1)$ *permutes* $v_{n_0} \otimes v_{n_0+1}$ *as* $v_{n_0+1} \otimes v_{n_0}$ *in the middle of consecutive tensor product* $\otimes\, v_i$.

(iii) *The dimension of the space of intertwining operators is* 2: $\dim \mathcal{I}(\Pi_n) = 2$.

We omit proofs of these lemmas, since they follow standard processes. Take an intertwining operator in Lemma 12.3 (ii) as

$$K(\tau_1) = c(H_n)^{-1} U = c(H_n)^{-1} I(\tau_1)$$
$$\text{with } c(H_n)^{-1} := \mu_{G_n}(H_n) = \binom{n}{n_0} = \binom{n}{n_1}.$$

LEMMA 12.4. *An intertwining operator* $\widetilde{U}$ *for* $\Pi_n$ *is defined as follows by an* $\mathcal{L}(V_n)$*-valued function*

$$K(g) = \begin{cases} c(H_n)^{-1}\pi_n(h_1) U \pi_n(h_2) & \text{for } g = h_1\tau_1 h_2 \in H_n\tau_1 H_n, \\ \mathbf{0} & \text{for } g \notin H_n \tau_1 H_n, \end{cases} \tag{12.6}$$

$$(\widetilde{U}\varphi)(g) = \int_{G_n} K(g{g'}^{-1})\, \varphi(g')\, d\mu_{G_n}(g'). \tag{12.7}$$

Let $s_i = (i\ \ i+1)$, $1 \le i \le n-1$, be simple reflexions of $\mathfrak{S}_n$.

LEMMA 12.5. (i) *The double coset* $\mathfrak{S}_{n_0,n_1}\tau_1\mathfrak{S}_{n_0,n_1}$ *consists of* $n_0 n_1$ *number of left* $\mathfrak{S}_{n_0,n_1}$*-cosets with representative elements* $(a\ \ b)$ *with* $a \in I_{n,0}$,

*$b \in I_{n,1}$. This double coset consists of elements $\sigma \in \mathfrak{S}_n$ such that its cycle decomposition contain cycles in $\mathfrak{S}_{n_0,n_1}$ and one of the cycles given as*

$$(a_1 \ a_2 \ \cdots \ a_p \ b_1 \ b_2 \ \cdots \ b_q), \ \ p \geq 1, \, q \geq 1, \quad \text{with } \ a_i \in I_{n,0}, \, b_j \in I_{n,1}.$$

*The order of the subset is $|\mathfrak{S}_{n_0,n_1}\tau_1\mathfrak{S}_{n_0,n_1}| = n_0!\, n_1! \cdot n_0 n_1$.*

(ii) *The subgroup $\mathfrak{S}_{n_0,n_1} \cap \tau_1\mathfrak{S}_{n_0,n_1}{\tau_1}^{-1}$ is equal to the centralizer $(\mathfrak{S}_{n_0,n_1})^{\tau_1}$ of $\tau_1$ in $\mathfrak{S}_{n_0,n_1}$, and is given as follows:*

$$\text{(12.8)} \qquad \big\{\sigma \in \mathfrak{S}_{n_0,n_1} \, ; \sigma(i) = i \ (i = n_0, n_0+1)\big\} \cong \mathfrak{S}_{n_0-1} \times \mathfrak{S}_{n_1-1}.$$

*A complete set of representatives of $\big(\mathfrak{S}_{n_0,n_1} \cap \tau_1\mathfrak{S}_{n_0,n_1}{\tau_1}^{-1}\big)\backslash\mathfrak{S}_{n_0,n_1}$ is given by the following set of products in $\mathfrak{S}_{n_0,n_1} \cong \mathfrak{S}_{n_0} \times \mathfrak{S}_{n_1}$:*

$$\text{(12.9)} \qquad \begin{aligned} &\big\{\mathbf{1}, \, s_{n_0-1}, \, s_{n_0-1}s_{n_0-2}\,, \ldots, \, s_{n_0-1}s_{n_0-2}\cdots s_1\big\} \\ &\quad \times \big\{\mathbf{1}, \, s_{n_0+1}\,, \, s_{n_0+1}s_{n_0+2}\,, \ldots, \, s_{n_0+1}s_{n_0+2}\cdots s_{n-1}\big\}. \end{aligned}$$

LEMMA 12.6. (i) *The subset $\mathfrak{S}_{n_0,n_1}\tau_1\mathfrak{S}_{n_0,n_1}\tau_1\mathfrak{S}_{n_0,n_1}$ consists of three $\mathfrak{S}_{n_0,n_1} \times \mathfrak{S}_{n_0,n_1}$ double cosets with the following representative elements:*

$$\mathbf{1}\,, \ \ \tau_1 = (n_0 \ \ n_0+1), \ \ \tau_1' := \tau_1\tau_2 = (n_0 \ \ n_0+1)(n_0-1 \ \ n_0+2).$$

(ii) $\big(\mathfrak{S}_{n_0,n_1}\tau_1\mathfrak{S}_{n_0,n_1}\big) \cap \big(\mathfrak{S}_{n_0,n_1}\tau_1\mathfrak{S}_{n_0,n_1}\tau_1\mathfrak{S}_{n_0,n_1}\big) = \mathfrak{S}_{n_0,n_1}\tau_1\mathfrak{S}_{n_0,n_1}$.

Using the combinatorial results such as Lemmas 12.5 and 12.6, we can determine the square $\widetilde{U}^2$ by elementary but rather lengthy calculations for convolution of integral kernels.

LEMMA 12.7. $\widetilde{U}^2 = n_0 n_1\, I + (n_0 - n_1)\,\widetilde{U}$ *or* $\big(\widetilde{U} - n_0 I\big)\big(\widetilde{U} + n_1 I\big) = 0$.

### 12.3. Virtual character associated to the intertwining operator $\widetilde{U}$

We know from Lemma 12.3 (iii) that $\Pi_n$ decomposes into two inequivalent irreducible components, denoted by $\Pi_{n,\gamma}$, $\gamma = 0, 1$, and accordingly the intertwining operator $\widetilde{U}$ is a linear combination of orthogonal projections $P_\gamma = P_{\Pi_{n,\gamma}}$ onto them:

$$\text{(12.10)} \qquad \Pi_n = \sum_{\gamma=0,1}^{\oplus} \Pi_{n,\gamma} = \Pi_{n,0} \oplus \Pi_{n,1}, \quad \widetilde{U} = d_0 P_0 + d_1 P_1 \ \ (d_0 \neq d_1).$$

Lemma 12.8. *The pair of coefficients* $(d_0, d_1)$ *is one of* $(n_0, -n_1)$ *and* $(-n_1, n_0)$.

The proof of the lemma comes from Lemma 12.7. Exchanging the names of irreducible components $\Pi_{n,0}$ and $\Pi_{n,1}$ if necessary, we may assume that $(d_0, d_1) = (n_0, -n_1)$.

For the trace character $\chi_{\Pi_n}(g) = \mathrm{tr}\big(\Pi_n(g)\big)$ and the virtual character $\chi_{\widetilde{U}\Pi_n}(g) := \mathrm{tr}\big(\widetilde{U}\Pi_n(g)\big)$ $(g \in G_n)$, we have

$$\chi_{\Pi_n} = \chi_{\Pi_{n,0}} + \chi_{\Pi_{n,1}}, \qquad \chi_{\widetilde{U}\Pi_n} = n_0\chi_{\Pi_{n,0}} - n_1\chi_{\Pi_{n,1}}, \tag{12.11}$$

$$\chi_{\Pi_{n,0}} = \frac{n_1\chi_{\Pi_n} + \chi_{\widetilde{U}\Pi_n}}{n}, \quad \chi_{\Pi_{n,1}} = \frac{n_0\chi_{\Pi_n} - \chi_{\widetilde{U}\Pi_n}}{n}. \tag{12.12}$$

Moreover, from (12.6)–(12.7), we have $\mathrm{tr}(\widetilde{U}) = 0$ and so $\chi_{\widetilde{U}\Pi_n}(e) = \mathrm{tr}(\widetilde{U}) = 0$.

Lemma 12.9. *For* $\Pi_n$, $\Pi_{n,0}$ *and* $\Pi_{n,1}$, *we have*

$$\begin{aligned}
\dim \Pi_n &= (\dim \zeta)^n \cdot |\mathfrak{S}_n/\mathfrak{S}_{n_0,n_1}| = (\dim \zeta)^n \cdot \binom{n}{n_0}, \\
\dim \Pi_{n,0} &= \frac{n_1}{n} \dim \Pi_n = (\dim \zeta)^n \cdot \binom{n-1}{n_0}, \\
\dim \Pi_{n,1} &= \frac{n_0}{n} \dim \Pi_n = (\dim \zeta)^n \cdot \binom{n-1}{n_1}.
\end{aligned}$$

### 12.4. Identification of irreducible components $\Pi_{n,0}$, $\Pi_{n,1}$

In [JK, Chapter 2, §2.8], the irreducible components of induced representations of $\mathfrak{S}_n$ from subgroups of type $\mathfrak{S}_{n_0,n_1}$ are studied. Let $\lambda^{n_0;r}$ and $\lambda^{n_1;c}$ be Young diagrams with one row of length $n_0$ and with one column of length $n_1$ respectively for which the corresponding representations are $\pi(\lambda^{n_0;r})$ = the trivial one, and $\pi(\lambda^{n_1;c}) = \mathrm{sgn}$. Applying the general theory to our case, we get the following.

Proposition 12.10. *Induced representation* $\mathrm{Ind}_{\mathfrak{S}_{n_0,n_1}}^{\mathfrak{S}_n} \pi(\lambda^{n_0;r}) \boxtimes \pi(\lambda^{n_1;c})$ *is decomposed into two irreducible components. Their Young diagrams* $\lambda_n^{(0)}$ *and* $\lambda_n^{(1)}$ *are obtained from* $\lambda^{n_0;r}$ *and* $\lambda^{n_1;c}$ *by connecting them in possible ways to get Young diagrams of hook type, so that*

$$\lambda_n^{(0)} = (n_0 + 1, 1, 1, \ldots, 1), \quad \lambda_n^{(1)} = (n_0, 1, 1, \ldots, 1, 1),$$

*where a Young diagram* $\lambda$ *is expressed by lengths of rows as* $\lambda = \big(r_1(\lambda), r_2(\lambda), \dots\big)$.

Frobenius 'Charakteristik' for them are of rank 1 and given respectively by

$$\binom{a_1}{b_1} = \binom{n_1 - 1}{n_0} \quad \text{and} \quad \binom{n_1}{n_0 - 1}. \tag{12.13}$$

In general, let $\binom{a_1}{b_1}$ be a Frobenius 'Charakteristik' of rank 1 of an IUR of $\mathfrak{S}_n$, and denote this IUR and its character by $\pi\binom{a_1}{b_1}$ and $\chi\binom{a_1}{b_1}$ respectively. Then, $a_1 + b_1 = n - 1$, and $\pi\binom{a_1}{b_1} = \mathrm{sgn} \otimes \pi\binom{b_1}{a_1}$ with one-dimensional character sgn, and by Frobenius dimension formula (5.22),

$$\dim \pi\binom{a_1}{b_1} = \binom{n-1}{a_1} = \binom{n-1}{b_1},$$
$$\therefore \ \dim \pi(\lambda_n^{(0)}) = \binom{n-1}{n_0}, \quad \dim \pi(\lambda_n^{(1)}) = \binom{n-1}{n_1}.$$

Moreover irreducible characters for 'Charakteristik' of rank 1 can be calculated as follows [Frob, §5]. For an element $\sigma \in \mathfrak{S}_n$, let its cycle decomposition be $\sigma = \sigma_1\sigma_2\cdots\sigma_m$, and put $n_\ell(\sigma)$ be the multiplicity of cycles of length $\ell \geq 2$. Then the set of these numbers define the equivalence class $[\sigma]$ of $\sigma$. Let $n_1(\sigma)$ be the number of cycles of length 1 (or of trivial cycles) and put with an indeterminate $x$

$$F(x) = F_{[\sigma]}(x) = (1-x)^{n_1(\sigma)-1} \prod_{\ell \geq 2} (1 - x^\ell)^{n_\ell(\sigma)},$$
$$\text{then} \ \ F(x) = \sum_{0 \leq a \leq n-1} (-x)^a \cdot \chi\binom{a}{n-1-a}(\sigma).$$

LEMMA 12.11. *Let* $\sigma = (p \ \ q)$ *be a transposition in* $\mathfrak{S}_n$. *Then the character value at* $\sigma$ *is given as follows*:

$$\chi\binom{a}{n-1-a}(\sigma) = \frac{(n-2)!\,(n-2a-1)}{a!\,(n-a-1)!}$$
$$= \dim \pi\binom{a}{n-1-a} \cdot \frac{n-2a-1}{n-1}.$$

Comparing dimensions in Lemma 12.9 and those in (12.4), we can identify irreducible components $\Pi_{n,0}$, $\Pi_{n,1}$ in case $n_0 \neq n_1$ as given in Proposition 12.13 below.

However, in case $n_0 = n_1$, two Frobenius 'Charakteristik' for $\Pi_{n,0}$, $\Pi_{n,1}$ or two Young diagrams show us $\Pi_{n,1} = \mathrm{sgn} \otimes \Pi_{n,0}$, and we need to check character values for a transposition $\sigma \in \mathfrak{S}_n$. We apply Lemma 12.11, and we get, for $g = (e_{D_n}, \sigma)$ with the identity element $e_{D_n} \in D_n(T)$, the value $\widetilde{\chi}_{\Pi_n}(g)$ from Proposition 11.2 and the one $\widetilde{\chi}_{\widetilde{U}\Pi_n}(g) = \chi_{\widetilde{U}\Pi_n}(g)/\dim \Pi_n$ from Theorem 13.7.

Lemma 12.12. (i) *For normalized characters, there hold*

$$\widetilde{\chi}_{\Pi_n} = \frac{n_1}{n}\,\widetilde{\chi}_{\Pi_{n,0}} + \frac{n_0}{n}\,\widetilde{\chi}_{\Pi_{n,1}}, \quad \widetilde{\chi}_{\widetilde{U}\Pi_n} = \frac{n_0 n_1}{n}\big(\widetilde{\chi}_{\Pi_{n,0}} - \widetilde{\chi}_{\Pi_{n,1}}\big),$$
$$\widetilde{\chi}_{\Pi_{n,0}} = \widetilde{\chi}_{\Pi_n} + \frac{1}{n_1}\,\widetilde{\chi}_{\widetilde{U}\Pi_n}, \qquad \widetilde{\chi}_{\Pi_{n,1}} = \widetilde{\chi}_{\Pi_n} - \frac{1}{n_0}\,\widetilde{\chi}_{\widetilde{U}\Pi_n}$$

(ii) *For $g = (e_{D_n}, \sigma)$ with a transposition $\sigma \in \mathfrak{S}_n$,*

$$\widetilde{\chi}_{\Pi_n}(g) = \frac{1}{\dim \zeta}\left(\frac{n_0(n_0-1)}{n(n-1)} - \frac{n_1(n_1-1)}{n(n-1)}\right), \quad \widetilde{\chi}_{\widetilde{U}\Pi_n}(g) = \frac{1}{\dim \zeta}\,\frac{2n_0 n_1}{n(n-1)};$$
$$\widetilde{\chi}_{\Pi_{n,0}} = \frac{1}{\dim \zeta}\,\frac{n_0 - n_1 + 1}{n-1}, \qquad \widetilde{\chi}_{\Pi_{n,1}} = \frac{1}{\dim \zeta}\,\frac{n_0 - n_1 - 1}{n-1}.$$

In this way, we arrive at the following identification.

Proposition 12.13. *Let $\eta_n = \boxtimes_{i \in \boldsymbol{I}_n} \zeta_i$, $\zeta_i = \zeta$ $(\forall i)$. Then irreducible components of $\Pi_n$ are identified as*

$$(12.14) \qquad \Pi_{n,0} \cong \mathrm{Ind}_{H_n}^{G_n}\big(\eta_n \boxdot \pi(\lambda_n^{(0)})\big), \quad \Pi_{n,1} \cong \mathrm{Ind}_{H_n}^{G_n}\big(\eta_n \boxdot \pi(\lambda_n^{(1)})\big).$$

## §13. Explicit determination of irreducible characters $\chi_{\Pi_{n,0}}, \chi_{\Pi_{n,1}}$

### 13.1. Integral formulas for the virtual character $\chi_{\widetilde{U}\Pi_n}$

Lemma 13.1. *For the virtual character $\chi_{\widetilde{U}\Pi_n}(g) := \mathrm{tr}\big(\widetilde{U}\Pi_n(g)\big)$ $(g \in G_n)$ associated to $\widetilde{U}$ for $\Pi_n = \mathrm{Ind}_{H_n}^{G_n} \pi_n$, we have $\chi_{\widetilde{U}\Pi_n} = n_0\chi_{\Pi_{n,0}} - n_1\chi_{\Pi_{n,1}}$ and*

$$\chi_{\widetilde{U}\Pi_n}(g) = \int_{(H_n \cap \tau_1 H_n \tau_1{}^{-1})\backslash G_n} \mathrm{tr}\big(\pi_n(g' g g'^{-1}\tau_1)U\big)\, d\mu_{(H_n \cap \tau_1 H_n \tau_1{}^{-1})\backslash G_n}(g')$$
$$= \frac{1}{|\mathfrak{S}_{n_0,n_1} \cap \tau_1 \mathfrak{S}_{n_0,n_1} \tau_1{}^{-1}|} \sum_{\sigma' \in \mathfrak{S}_n} \mathrm{tr}\left(\pi_n(\sigma' g \sigma'^{-1}\tau_1) I(\tau_1)\right),$$

*where* $d\mu_{(H_n \cap \tau_1 H_n \tau_1{}^{-1})\backslash G_n}$ *is an invariant measure on* $(H_n \cap \tau_1 H_n \tau_1{}^{-1})\backslash G_n$ *such that each point has unit measure, and* $\pi_n(g'') = 0$ *by definition if* $g'' \notin H_n$.

Noting that

$$\frac{|(H_n \cap \tau_1 H_n \tau_1{}^{-1})\backslash G_n|}{|H_n\backslash G_n|} = \frac{|\mathfrak{S}_{n_0,n_1}|}{|(\mathfrak{S}_{n_0,n_1} \cap \tau_1 \mathfrak{S}_{n_0,n_1} \tau_1{}^{-1})|} = n_0 n_1,$$

we have similarly to the normalized character $\widetilde{\chi}_{\Pi_n} = \chi_{\Pi_n}/\dim \Pi_n$ the following formula for 'normalized' virtual character $\chi_{\widetilde{U}\Pi_n}(g)/\dim \Pi_n$:

$$\begin{aligned}(13.1) \qquad \widetilde{\chi}_{\Pi_n}(g) = \frac{\chi_{\widetilde{U}\Pi_n}(g)}{\dim \Pi_n} &= n_0 n_1 \cdot \int_{G_n} \frac{\mathrm{tr}\big(\pi_n(g' g g'^{-1} \tau_1) U\big)}{\dim \pi_n}\, d\mu_{G_n}(g') \\ &= \frac{n_0 n_1}{|\mathfrak{S}_n|} \sum_{\sigma' \in \mathfrak{S}_n} \frac{\mathrm{tr}\big(\pi_n(\sigma' g \sigma'^{-1} \tau_1) I(\tau_1)\big)}{\dim \pi_n} \qquad (g \in G_n),\end{aligned}$$

where $\dim \Pi_n = \big|\mathfrak{S}_{n_0,n_1}\backslash \mathfrak{S}_n\big| \cdot \dim \pi_n$, $\dim \pi_n = (\dim \zeta)^n$.

LEMMA 13.2. *For a* $g = (d, \sigma) \in G_n$ *with* $d = (t_i)_{1 \le i \le n}$, *assume that* $g\tau_1 = (d, \sigma\tau_1)$ *is in* $H_n$. *We have* $\sigma\tau_1 = \kappa'_0 \kappa'_1$ *with* $\kappa'_0 \in \mathfrak{S}_{I_{n,0}}$, $\kappa'_1 \in \mathfrak{S}_{I_{n,1}}$. *Put* $\mathrm{sgn}_1(\sigma\tau_1) := \mathrm{sgn}(\kappa'_1)$ *by definition, and let* $I(\sigma')$ *for* $\sigma' \in \mathfrak{S}_n$ *be as in* (3.2), *then*

$$\pi_n(g\tau_1) I(\tau_1) = \eta_n(d) \cdot I(\kappa'_0) I(\kappa'_1) \cdot \mathrm{sgn}_1(\sigma\tau_1).$$

*Moreover let* $\kappa'_0 = \prod_{j \in J'_0} \sigma_j$, $\kappa'_1 = \prod_{j \in J'_1} \sigma_j$, *be cycle decompositions of* $\kappa'_0$, $\kappa'_1$ *respectively, then*

$$(13.2) \quad \frac{\mathrm{tr}\big(\pi_n(g\tau_1) I(\tau_1)\big)}{\dim \pi_n} = \prod_{q \in Q'} \frac{\chi_\zeta(t_q)}{\dim \zeta} \cdot \prod_{j \in J'_0 \sqcup J'_1} \frac{\chi_\zeta\big(P_{\sigma_j}(d_j)\big)}{(\dim \zeta)^{\ell(\sigma_j)}} \cdot \prod_{j \in J'_1} \mathrm{sgn}(\sigma_j),$$

*where* $Q' = \boldsymbol{I}_n \setminus \bigsqcup_{j \in J'_0 \sqcup J'_1} K_j$ *with* $K_j = \mathrm{supp}(\sigma_j)$, *and* $d_j = (t_i)_{i \in K_j}$.

Here, for the calculation of trace, we utilized the following elementary general lemma.

LEMMA 13.3. *Let* $\sigma = (1\ 2\ \cdots\ \ell)$ *be a cycle and* $\zeta_i = \zeta$ *for* $i \in K := \{1, 2, \ldots, \ell\} = \mathrm{supp}(\sigma)$. *On the tensor product space* $\bigotimes_{i \in K} V(\zeta_i) = V(\zeta_1) \otimes \cdots \otimes V(\zeta_\ell)$, *the operator* $I(\sigma)$ *is defined as* $I(\sigma)\big(\bigotimes_{i \in K} v_i\big) := \bigotimes_{i \in K} v_{\sigma^{-1}(i)}\big)$ $(v_i \in V(\zeta_i))$. *Then*

$$(13.3) \qquad \mathrm{tr}\big(\textstyle\bigotimes_{i \in K} \zeta_i(t_i) \cdot I(\sigma)\big) = \chi_\zeta(P_\sigma(d)), \quad d = (t_i)_{i \in K} \in D_K(T).$$

*Remark* 13.1. It follows from (4.2) that, for the character of $\Pi_n = \mathrm{Ind}_{H_n}^{G_n} \pi_n$, $\chi_{\Pi_n}(g) \neq 0$ only when $g$ is conjugate to an element of $H_n$. On the other hand, we see from (13.1) that $\chi_{\widetilde{U}\Pi_n}(g) \neq 0$ only when $g$ is conjugate to an element of $H_n\tau_1$. By Lemma 13.4 below, we see that every element $g \in G_n$ except elements of $D_n$ is conjugate under $G_n$ to an element of $H_n\tau_1$.

### 13.2. Explicit calculation of virtual characters $\chi_{\widetilde{U}\Pi_n}(g) = \mathrm{tr}(\widetilde{U}\Pi_n(g))$

For a $g = (d, \sigma) \in G_n = \mathfrak{S}_n(T)$, let

$$g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r} g_1 g_2 \cdots g_m, \quad \xi_q = \big(t_q, (q)\big), \quad g_j = (d_j, \sigma_j), \tag{13.4}$$

be a standard decomposition and put $Q = \{q_1, q_2, \ldots, q_r\}$, $J = \{1, 2, \ldots, m\}$. For $\sigma' \in \mathfrak{S}_n$, we study the condition that $\sigma' g \sigma'^{-1}\tau_1 \in H_n$. This condition is equivalent to $\sigma'\sigma\sigma'^{-1} \in \mathfrak{S}_{n_0,n_1}\tau_1$ since $\tau_1^2 = \mathbf{1}$. Put

$$I'_{n,0} := I_{n,0} \setminus \{n_0\} = \{1, 2, \ldots, n_0 - 1\},$$
$$I'_{n,1} := I_{n,1} \setminus \{n_0{+}1\} = \{n_0 + 2, \ldots, n - 1, n\}.$$

LEMMA 13.4. *For $\sigma = \sigma_1\sigma_2\cdots\sigma_m$, the condition $\sigma'\sigma\sigma'^{-1} = \sigma'_1\sigma'_2\cdots\sigma'_m \in \mathfrak{S}_{n_0,n_1}\tau_1$ for a $\sigma' \in \mathfrak{S}_n$, where $\sigma'_j := \sigma'\sigma_j\sigma'^{-1}$, is equivalent to the following: there exists a subset $J_1 \subsetneq J$ and a $j' \in J \setminus J_1$ such that*

$$\sigma'_{j'} = (a_1\ a_2\ \cdots\ a_u\ n_0\ b_1\ b_2\ \cdots\ b_v\ n_0 + 1)$$
$$\text{with } a_i \in I'_{n,0},\ b_i \in I'_{n,1},\ u \geq 0,\ v \geq 0, \ \text{ or}$$
$$\sigma'_{j'}\tau_1 = (a_1\ a_2\ \cdots\ a_u\ n_0)(b_1\ b_2\ \cdots\ b_v\ n_0 + 1);$$
$$\prod_{j \in J_0} \sigma'_j \in \mathfrak{S}_{I'_{n,0}}, \quad \prod_{j \in J_1} \sigma'_j \in \mathfrak{S}_{I'_{n,1}}, \quad \text{with } J_0 = J \setminus (\{j'\} \sqcup J_1).$$

*Every non-trivial element $\sigma \in \mathfrak{S}_n$ is conjugate to an element in $\mathfrak{S}_{n_0,n_1}\tau_1$.*

Proof is omitted. Put

$$\kappa_0 = (a_1\ a_2\ \cdots\ a_u\ n_0), \quad \kappa_1 = (b_1\ b_2\ \cdots\ b_v\ n_0 + 1), \tag{13.5}$$

then, $\sigma'_{j'}\tau_1 = \kappa_0\kappa_1$, $\kappa_0 \in \mathfrak{S}_{I_{n,0}}$, $\kappa_1 \in \mathfrak{S}_{I_{n,1}}$, and $\kappa_0 = \mathbf{1}$ if $u = 0$, and $\kappa_1 = \mathbf{1}$ if $v = 0$.

From this lemma we see that any conjugacy class of $\mathfrak{S}_n$ except the class of the identity element $\mathbf{1}$ has a representative in $\mathfrak{S}_{n_0,n_1}\tau_1$.

For $g = (d, \sigma)$ above, let $d = (t_i)_{i \in \boldsymbol{I}_n}$, and $K_j = \mathrm{supp}(g_j) = \mathrm{supp}(\sigma_j)$, $d_j = (t_i)_{i \in K_j}$. The condition $\sigma' g \sigma'^{-1} \tau_1 \in H_n$ for $\sigma' \in \mathfrak{S}_n$ is equivalent to $\sigma' \sigma \sigma'^{-1} \tau_1 \in \mathfrak{S}_{n_0, n_1}$, and this can be written down by Lemma 13.4 as follows.

LEMMA 13.5. *Let $\sigma' \in \mathfrak{S}_n$. Then, $\sigma' \sigma \sigma'^{-1} \tau_1 \in \mathfrak{S}_{n_0,n_1}$ if and only if there exists a partition of $J$ as $J = \{j'\} \sqcup J_0 \sqcup J_1$ such that, with $\sigma'_j = \sigma' \sigma_j \sigma'^{-1}$,*

(S1) $\sigma'_{j'} = (a_1\ a_2\ \cdots\ a_u\ n_0\ b_1\ b_2\ \cdots\ b_v\ n_0+1)$ *with* $u \geq 0$, $v \geq 0$, *and* $a_i \in I'_{n,0} := I_{n,0} \setminus \{n_0\}$, $b_i \in I'_{n,1} := I_{n,1} \setminus \{n_0 + 1\}$;

(S2) $\sigma'_j \in \mathfrak{S}_{I'_{n,0}}$ $(j \in J_0)$, $\sigma'_j \in \mathfrak{S}_{I'_{n,1}}$ $(j \in J_1)$.

The condition (S2) is equivalent to

(S2′) $\sigma'(K_j) \subset I'_{n,0}$ $(j \in J_0)$, $\sigma'(K_j) \subset I'_{n,1}$ $(j \in J_1)$.

LEMMA 13.6. *For $g = (d, \sigma) \in G_n$, and $\sigma' \in \mathfrak{S}_n$, with the same notations as above,*

$$\begin{aligned} \pi_n(\sigma' g \sigma'^{-1} \tau_1) I(\tau_1) &= \pi_n(\sigma'(d))\, I(\sigma' \sigma \sigma'^{-1} \tau_1) I(\tau_1) \cdot \mathrm{sgn}_1(\sigma' \sigma \sigma'^{-1} \tau_1) \\ &= \pi_n(\sigma'(d))\, I(\sigma' \sigma \sigma'^{-1}) \cdot \mathrm{sgn}(\kappa_1) \prod_{j \in J_1} \mathrm{sgn}(\sigma_j). \end{aligned}$$

$$\frac{\mathrm{tr}\big(\pi_n(\sigma' g \sigma'^{-1} \tau_1) I(\tau_1)\big)}{\dim \pi_n} = \prod_{q \in Q} \frac{\chi_\zeta(t_q)}{\dim \zeta} \cdot \prod_{j \in J} \frac{\chi_\zeta\big(P_{\sigma_j}(d_{K_j})\big)}{(\dim \zeta)^{\ell(\sigma_j)}} \cdot (-1)^v \prod_{j \in J_1} \mathrm{sgn}(\sigma_j).$$

Noting that $\dim \Pi_n = \big|\mathfrak{S}_{n_0,n_1} \backslash \mathfrak{S}_n\big| \cdot \dim \pi_n$, we get an explicit formula for the normalized virtual character $\widetilde{\chi}_{\widetilde{U}\Pi_n} = \chi_{\widetilde{U}\Pi_n} / \dim \Pi_n$.

THEOREM 13.7. *Take $g = (d, \sigma) \in G_n$ and let its standard decomposition be as in* (13.4)*. If $g$ is in $D_n$ or $\sigma = \mathbf{1}$, then $g$ is not conjugate under $G_n$ to an element of $H_n \tau_1$, and $\widetilde{\chi}_{\widetilde{U}\Pi_n}(g) = 0$. If $g \in G_n \setminus D_n$, then*

$$\widetilde{\chi}_{\widetilde{U}\Pi_n}(g) = \sum_{J_1 \subsetneq J} c''(\sigma, J_1) \cdot \prod_{q \in Q} \frac{\chi_\zeta(t_q)}{\dim \zeta} \cdot \prod_{j \in J} \frac{\chi_\zeta\big(P_{\sigma_j}(d_{K_j})\big)}{(\dim \zeta)^{\ell(\sigma_j)}} \cdot \prod_{j \in J_1} \mathrm{sgn}(\sigma_j),$$

$$c''(\sigma, J_1) = \frac{(n - |\sigma|)!}{n!} \sum_{j' \in J \setminus J_1} d(j', J_1), \quad J_0 := (J \setminus J_1) \setminus \{j'\},$$

*in the case where* $\sum_{j\in J_1}\ell_j < n_1$, $\sum_{j\in J_0}\ell_j < n_0$,

$$\begin{aligned} d(j',J_1) = \ell_{j'}\cdot \sum_{u+v=\ell_{j'}-2:\,u,v\geq 0} &(-1)^v \\ \times \Big\{ n_0(n_0-1)(n_0-2)\cdots\Big(n_0-\sum_{j\in J_0}\ell_j-u\Big) \\ \times\, n_1(n_1-1)(n_1-2)\cdots\Big(n_1-\sum_{j\in J_1}\ell_j-v\Big)\Big\}, \end{aligned}$$

*otherwise* $d(j',J_1)=0$.

Note that the summation for $d(j',J_1)$ actually runs over $(u,v)$ satisfying

$$(\clubsuit)\qquad u+v=\ell_{j'}-2,\ \ 0\leq u< n_0-\sum_{j\in J_0}\ell_j\,,\ \ 0\leq v<n_1-\sum_{j\in J_1}\ell_j\,.$$

However, outside of this condition the corresponding summands are automatically equal to zero. When $u$ (and $v$) can run over all $0,1,\ldots,\ell_{j'}-2$, by using a formula

$$\sum_{0\leq r\leq m}(-1)^r\binom{n}{r}=(-1)^m\binom{n-1}{m},\quad \text{and putting }\ \binom{n}{a}=0\ \ (a<0),$$

we can reduce the expression for $d(j',J_1)$ in a little more simpler form but not so much.

Lemma 13.8. *Suppose* $n_0-\sum_{j\in J_0}\ell_j\geq \ell_{j'}-1$, $n_1-\sum_{j\in J_1}\ell_j\geq\ell_{j'}-1$, *then*

$$\begin{aligned} d(j',J_1) &= \frac{n_0!n_1!}{(n-|\sigma|)!}\,\ell_{j'} \\ &\quad\times\left\{\binom{n-|\sigma|}{n_0-\sum_{j\in J_0}\ell_j-\ell_{j'}}+(-1)^{\ell_{j'}}\binom{n-|\sigma|}{n_0-\sum_{j\in J_0}\ell_j-1}\right\} \\ &= \frac{n_0!n_1!}{(n-|\sigma|)!}\,\ell_{j'} \\ &\quad\times\left\{\binom{n-|\sigma|}{n_0-\sum_{j\in J_0}\ell_j-\ell_{j'}}-\operatorname{sgn}(\sigma_{j'})\binom{n-|\sigma|}{n_1-\sum_{j\in J_1}\ell_j-\ell_{j'}+1}\right\}. \end{aligned}$$

## §14. Limits and asymptotic behavior of the irreducible characters $\widetilde{\chi}_{\Pi_{n,0}}$, $\widetilde{\chi}_{\Pi_{n,1}}$

By Lemma 12.12 (i), we have for the virtual character $\widetilde{\chi}_{\widetilde{U}\Pi_n}$,

$$\widetilde{\chi}_{\widetilde{U}\Pi_n} = \frac{\chi_{\widetilde{U}\Pi_n}}{\dim \Pi_n} = \frac{n_0 n_1}{n} \cdot \big(\widetilde{\chi}_{\Pi_{n,0}} - \widetilde{\chi}_{\Pi_{n,1}}\big).$$

We get from the formula for $\widetilde{\chi}_{\widetilde{U}\Pi_n}$ in Theorem 13.7, an asymptotic evaluation of the difference $\widetilde{\chi}_{\Pi_{n,0}} - \widetilde{\chi}_{\Pi_{n,1}}$ as follows.

THEOREM 14.1. *Assume $n_0/n \to B_0$, $n_1/n \to B_1$ $(n \to \infty)$. Then $B_0 + B_1 = 1$, and we have as a limit a continuous invariant class function on $G = \mathfrak{S}_\infty(T)$ as follows. On the subgroup $D \subset G$, the virtual character $\chi_{\widetilde{U}\Pi_n}$ is always zero. For $g \in G \setminus D$, let its standard decomposition be as in* (13.4)*, then*

$$\lim_{n\to\infty} \frac{\chi_{\widetilde{U}\Pi_n}(g)}{\dim \Pi_n} = \lim_{n\to\infty} \frac{n_0 n_1}{n} \big(\widetilde{\chi}_{\Pi_{n,0}} - \widetilde{\chi}_{\Pi_{n,1}}\big)$$
$$= c_J(B_0, B_1) \cdot \prod_{q\in Q} \frac{\chi_\zeta(t_q)}{\dim \zeta} \cdot \prod_{j\in J} \frac{\chi_\zeta\big(P_{\sigma_j}(d_{K_j})\big)}{(\dim \zeta)^{\ell(\sigma_j)}},$$

$$c_J(B_0, B_1) = \sum_{\{j'\}\sqcup J_0 \sqcup J_1 = J} \ell_{j'}\big(B_0^{\ell_{j'}} B_1 - B_0 B_1^{\ell_{j'}} \operatorname{sgn}(\sigma_{j'})\big) \times \prod_{j\in J_0} B_0^{\ell_j} \prod_{j\in J_1} B_1^{\ell_j} \operatorname{sgn}(\sigma_j).$$

For the normalized character $\widetilde{\chi}_{\Pi_n}$, we have

$$\widetilde{\chi}_{\Pi_n} = \frac{\dim \Pi_{n,0}}{\dim \Pi_n} \widetilde{\chi}_{\Pi_{n,0}} + \frac{\dim \Pi_{n,1}}{\dim \Pi_n} \widetilde{\chi}_{\Pi_{n,1}} = \frac{n_1}{n} \cdot \widetilde{\chi}_{\Pi_{n,0}} + \frac{n_0}{n} \cdot \widetilde{\chi}_{\Pi_{n,1}}. \tag{14.1}$$

Thus we can determine finally the limits of normalized characters of two irreducible components $\Pi_{n,0}$, $\Pi_{n,1}$ of $\Pi_n$, using Theorem 14.1.

THEOREM 14.2. *Assume that $n_0/n \to B_0$, $n_1/n \to B_1$. Then $B_0 + B_1 = 1$ and*

$$\lim_{n\to\infty} \widetilde{\chi}_{\Pi_{n,0}} = \lim_{n\to\infty} \widetilde{\chi}_{\Pi_{n,1}} = \lim_{n\to\infty} \widetilde{\chi}_{\Pi_n}.$$

We obtain also another evaluation of the difference $\widetilde{\chi}_{\Pi_{n,0}} - \widetilde{\chi}_{\Pi_{n,1}}$ as follows, and this, together with Proposition 11.2, Theorem 11.3 and (14.1), will contribute to evaluate fluctuations of each $\widetilde{\chi}_{\Pi_{n,0}}$, $\widetilde{\chi}_{\Pi_{n,1}}$ around their common limit.

THEOREM 14.3. *The difference* $\widetilde{\chi}_{\Pi_{n,0}} - \widetilde{\chi}_{\Pi_{n,1}}$ *is evaluated as*

$$\sup_{g\in G_n} \left|\widetilde{\chi}_{\Pi_{n,0}}(g) - \widetilde{\chi}_{\Pi_{n,1}}(g)\right| \le \frac{|\operatorname{supp}(\sigma)|\big(|\operatorname{supp}(\sigma)|-1\big)}{n-1}. \tag{14.2}$$

*Proof.* From (13.1) and Lemma 13.2, we have for a $g=(d,\sigma)\in G_n$,

$$\begin{aligned}\left|\widetilde{\chi}_{\Pi_{n,0}}(g) - \widetilde{\chi}_{\Pi_{n,1}}(g)\right| &= \frac{n}{n_0 n_1}\frac{\left|\chi_{\widetilde{U}\Pi_n}(g)\right|}{\dim \Pi_n} \\ &\le \frac{n}{|\mathfrak{S}_n|}\sum_{\sigma'\in\mathfrak{S}_n}\frac{\left|\operatorname{tr}\big(\pi_n(\sigma' g {\sigma'}^{-1}\tau_1)I(\tau_1)\big)\right|}{\dim \pi_n} \\ &\le \frac{1}{(n-1)!}\cdot\left|\left\{\sigma'\in\mathfrak{S}_n\,;\,\sigma'\sigma{\sigma'}^{-1}\tau_1\in\mathfrak{S}_{I_{n,0}}\times\mathfrak{S}_{I_{n,1}}\right\}\right|.\end{aligned}$$

Then, by (S1) in Lemma 13.5, we have an upper bound for the numerator in the right hand side as

$$\le \Big(\sum_{1\le j'\le m}\ell_{j'}(\ell_{j'}-1)\Big)\cdot(n-2)! \le |\operatorname{supp}(\sigma)|\big(|\operatorname{supp}(\sigma)|-1\big)\cdot(n-2)!\,.$$

□

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