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THE TUMURA–CLUNIE THEOREM IN SEVERAL COMPLEX VARIABLES

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Abstract

It is a well-known result that if a nonconstant meromorphic function f on $\mathbb{C}$ and its lth derivative $f^{(l)}$ have no zeros for some $l \geq 2$, then f is of the form $f(z) = \exp(Az + B)$ or $f(z) = (Az + B)^{-n}$ for some constants A, B. We extend this result to meromorphic functions of several variables, by first extending the classic Tumura–Clunie theorem for meromorphic functions of one complex variable to that of meromorphic functions of several complex variables using Nevanlinna theory.

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1. Introduction

Let $\mathbb{Z}_+$ denote the set of nonnegative integers. For $z = (z_1, \ldots, z_m) \in \mathbb{C}^m$, $\mathbf{i} = (i_1, \ldots, i_m) \in \mathbf{Z}_+^m$, we write

$$\partial_{z_k} = \frac{\partial}{\partial z_k},\ k = 1, \ldots, m; \quad \partial^{\mathbf{i}} = \partial_z^{\mathbf{i}} = \partial_{z_1}^{i_1} \cdots \partial_{z_m}^{i_m}; \quad |\mathbf{i}| = i_1 + \cdots + i_m.$$

In this paper, we are interested in the following problem.

Conjecture 1.1. If f is a meromorphic function in $\mathbb{C}^m$ such that f and $\partial^{\mathbf{l}} f$ have no zeros for some $\mathbf{l} = (l_1, \ldots, l_m) \in \mathbb{Z}_+^m$ with $l_k \geq 2$ $(1 \leq k \leq m)$ and such that the set of poles of f is algebraic, then there exists a partition

$$\{1, \ldots, m\} = I_0 \cup I_1 \cup \cdots \cup I_k$$

such that $I_i \cap I_j = \emptyset$ $(i \neq j)$, and

$$f(z_1, \ldots, z_m) = \exp\Big(\sum_{i \in I_0} A_i z_i + B_0\Big) \prod_{j=1}^{k} \Big(\sum_{i \in I_j} A_i z_i + B_j\Big)^{-n_j},$$

where A_i, B_j are constants with $A_i \neq 0$, and n_j are positive integers.

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This is open if $m > 1$. When $m = 1$, the conclusion of Conjecture 1.1 was obtained by Tumura [12], and Hayman [5] gave a proof for the case $l = l_m = 2$. Later, as a correction of the gap in Tumura's proof, Clunie [1] gave a valid proof of the assertion for any $l > 1$ (or see [6]). If there is no finiteness assumption on the poles of f, a proof was given by Frank [2] in 1976 for $l = l_m \geq 3$ (see also Frank *et al.* [3]) and Langley [9] in 1993 for $l = l_m = 2$.

Let f be a nonconstant meromorphic function in $\mathbb{C}^m$. We shall be concerned largely with meromorphic functions $h = P(f, \partial^{\mathbf{i}_1} f, \dots, \partial^{\mathbf{i}_k} f)$ which are polynomials in f and the partial derivatives $\partial^{\mathbf{i}_1} f, \dots, \partial^{\mathbf{i}_k} f$ of f with meromorphic coefficients a of the form

$$\| \quad T(r, a) = o(T(r, f)), \tag{1.1}$$

where $T(r, f)$ is *Nevanlinna's characteristic function* of f, and where the symbol $\|$ means that the relation holds outside a set of r of finite linear measure. Such functions h will be called *differential polynomials* in f. The degree of the polynomial $P(x_0, x_1, \dots, x_k)$ is called the degree of h. To study Conjecture 1.1, the following result will play a crucial role.

THEOREM 1.2. *Suppose that f is meromorphic and not constant in $\mathbb{C}^m$, that*

$$g = f^n + P_{n-1}(f), \tag{1.2}$$

where $P_{n-1}(f)$ is a differential polynomial of degree at most $n - 1$ in f, and that

$$\| \quad N(r, f) + N\left(r, \frac{1}{g}\right) = o(T(r, f)),$$

where $N(r, f)$ is Nevanlinna's valence function of f for poles. Then

$$g = \left(f + \frac{a}{n}\right)^n,$$

where a is a meromorphic function of the form (1.1) in $\mathbb{C}^m$ determined by the terms of degree $n - 1$ in $P_{n-1}(f)$ and by g.

When $m = 1$, Theorem 1.2 is due to Hayman [6, Theorem 3.9]. By using Theorem 1.2, we can give a proof of Conjecture 1.1, under a condition on the nonvanishing of the partial derivatives of order greater than 1 that differs from the one posed in the conjecture. This is the following theorem.

THEOREM 1.3. *If f is a meromorphic function in $\mathbb{C}^m$ such that f, $\partial_{z_1}^{l_1} f$, $\dots$, $\partial_{z_m}^{l_m} f$ have no zeros for some $l_k \geq 2$ $(1 \leq k \leq m)$ and such that the set of poles of f is algebraic, then there exists a partition*

$$\{1, \dots, m\} = I_0 \cup I_1 \cup \cdots \cup I_k$$

such that $I_i \cap I_j = \emptyset$ $(i \neq j)$, and

$$f(z_1, \dots, z_m) = \exp\left(\sum_{i \in I_0} A_i z_i + B_0\right) \prod_{j=1}^{k} \left(\sum_{i \in I_j} A_i z_i + B_j\right)^{-n_j},$$

where A_i, B_j are constants with $A_i \neq 0$, and n_j are nonnegative integers.

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In particular, if f is entire then the function f in Theorem 1.3 has only an exponential form

$$f(z_1, \dots, z_m) = \exp(A_1 z_1 + \cdots + A_m z_m + B_0).$$

We shall use the methods developed in [6–8] and the generalised Clunie lemma (Lemma 2.1, below) to prove the main results.

2. Proof of Theorem 1.2

The proofs of Theorems 1.2 and 1.3 are based on the following *generalised Clunie lemma*.

Lemma 2.1. Let f be a nonconstant meromorphic function on $\mathbb{C}^m$. Take a positive integer n and take polynomials of f and its partial derivatives:

$$P(f) = \sum_{\mathbf{p}\in I} a_{\mathbf{p}} f^{p_0} (\partial^{\mathbf{i}_1} f)^{p_1} \cdots (\partial^{\mathbf{i}_l} f)^{p_l}, \quad \mathbf{p} = (p_0, \dots, p_l) \in \mathbb{Z}_+^{l+1}, \tag{2.1}$$

$$Q(f) = \sum_{\mathbf{q}\in J} c_{\mathbf{q}} f^{q_0} (\partial^{\mathbf{j}_1} f)^{q_1} \cdots (\partial^{\mathbf{j}_s} f)^{q_s}, \quad \mathbf{q} = (q_0, \dots, q_s) \in \mathbb{Z}_+^{s+1}, \tag{2.2}$$

and

$$B(f) = \sum_{k=0}^{n} b_k f^k, \tag{2.3}$$

where I, J are finite sets of distinct elements and $a_{\mathbf{p}}, c_{\mathbf{q}}, b_k$ are meromorphic functions on $\mathbb{C}^m$ with $b_n \not\equiv 0$. Assume that f satisfies the equation

$$B(f)Q(f) = P(f) \tag{2.4}$$

such that $P(f)$, $Q(f)$ and $B(f)$ are differential polynomials, that is, their coefficients a have property (1.1). If $\deg(P(f)) \le n = \deg(B(f))$, then

$$\| \quad m(r, Q(f)) = o(T(r, f)).$$

For the case $m = 1$, see [6, Lemma 3.3]. We refer the reader to [7, 8] for some special cases of Lemma 2.1, where $P(f)$ is only a polynomial in f. A general proof can be found in [10].

Now we begin the proof of Theorem 1.2. Note that by (1.2) g can have poles only at poles of f or of the coefficients a_ν of $P_{n-1}(f)$. Let l be the order of the highest partial derivatives $\partial^{\mathbf{i}} f$ of f occurring on the right-hand side of (1.2). At a pole of f of order p, $\partial^{\mathbf{i}} f$ generically has a pole of order at most

$$p + l \le (l + 1)p,$$

and so g has a pole of order at most $n(l + 1)p + k$, where k is the sum of the orders of the poles of all the coefficients a_ν. Thus

$$N(r, g) \le n(l + 1)N(r, f) + \sum N(r, a_\nu),$$

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and so

$$\| \quad N(r,g) = o(T(r,f))$$

by hypothesis. *Nevanlinna's proximity function* of g satisfies

$$\| \quad m(r,g) \le O\Big(\sum_{|\mathbf{i}|\le l} m(r,\partial^{\mathbf{i}} f) + \sum_{\nu} m(r,a_\nu)\Big) = O(T(r,f)),$$

so that

$$\| \quad T(r,g) = m(r,g) + N(r,g) = O(T(r,f)).$$

Note that

$$\| \quad N\Big(r,\frac{\partial_{z_i} g}{g}\Big) \le N(r,g) + N\Big(r,\frac{1}{g}\Big) = o(T(r,f)).$$

We have

$$\| \quad T\Big(r,\frac{\partial_{z_i} g}{g}\Big) = m\Big(r,\frac{\partial_{z_i} g}{g}\Big) + N\Big(r,\frac{\partial_{z_i} g}{g}\Big) = o(T(r,f)), \tag{2.5}$$

where, by using the lemma of logarithmic derivative (see [7]), we also have

$$\| \quad m\Big(r,\frac{\partial_{z_i} g}{g}\Big) = o(T(r,f)).$$

We now differentiate (1.2) and obtain

$$\partial_{z_i} g = n f^{n-1}\partial_{z_i} f + Q_i(f) \tag{2.6}$$

for each $i = 1,\dots,m$, where $Q_i(f) = \partial_{z_i} P_{n-1}(f)$ is a differential polynomial in f of degree at most $n-1$. We multiply (1.2) by $\partial_{z_i} g/g$ and subtract from (2.6). This gives

$$f^{n-1}\Big(n\partial_{z_i} f - \frac{\partial_{z_i} g}{g} f\Big) + L_{n-1}(f) = 0. \tag{2.7}$$

The differential polynomial $L_{n-1}(f)$ satisfies the conditions of the generalised Clunie lemma. Thus we deduce from (2.7) that

$$\| \quad m\Big(r, n\partial_{z_i} f - \frac{\partial_{z_i} g}{g} f\Big) = o(T(r,f)).$$

Again the function

$$F_i = \partial_{z_i} f - \frac{1}{n}\frac{\partial_{z_i} g}{g} f$$

has poles only at poles of f or $\partial_{z_i} g/g$ and so we have

$$\| \quad N(r,F_i) \le 2N(r,f) + N\Big(r,\frac{\partial_{z_i} g}{g}\Big) = o(T(r,f)),$$

and hence

$$\| \quad T(r,F_i) = m(r,F_i) + N(r,F_i) = o(T(r,f)).$$

If h is defined by the equations

$$\frac{\partial_{z_i} h}{h} = \frac{1}{n}\frac{\partial_{z_i} g}{g}, \quad i = 1, \ldots, m,$$

we have just proved that

$$\partial_{z_i} f = \psi_i f + F_i,$$

where

$$\psi_i = \frac{1}{n}\frac{\partial_{z_i} g}{g} = \frac{\partial_{z_i} h}{h}.$$

We deduce that

$$\partial_{z_j}\partial_{z_i} f = f\partial_{z_j}\psi_i + \psi_i\partial_{z_j} f + \partial_{z_j} F_i = (\partial_{z_j}\psi_i + \psi_i\psi_j) f + \partial_{z_j} F_i + \psi_i F_j.$$

If we define $\psi = \log h$, by induction we obtain

$$\partial^{\mathbf{i}} f = L_{\mathbf{i}}(\psi) f + F_{\mathbf{i}},$$

where $\mathbf{i} = (i_1, \ldots, i_m) \in \mathbb{Z}_+^m$, $L_{\mathbf{i}}(\psi)$ is a certain differential polynomial in ψ which is independent of F_i, and

$$\| \quad T(r, F_{\mathbf{i}}) = o(T(r, f)),$$

and further $F_{\mathbf{i}} = 0$ if $F_i = 0$ for each i. Writing h instead of f so that $\partial_{z_i} h = \psi_i h$, we deduce that

$$\partial^{\mathbf{i}} h = L_{\mathbf{i}}(\psi) h, \tag{2.8}$$

so that

$$\partial^{\mathbf{i}} f = \frac{\partial^{\mathbf{i}} h}{h} f + F_{\mathbf{i}}.$$

We deduce that if $n_0 + n_1 + \cdots + n_l = n - 1$,

$$\begin{aligned} f^{n_0}(\partial^{\mathbf{i}_1} f)^{n_1} \cdots (\partial^{\mathbf{i}_l} f)^{n_l} &= f^{n-1}\left(\frac{\partial^{\mathbf{i}_1} h}{h}\right)^{n_1} \cdots \left(\frac{\partial^{\mathbf{i}_l} h}{h}\right)^{n_l} + P_{n-2}(f) \\ &= \left(\frac{f}{h}\right)^{n-1} h^{n_0}(\partial^{\mathbf{i}_1} h)^{n_1} \cdots (\partial^{\mathbf{i}_l} h)^{n_l} + P_{n-2}(f), \end{aligned}$$

where $P_{n-2}(f)$ is a differential polynomial in f of degree at most $n-2$. Therefore, if $\pi_{n-1}(f)$ is a homogeneous differential polynomial of degree $n-1$ in f,

$$\pi_{n-1}(f) = \left(\frac{f}{h}\right)^{n-1} \pi_{n-1}(h) + P_{n-2}(f). \tag{2.9}$$

Also it follows from (2.5) and (2.8) that

$$\| \quad T\left(r, \frac{\partial^{\mathbf{i}} h}{h}\right) = T(r, L_{\mathbf{i}}(\psi)) = o(T(r, f))$$

and so

$$\| \quad T(r, h^{1-n}\pi_{n-1}(h)) = o(T(r, f)).$$

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By using (2.9), we can rewrite (1.2) in the form

$$g = f^n + af^{n-1} + P_{n-2}(f),$$

where $P_{n-2}(f)$ is a differential polynomial in f of degree at most $n-2$, and

$$a = \frac{\pi_{n-1}(h)}{h^{n-1}}$$

satisfies property (1.1). Further, we may write this as

$$g = H^n + P_{n-2}(H), \tag{2.10}$$

where $P_{n-2}(H)$ is a differential polynomial in H of degree at most $n-2$, and

$$H = f + \frac{a}{n}.$$

We can now again differentiate (2.10) and eliminate g. We obtain the analogue of (2.7), namely

$$H^{n-1}G_i = L_{n-2}(f), \tag{2.11}$$

where we note that this time the differential polynomial L_{n-2} has degree at most $n-2$, and

$$G_i = \partial_{z_i}H - \frac{1}{n}\frac{\partial_{z_i}g}{g}H.$$

By using the generalised Clunie lemma with (2.11) twice, we deduce as before that

$$\| \quad T(r, G_i) = o(T(r, f))$$

and

$$\| \quad T(r, HG_i) = o(T(r, f)).$$

Hence if G_i is not identically zero we deduce that

$$\| \quad T(r, H) \le T(r, HG_i) + T\Big(r, \frac{1}{G_i}\Big) = o(T(r, f)).$$

This gives a contradiction since then also

$$\| \quad T(r, f) \le T(r, H) + T\Big(r, \frac{a}{n}\Big) + O(1) = o(T(r, f)).$$

Thus G_i must be identically zero for each $i \in \{1, \ldots, m\}$ and

$$n\frac{\partial_{z_i}H}{H} = \frac{\partial_{z_i}g}{g}, \quad g = cH^n,$$

where c is a constant.

We can finally prove that $c = 1$. For otherwise we should deduce from (2.10) that

$$(1-c)H^n + P_{n-2}(H) = 0.$$

A further application of the generalised Clunie lemma now yields that

$$\| \quad m(r,(1-c)H) = o(T(r,H)) = o(T(r,f)),$$

and since

$$\| \quad N(r,H) \le N(r,f) + N\Big(r,\frac{a}{n}\Big) = o(T(r,f)),$$

by hypothesis, this yields

$$\| \quad T(r,H) = o(T(r,f))$$

giving a contradiction as before. Thus $c = 1$ and Theorem 1.2 is proved.

3. Proof of Theorem 1.3

According to Hartogs' theorem, a function f defined on $\mathbb{C}^m$ is meromorphic on $\mathbb{C}^m$ if and only if, for each $j \in \{1,\ldots,m\}$, f is meromorphic for z_j when the variables $z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_m$ are fixed. Thus we can prove Theorem 1.3 by induction on the number m of variables. When $m = 1$, Theorem 1.3 is just the result of Tumura [12] and Clunie [1] (or see [6, Theorem 3.8]). Now we fix $m \ge 2$ and assume that the conclusion in Theorem 1.3 holds for variables of number at most $m-1$.

First of all, we claim that $\partial_{z_i} f/f$ $(1 \le i \le m)$ are all rational. Assume, to the contrary, that one of them, say $\partial_{z_m} f/f$, is transcendental. Applying Theorem 1.3 for $m-1$ variables $z_1, \ldots, z_{m-1}$, there exists a partition

$$\{1,\ldots,m-1\} = I_0 \cup I_1 \cup \cdots \cup I_k$$

with $I_i \cap I_j = \emptyset$ $(i \ne j)$ such that

$$f(z_1,\ldots,z_m) = \exp\Big(\sum_{i\in I_0}\tilde{A}_i(z_m)z_i + \tilde{B}_0(z_m)\Big)\prod_{j=1}^{k}\Big(\sum_{i\in I_j}\tilde{A}_i(z_m)z_i + \tilde{B}_j(z_m)\Big)^{-n_j}, \tag{3.1}$$

where $\tilde{A}_i(z_m)$ $(\not\equiv 0)$ and $\tilde{B}_j(z_m)$ are entire functions of z_m because f is a nonvanishing meromorphic function on $\mathbb{C}^m$. Set

$$F = \frac{\partial_{z_m} f}{f}.$$

Then

$$\partial_{z_m}^2 f = F\partial_{z_m} f + f\partial_{z_m} F = (F^2 + \partial_{z_m} F)f.$$

For $n \ge 1$ we deduce inductively (see [6, Lemma 3.5]) that

$$\frac{\partial_{z_m}^n f}{f} = F^n + \frac{n(n-1)}{2}F^{n-2}\partial_{z_m} F + a_n F^{n-3}\partial_{z_m}^2 F + b_n F^{n-4}(\partial_{z_m} F)^2 + P_{n-3}(F), \tag{3.2}$$

where

$$a_n = \tfrac{1}{6}n(n-1)(n-2), \quad b_n = \tfrac{1}{8}n(n-1)(n-2)(n-3),$$

and $P_{n-3}(F)$ is a differential polynomial with constant coefficients, which vanishes identically for $n \le 3$ and has degree $n-3$ when $n > 3$.

Since the set of poles of f is algebraic, we have

$$N(r, f) = O(\log r)$$

(see [4] or [11]) and hence

$$N(r, F) = N\Big(r, \frac{\partial_{z_m} f}{f}\Big) \le 2N(r, f) = O(\log r).$$

Since F is transcendental, which means (see [4] or [11])

$$\lim_{r\to\infty} \frac{T(r, F)}{\log r} = \infty,$$

then the function

$$g = \frac{\partial^l_{z_m} f}{f}$$

with $l = l_m$ satisfies

$$\| \quad N(r, F) + N\Big(r, \frac{1}{g}\Big) = o(T(r, F)).$$

Thus Theorem 1.2 shows that $g = \psi^l$, where $\psi = F + a/l$. The case which is relevant to Theorem 1.2 is that in which $g = F^l + P_{l-1}(F)$, where

$$P_{l-1}(F) = \frac{l(l-1)}{2} F^{l-2}\partial_{z_m} F + a_l F^{l-3}\partial^2_{z_m} F + b_l F^{l-4}(\partial_{z_m} F)^2 + P_{l-3}(F).$$

In this case

$$h^{l-1} a = \pi_{l-1}(h) = \frac{l(l-1)}{2} h^{l-2}\partial_{z_m} h,$$

that is,

$$a = \frac{l(l-1)}{2}\frac{\partial_{z_m} h}{h} = \frac{l-1}{2}\frac{\partial_{z_m} g}{g};$$

see the proof of Theorem 1.2 for the last relation and definitions of h and $\pi_{l-1}(h)$. Hence,

$$\psi = F + \frac{l-1}{2l}\frac{\partial_{z_m} g}{g} = F + \frac{l-1}{2}\frac{\partial_{z_m}\psi}{\psi}.$$

Set

$$\alpha = \frac{l-1}{2}\frac{\partial_{z_m}\psi}{\psi}.$$

Then

$$\partial_{z_m}\psi = \frac{2\alpha}{l-1}\psi, \quad \partial^2_{z_m}\psi = \Big(\frac{4\alpha^2}{(l-1)^2} + \frac{2\partial_{z_m}\alpha}{l-1}\Big)\psi,$$

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and hence

$$
\begin{aligned}
F &= \psi - \alpha, \\
\partial_{z_m} F &= \partial_{z_m}\psi - \partial_{z_m}\alpha = \frac{2\alpha}{l-1}\psi - \partial_{z_m}\alpha, \\
\partial^2_{z_m} F &= \partial^2_{z_m}\psi - \partial^2_{z_m}\alpha = \Big(\frac{4\alpha^2}{(l-1)^2} + \frac{2\partial_{z_m}\alpha}{l-1}\Big)\psi - \partial^2_{z_m}\alpha,
\end{aligned}
$$

and so on. Thus, if $n = l = l_m \geq 2$, we obtain from (3.2) that

$$
\begin{aligned}
\psi^l = (\psi - \alpha)^l &+ \frac{l(l-1)}{2}(\psi - \alpha)^{l-2}\Big(\frac{2\alpha}{l-1}\psi - \partial_{z_m}\alpha\Big) \\
&+ \Big(a_l\Big(\frac{4\alpha^2}{(l-1)^2} + \frac{2\partial_{z_m}\alpha}{l-1}\Big) + b_l\frac{4\alpha^2}{(l-1)^2}\Big)\psi^{l-2} + Q_{l-3}(\psi),
\end{aligned}
\tag{3.3}
$$

where $Q_{l-3}(\psi)$ is a differential polynomial of degree at most $l-3$ in ψ. In fact, the coefficients of Q_{l-3} are polynomials in α and its derivatives on z_m and the poles of α occur at the zeros and poles of ψ, that is, of g, and so by hypothesis

$$N(r,\alpha) = O(\log r) = o(T(r,F)),$$

and also

$$\| \quad m(r,\alpha) = m\Big(r, \frac{\partial_{z_m}\psi}{\psi}\Big) + O(1) = m\Big(r, \frac{\partial_{z_m} g}{g}\Big) + O(1) = o(T(r,g)) = o(T(r,F)).$$

Therefore,

$$\| \quad T(r,\alpha) = o(T(r,F)). \tag{3.4}$$

We collect terms in the powers of ψ in (3.3) and note that terms of degree l and $l-1$ are eliminated. The equation becomes

$$a_0\psi^{l-2} + (\text{terms of degree at most } l-3) = 0, \tag{3.5}$$

where

$$
\begin{aligned}
a_0 &= \frac{l(l-1)}{2}(\alpha^2 - \partial_{z_m}\alpha) - l(l-2)\alpha^2 + a_l\Big(\frac{4\alpha^2}{(l-1)^2} + \frac{2\partial_{z_m}\alpha}{l-1}\Big) + b_l\frac{4\alpha^2}{(l-1)^2} \\
&= \frac{l(l+1)}{6}\Big(\frac{\alpha^2}{l-1} - \partial_{z_m}\alpha\Big).
\end{aligned}
$$

If $l = 2$, we see at once that $a_0\psi^{l-2} = 0$, so that $a_0 = 0$. If $l > 2$, we apply the generalised Clunie lemma with (3.5), and deduce that

$$\| \quad m(r, a_0\psi) = o(T(r,\psi)) = o(T(r,F)).$$

Since by hypothesis

$$\| \quad N(r,\psi) = O\{N(r,g)\} = o(T(r,F)),$$

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we deduce that

$$\| \quad T(r, a_0\psi) = o(T(r, F))$$

and hence, if a_0 is not identically zero, by using (3.4),

$$\| \quad T(r, \psi) = o(T(r, F)),$$

$$\| \quad T(r, F) \le T(r, \psi) + T\left(r, \frac{\partial_{z_m} g}{g}\right) + O(1) = o(T(r, F)),$$

giving a contradiction. Thus in any case

$$a_0 = \frac{l(l+1)}{6}\left(\frac{\alpha^2}{l-1} - \partial_{z_m}\alpha\right) = 0.$$

This gives on integration either $\alpha = 0$ or

$$\frac{\partial_{z_m}\alpha}{\alpha^2} = \frac{1}{l-1}, \quad \frac{1}{\alpha} = \frac{c_1 - z_m}{l-1},$$
$$\alpha = \frac{l-1}{2}\frac{\partial_{z_m}\psi}{\psi} = \frac{l-1}{c_1 - z_m},$$
$$\psi = c_2(c_1 - z_m)^{-2}, \quad c_2 \not\equiv 0,$$
$$\frac{\partial_{z_m} f}{f} = F = \psi - \frac{l-1}{2}\frac{\partial_{z_m}\psi}{\psi} = c_2(c_1 - z_m)^{-2} - \frac{l-1}{c_1 - z_m},$$
$$f(z) = c_3(c_1 - z_m)^{l-1}\exp(c_2(c_1 - z_m)^{-1}), \quad c_3 \not\equiv 0,$$

where c_1, c_2 and c_3 are entire functions that are independent of z_m. Clearly this function f cannot be meromorphic in $\mathbb{C}^m$. Thus α must be identically zero. It follows that ψ is independent of z_m, and so is F. Note that

$$\begin{aligned} F(z_1, \ldots, z_m) &= \sum_{i \in I_0} \tilde{A}_i'(z_m) z_i + \tilde{B}_0'(z_m) \\ &\quad - \sum_{j=1}^{k} n_j \left(\sum_{i \in I_j} \tilde{A}_i(z_m) z_i + \tilde{B}_j(z_m)\right)^{-1} \left(\sum_{i \in I_j} \tilde{A}_i'(z_m) z_i + \tilde{B}_j'(z_m)\right) \end{aligned} \tag{3.6}$$

is independent of z_m, and so $\tilde{A}_i'(z_m)$, $\tilde{B}_j'(z_m)$ are constants. Therefore F is rational. This is a contradiction, and so our claim is proved.

Since $\partial_{z_i} f/f$ $(1 \le i \le m)$ are all rational, so in particular is F, and, by using (3.2), then $\partial_{z_m}^l f/f$ is rational. Thus, writing f in the form (3.1) again, by the induction assumptions we find from (3.6) that $\tilde{A}_i'(z_m)$, $\tilde{B}_j'(z_m)$ are polynomials. By using the relation

$$\frac{\partial_{z_m}^l f}{f} = F^l + P_{l-1}(F),$$

we see that if $\sum_{i \in I_0} \tilde{A}_i'(z_m) z_i + \tilde{B}_0'(z_m)$ is not constant,

$$\frac{\partial_{z_m}^l f}{f}(z_1, \ldots, z_m) \sim F^l \sim \left(\sum_{i \in I_0} \tilde{A}_i'(z_m) z_i + \tilde{B}_0'(z_m)\right)^l$$

as $z_m \to \infty$. It follows that $\partial^l_{z_m} f = 0$ somewhere in $\mathbb{C}^m$, giving a contradiction. Thus we have

$$\tilde{A}'_i(z_m) \equiv 0 \ (i \in I_0), \quad \tilde{B}'_0(z_m) = \hat{A}_0 = \text{constant},$$

and so there exist constants A_i, B_0 such that

$$\tilde{A}_i(z_m) = A_i \ (i \in I_0), \quad \tilde{B}_0(z_m) = \hat{A}_0 z_m + B_0.$$

If $\hat{A}_0 \neq 0$, and F is not constant, we see that

$$F(z) = \hat{A}_0, \quad \partial^n_{z_m} F = 0 \ (n \geq 1)$$

at $z_m = \infty$. Now we see that on the right-hand side of (3.2) with $n = l$ all the terms except the first vanish at $z_m = \infty$, so that $\partial^l_{z_m} f / f = \hat{A}^l_0$ at $z_m = \infty$, and $\partial^l_{z_m} f$ must again have a zero in $\mathbb{C}^m$. This is a contradiction. Thus if $\hat{A}_0 \neq 0$, it follows that F must be constant, and so

$$\tilde{A}'_i(z_m) \equiv 0 \ (i \in I_j), \quad \tilde{B}'_j(z_m) \equiv 0$$

for each $j = 1, \ldots, k$, that is,

$$\tilde{A}_i(z_m) = A_i = \text{constant} \quad (i \in I_j), \quad \tilde{B}_j(z_m) = B_j = \text{constant}.$$

Set $\tilde{I}_0 = I_0 \cup \{m\}$. Then the partition

$$\{1, \ldots, m\} = \tilde{I}_0 \cup I_1 \cup \cdots \cup I_k$$

has the property in Theorem 1.3.

Finally, if $\hat{A}_0 = 0$ then

$$f(z_1, \ldots, z_m) = \frac{1}{Q(z_1, \ldots, z_m)} \exp\Big(\sum_{i \in I_0} A_i z_i + B_0\Big),$$

where

$$Q(z_1, \ldots, z_m) = \prod_{j=1}^{k} \Big(\sum_{i \in I_j} \tilde{A}_i(z_m) z_i + \tilde{B}_j(z_m)\Big)^{n_j}.$$

For $i \in I_j$ $(1 \leq j \leq k)$, it is easy to see that

$$\tilde{A}_i(z_m) = A_i = \text{constant}$$

since $\partial^{l_i}_{z_i} f$ has no zeros. Set

$$\deg(\tilde{B}_j) = p_j$$

and consider the polynomial

$$Q_1(z_m) = \prod_{j \in J} \Big(\sum_{i \in I_j} A_i z_i + \tilde{B}_j(z_m)\Big)^{n_j}$$

in z_m, where

$$J = \{j \mid 1 \leq j \leq k, \ p_j \geq 1\}.$$

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We choose $(z_1, \dots, z_{m-1}) \in \mathbb{C}^{m-1}$ such that the polynomial in z_m,

$$R_j(z_m) = \sum_{i \in I_j} A_i z_i + \tilde{B}_j(z_m) \quad (j \in J),$$

has distinct zeros. Thus if $Q_1(z_m)$ has degree n, $f_1(z_m) = f(z_1, \dots, z_m)$ has a zero of order n at $z_m = \infty$ and no finite zeros. Suppose that $f_1(z_m)$ has distinct poles of multiplicity q_ν for $\nu = 1, \dots, N$. Then

$$q_1 + \cdots + q_N = n.$$

Also, $f_1^{(l)}(z_m)$ has poles of multiplicity $q_\nu + l$, so that altogether $f_1^{(l)}(z_m)$ has

$$(q_1 + l) + \cdots + (q_N + l) = n + lN$$

poles. Also $f_1^{(l)}(z_m)$ has a zero of order $n + l$ at $z_m = \infty$. Thus $f_1^{(l)}(z_m)$ has $l(N-1)$ finite zeros, and so $N = 1$ since $f_1^{(l)}(z_m) = \partial_{z_m}^l f$ has no finite zeros. Thus J contains only one element, say $J = \{1\}$, which means that

$$\tilde{B}_j(z_m) = B_j = \text{constant } (2 \le j \le k).$$

Since $f_1(z_m)$ has only one pole, it follows that when $I_1 = \emptyset$,

$$Q_1(z_m) = \tilde{B}_1(z_m)^{n_1} = (A_m z_m + B_1)^n,$$

where A_m $(\neq 0)$, B_1 are constant, and so the partition

$$\{1, \dots, m\} = I_0 \cup \tilde{I}_1 \cup \cdots \cup I_k$$

has the property in Theorem 1.3, where $\tilde{I}_1 = \{m\}$.

When $I_1 \neq \emptyset$, and $p_1 \ge 2$, we may choose $(z_1, \dots, z_{m-1}) \in \mathbb{C}^{m-1}$ as before such that $R_1(z_m)$ has at least two distinct zeros, and hence $f_1(z_m)$ has at least two distinct poles. This is a contradiction. Thus $p_1 = 1$, and so there exist constants A_m $(\neq 0)$, B_1 such that

$$\tilde{B}_1(z_m) = A_m z_m + B_1.$$

Hence the partition

$$\{1, \dots, m\} = I_0 \cup \tilde{I}_1 \cup \cdots \cup I_k$$

has the property in Theorem 1.3, where $\tilde{I}_1 = I_1 \cup \{m\}$, and we obtain the conclusion in Theorem 1.3, which also completes the proof by induction.

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