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ON SECOND-ORDER DIFFERENTIAL OPERATORS WITH BOHR-NEUGEBAUER TYPE PROPERTY

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ABSTRACT. Let B be a bounded linear operator having domain and range in a Banach space. If the second-order differential operator d^2/dt^2-B has a Bohr-Neugebauer type property for Bochner almost periodic functions, then any Stepanov-bounded solution of the differential equation $(d^2/dt^2)u(t)-Bu(t)=g(t)$ is Bochner almost periodic, with $g(t)$ being a Stepanov-almost periodic continuous function.

1. Suppose X is a Banach space and J is the interval $-\infty<t<\infty$. A function $f\in L^p_{loc}(J;X)$ with $1\le p<\infty$ is said to be Stepanov-bounded or S^p-bounded on J if

$$\|f\|_{S^p}=\sup_{t\in J}\left[\int_t^{t+1}\|f(s)\|^p\,ds\right]^{1/p}<\infty \tag{1.1}$$

(for the definitions of (Bochner or strong) almost periodicity and S^p-almost periodicity, see pp. 3 and 77, Amerio-Prouse [1]).

Suppose that B is a bounded linear operator having domain and range in X. We say that the second-order differential operator d^2/dt^2-B has Bohr-Neugebauer property if, for any almost periodic X-valued function $f(t)$, any bounded (on J) solution of the equation

$$\frac{d^2}{dt^2}u(t)-Bu(t)=f(t)\quad\text{on}\quad J \tag{1.2}$$

is almost periodic.

The object of this paper is to establish the following result.

THEOREM. *For a bounded linear operator B with domain $D(B)$ and range $R(B)$ in a Banach space X, let the differential operator d^2/dt^2-B be such that, for any almost periodic X-valued function $f(t)$, any S^p-bounded solution $u:J\to D(B)$ of the equation* (1.2) *is S^1-almost periodic. If $p>1$, then, for any S^1-almost periodic continuous X-valued function $g(t)$, any S^p-bounded solution $u:J\to D(B)$ of the equation*

$$\frac{d^2}{dt^2}u(t)-Bu(t)=g(t)\quad\text{on}\quad J \tag{1.3}$$

is almost periodic.

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2. We shall require the following result.

LEMMA (A). *If a differentiable function $h:J\to X$ is S^1-almost periodic, and if h' is uniformly continuous on J, then h and h' are both almost periodic from J to X.*

Proof. See Remark (iii) of Rao-Hengartner [5].

3. **Proof of Theorem.** By (1.3), we have the representation

$$u'(t) = u'(0)+\int_0^t Bu(s)\,ds+\int_0^t g(s)\,ds \quad \text{on} \quad J. \tag{3.1}$$

From this representation, we can show that $u'(t)$ is uniformly continuous on J (see Rao [4]).

Now consider a sequence $\{\rho_n(t)\}_{n=1}^{\infty}$ of non-negative continuous functions on J such that

$$\rho_n(t) = 0 \quad \text{for} \quad |t| \geq n^{-1}, \int_{-n^{-1}}^{n^{-1}} \rho_n(t)\,dt = 1. \tag{3.2}$$

The convolution between u and ρ_n is defined by

$$(u * \rho_n)(t) = \int_J u(t-s)\rho_n(s)\,ds = \int_J u(s)\rho_n(t-s)\,ds. \tag{3.3}$$

From (1.3), it follows that

$$\frac{d^2}{dt^2}(u*\rho_n)(t)-B(u*\rho_n)(t) = (g*\rho_n)(t) \quad \text{on} \quad J. \tag{3.4}$$

As shown in Rao [4], $(u*\rho_n)(t)$ is bounded on J for all $n\geq 1$ and $(g*\rho_n)(t)$ is almost periodic from J to X for all $n\geq 1$.

Therefore it follows from our assumption on the operator d^2/dt^2-B that $(u*\rho_n)(t)$ is S^1-almost periodic from J to X for all $n\geq 1$. The only use made of this assumption in this paper is to guarantee that $(u*\rho_n)(t)$ is almost periodic in the Stepanov sense.

Further, by the uniform continuity of $u'(t)$ on J, we can show that $(u'*\rho_n)(t)$ is uniformly continuous on J.

Now we have

$$(u*\rho_n)'(t) = (u'*\rho_n)(t) \quad \text{on} \quad J. \tag{3.5}$$

Hence $(u*\rho_n)'(t)$ is uniformly continuous on J. Thus, by Lemma (A), $(u*\rho_n)(t)$ and $(u'*\rho_n)(t)$ are both almost periodic from J to X for all $n\geq 1$.

Again by the uniform continuity of $u'(t)$ on J, the sequence of convolutions $(u'*\rho_n)(t)$ converges to $u'(t)$ as $n\to\infty$, uniformly on J. So $u'(t)$ is almost periodic from J to X, and hence is bounded on J. Therefore $u(t)$ is uniformly continuous on J. Consequently, $(u*\rho_n)(t)\to u(t)$ as $n\to\infty$, uniformly on J. So $u(t)$ is almost periodic from J to X, which completes the proof of the theorem.

4. Notes. (i) For $p=1$, our Theorem remains valid for any S^1-bounded uniformly continuous solution of the equation (1.3).

(ii) Suppose X is a separable Hilbert space, and consider the second-order operator differential equation

$$\frac{d^2}{dt^2}u(t)-Bu(t)=f(t) \quad \text{on} \quad J, \quad \text{where} \quad f: J \to X \quad \text{is}$$

an almost periodic function and B is a completely continuous linear operator in X commuting with its adjoint (see p. 258, Bochner-Neumann [2]). Then, by Theorem 1 of Cooke [3], the operator d^2/dt^2-B has Bohr-Neugebauer property. Now suppose that $u(t)$ is an S^p-bounded solution $(1<p<\infty)$ of the above differential equation.

If we replace g by f in the proof of our Theorem, then, by the Bohr-Neugebauer property of the operator d^2/dt^2-B, it follows that $u(t)$ is almost periodic from J to X. Thus, in this case, the operator d^2/dt^2-B satisfies the assumption of our Theorem for $p>1$.

(iii) Now suppose X is a Hilbert space and B is a bounded linear operator in X with $B\geq 0$. Then the operator d^2/dt^2-B has Bohr-Neugebauer property (see Zaidman [6]). Consequently, the operator d^2/dt^2-B satisfies the hypothesis of our Theorem for $p>1$.

(iv) Finally, suppose X is a reflexive Banach space and $B=0$. Given an almost periodic X-valued function $f(t)$, suppose $u(t)$ is a bounded solution of the differential equation

$$\frac{d^2}{dt^2}u(t)=f(t) \quad \text{on} \quad J. \tag{4.1}$$

Then we have the representation

$$u'(t)=u'(0)+\int_0^t f(s)\,ds \quad \text{on} \quad J. \tag{4.2}$$

By Lemma 2 of Cooke [3], it follows from (4.1) that $u'(t)$ is bounded on J. Consequently, by (4.2), $u'(t)$ is almost periodic from J to X (see Amerio-Prouse [1], p. 55 and Authors' Remark on p. 82). Therefore $u(t)$ is also almost periodic from J to X. Hence the operator d^2/dt^2 has Bohr-Neugebauer property.

Now, given an S^1-almost periodic continuous X-valued function $g(t)$, suppose $u(t)$ is an S^p-bounded solution $(1\leq p<\infty)$ of the differential equation

$$\frac{d^2}{dt^2}u(t)=g(t) \quad \text{on} \quad J. \tag{4.3}$$

From (4.3), it follows that

$$\frac{d^2}{dt^2}(u*\rho_n)(t)=(g*\rho_n)(t) \quad \text{on} \quad J, \tag{4.4}$$

where $\{\rho_n(t)\}_{n=1}^{\infty}$ is the sequence defined in the proof of our Theorem.

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Then $(u * \rho_n)(t)$ is bounded on J and $(g * \rho_n)(t)$ is almost periodic from J to X. As shown above, $(u * \rho_n)(t)$ and $(u * \rho_n)'(t)=(u' * \rho_n)(t)$ are both almost periodic from J to X.

By (4.3), it follows from Theorem 8, p. 79, Amerio-Prouse [1] that $u'(t)$ is uniformly continuous on J. So $(u' * \rho_n)(t) \to u'(t)$ as $n \to \infty$, uniformly on J. Hence $u'(t)$ is almost periodic from J to X. Therefore $u(t)$ is uniformly continuous on J, and hence $(u * \rho_n)(t) \to u(t)$ as $n \to \infty$, uniformly on J. Consequently, $u(t)$ is almost periodic from J to X. So the operator d^2/dt^2 satisfies the assumption of our Theorem for $1 \leq p < \infty$.

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