8
Analysis of polarized states: polarimetry

In the previous chapters we have dealt with the production of the polarized states that serve as initial states in reactions. Here we turn to the measurement of the state of polarization of an ensemble of particles, i.e. to polarimetry.

In the analysis of the state of polarization we may be dealing with stable or unstable particles. If the particles are stable it may be possible to rely on well-understood reactions, such as those of QED, to achieve the polarization analysis, via, e.g. Coulomb interference or scattering off a laser beam. Or, if this is impracticable, it is sometimes possible to use a double-scattering technique even if the reaction mechanism is unknown. The only assumption needed for this is time-reversal invariance. If the particles are unstable their decay angular distribution gives information on their state of polarization prior to decay. This is not surprising if the decay is electromagnetic, so that the decay amplitudes are precisely known. What is remarkable, however, is that even when the decay mechanism is *not known* certain decays are 'magic' and still provide information on the polarization state of the decaying particle. Examples are $\rho \to \pi\pi$, $\omega \to \gamma\pi$, $D^* \to \gamma D$, $\psi \to \rho\pi$, $a_2 \to \rho\pi$ etc.

For electron beams, where we can rely on QED, it has been possible to construct very accurate and rapidly acting polarimeters.

One of the most interesting challenges at the moment is to construct efficient high energy proton polarimeters for use at RHIC, UNK and possibly at Fermilab. We shall discuss some of the current ideas in this field.

We shall also give a general treatment of the measurement of the density matrix from sequential scattering and resonance decays. The approach is remarkably simple and powerful and applies to the decay of a resonance of arbitrary spin.

8.1 Stable particles

Here we are primarily concerned with spin-1/2 particles, electrons, protons and neutrons. We consider separately the following cases.

(a) The reaction mechanism is known or essentially understood; this inevitably means electromagnetic or electroweak interactions.
(b) The reaction amplitudes cannot be calculated from first principles, as is the case for strong interactions.

The fundamental ingredient is the fact that differential cross-sections display azimuthal asymmetries if the initial state is polarized. We consider a fixed axis system with a polarized beam A moving along OZ and unpolarized target B. Then for a $2 \to 2$ reaction $AB \to CD$ one has from subsection 5.4.2

$$\frac{d^2\sigma}{dtd\phi} = \frac{1}{2\pi}\frac{d\sigma}{dt}\sum_{l,m}(2l+1)t^l_m(A)(lm;0,0|0,0;0,0)e^{-im\phi} \tag{8.1.1}$$

where C has polar angles θ, ϕ.

For an inclusive reaction $AB \to CX$, from Section 5.8 one has

$$\frac{d^3\sigma}{dtd\phi dM_X^2} = \frac{1}{2\pi}\frac{d^2\sigma}{dtdM_X^2}\sum_{l,m}(2l+1)t^l_m(A) \times (lm;0,0|0,0;0,0)^{\text{inc}}e^{-im\phi} \tag{8.1.2}$$

Thus if the 'lm analysing powers' $(l,m;0,0|0,0;0,0)$ for the reaction are known or can be calculated one can learn about the polarization state of the beam from the ϕ-dependence of the differential cross-section.

For spin-1/2 *beam* particles, with spin-polarization vector $\boldsymbol{\mathcal{P}}$ and with parity-conserving reactions, (8.1.1) and (8.1.2) simplify to (using (5.6.5) and (3.1.35))

$$\frac{d^2\sigma}{dtd\phi} = \frac{1}{2\pi}\frac{d\sigma}{dt}\left[1 + A(t)(\mathscr{P}_y\cos\phi - \mathscr{P}_x\sin\phi)\right] \tag{8.1.3}$$

$$\frac{d^3\sigma}{dtd\phi dM_X^2} = \frac{1}{2\pi}\frac{d^2\sigma}{dtdM_X^2}\left[1 + A_{\text{inc}}(t)(\mathscr{P}_y\cos\phi - \mathscr{P}_x\sin\phi)\right] \tag{8.1.4}$$

where the As are the analysing powers of the reactions for particle A.

In the above we have chosen an arbitrary reference frame with the beam A arriving along OZ and B either at rest or moving along the negative Z-axis. The combination $\mathscr{P}_y\cos\phi - \mathscr{P}_x\sin\phi$ is just $\boldsymbol{\mathcal{P}}\cdot\hat{\mathbf{n}}$ where $\hat{\mathbf{n}}$ is a unit normal to the scattering plane, i.e. $\hat{\mathbf{n}}$ is along $\mathbf{p}_A \times \mathbf{p}_C$. Thus a measurement of the ϕ-dependence gives us the components of $\boldsymbol{\mathcal{P}}$ perpendicular to the collision plane.

For the case of the collision of two spin-1/2 particles whose spin-polarization vectors are unknown, the more general results (5.6.12) or (5.6.20) can be used to measure their polarizations, providing, of course, that we know the values of the various generalized analysing powers.

8.1.1 Reaction mechanism understood

When two hadrons interact, their interaction is controlled by a mixture of strong (nuclear) and electromagnetic forces, and in an exact treatment one would add together the nuclear and electromagnetic hamiltonians. Generally the nuclear forces totally dominate, but there are certain kinematical regions where the long range of the electromagnetic forces leads to transition amplitudes that grow rapidly and eventually exceed the nuclear amplitudes. Of particular interest is the region of small momentum transfer, where, for example, the one-photon exchange amplitudes diverge as $t \to 0$.

There is thus a region of very small t (typically $\approx 10^{-3}$ $(\mathrm{GeV}/c)^2$ at high energies) where the known electromagnetic and the nuclear amplitudes are comparable. Although the nuclear amplitudes cannot be calculated from first principles they are expected to have smooth finite limits as $t \to 0$. Moreover any significant variation with t is expected to occur only for scales of order of a typical hadron mass squared, so that, to a first approximation, we can use just their values in the forward direction $t = 0$.

In summary a knowledge of the electromagnetic amplitudes together with some limited information on the forward nuclear amplitudes may yield enough information to estimate the analysing power of the reaction, at least for very small t. However, it will be seen that in situations involving hadrons it is perhaps an overstatement to claim that the reactions are truly understood.

(*i*) *Electromagnetic–hadronic interference in proton–proton scattering*

Interference at very small angles between the electromagnetic (EM) and hadronic contributions to the scattering amplitudes has long been used as a tool in the study of the phase of the hadronic amplitude. This only utilizes the interference between the hadronic forces and the longest-range part of the EM interaction, namely the Coulomb force. But at high energies magnetic effects become important and we expect to find that EM contributions to helicity-flip amplitudes gives rise to spin-dependent interference phenomena.

Here we shall focus only on the most dominant effects, and we shall approximate the amplitudes as a sum of the one-photon exchange and nuclear amplitudes, as shown in Fig. 8.1. For a detailed treatment and a

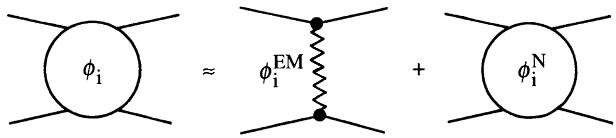

Fig. 8.1 Approximate form of the proton–proton amplitude as a sum of one-photon-exchange and nuclear amplitudes.

discussion of possible inaccuracies in this simple approach, the reader is referred to Buttimore, Gotsman and Leader (1978) and to Leader (1997).

With our normalization the most singular EM contributions to the $pp \to pp$ amplitudes are, for $s \gg m^2$, $t \to 0$,

$$\begin{aligned} \phi_1^{\mathrm{EM}} \equiv H^{\mathrm{EM}}_{++;++} \approx \frac{\sqrt{4\pi\alpha}}{t} \qquad \phi_3^{\mathrm{EM}} \equiv H^{\mathrm{EM}}_{+-;+-} \approx \frac{\sqrt{4\pi\alpha}}{t} \\ \phi_5^{\mathrm{EM}} \equiv H^{\mathrm{EM}}_{++;+-} \approx -\frac{\sqrt{4\pi\alpha}}{\sqrt{-t}}\frac{\alpha\kappa}{2m} \end{aligned} \tag{8.1.5}$$

where κ is the *anomalous* magnetic moment of the proton in units of the proton magneton. The EM contributions to

$$\phi_2 \equiv H_{++;--} \qquad \text{and} \qquad \phi_4 \equiv H_{+-;-+} \tag{8.1.6}$$

are non-singular as $t \to 0$.

The nuclear (N) contributions to all ϕ_i are non-singular as $t \to 0$. Indeed from (4.3.1) we expect

$$\phi^{\mathrm{N}}_{1,2,3} \approx \text{constant} \qquad \phi_5^{\mathrm{N}} \approx \sqrt{-t} \qquad \phi_4^{\mathrm{N}} \approx t. \tag{8.1.7}$$

It is generally supposed, upon the basis of models and some rather sparse low energy data, that the double-flip amplitude ϕ_2^{N} is negligible at high energies. Moreover, to a good approximation the non-flip amplitudes are imaginary, so can be estimated for *very* small t via the optical theorem (see eqn (5.1.4)). Thus, for very small t,

$$\begin{aligned} \phi_1(t) + \phi_3(t) &\approx \phi_1(0) + \phi_3(0) \\ &\approx i\,\mathrm{Im}\,[\phi_1(0) + \phi_3(0)] = \frac{i}{\sqrt{4\pi}}\sigma_{\mathrm{tot}}. \end{aligned} \tag{8.1.8}$$

With the above approximations the differential cross-section is given by (see eqn (4.1.4))

$$\frac{d\sigma}{dt} \approx \frac{1}{2}\left(|\phi_1|^2 + |\phi_3|^2\right) \approx 4\pi\left[\frac{\alpha^2}{t^2} + \left(\frac{\sigma_{\mathrm{tot}}}{8\pi}\right)^2\right]. \tag{8.1.9}$$

We see that the nuclear and electromagnetic contributions are comparable for

$$|t| \approx |t_C| \equiv \frac{8\pi\alpha}{\sigma_{\text{tot}}}. \tag{8.1.10}$$

For $\sigma_{\text{tot}} \geq 40$ mb we get electromagnetic dominance for

$$|t| < 2 \times 10^{-3}\ (\text{GeV}/c)^2. \tag{8.1.11}$$

The analysing power to be used in (8.1.3) is given by (see Table A10.5)

$$A \approx -\frac{2\ \text{Im}\,[\phi_5^*(\phi_1 + \phi_3)]}{|\phi_1|^2 + |\phi_3|^2} \tag{8.1.12}$$

Using (8.1.5), (8.1.7) and (8.1.9) we can eventually write, for very small t,

$$A(t) \approx A_{\text{max}} \left(\frac{4\,(t/t_{\text{max}})^{3/2}}{3\,(t/t_{\text{max}})^2 + 1} \right) \tag{8.1.13}$$

which has a maximum value $A = A_{\text{max}}$ at

$$t = t_{\text{max}} = -\frac{8\sqrt{3}\pi\alpha}{\sigma_{\text{tot}}} = -\sqrt{3}|t_C|. \tag{8.1.14}$$

The maximum value is

$$A_{\text{max}} = \frac{\kappa\sqrt{-3t_{\text{max}}}}{4m}. \tag{8.1.15}$$

For a proton beam with Lab momentum $p_L \approx 200$ GeV$/c$ and taking $\sigma_{\text{tot}} \sim 100($ GeV$/c)^2$ one has $t_{\text{max}} \approx -3\times 10^{-3}($ GeV$/c)^2$ and $A_{\text{max}} \approx 4.6\%$. $A(t)$ and $d\sigma/d\Omega$ are shown in Fig. 8.2.

We see that $A(t)$ is generally small and decreases rapidly with t. Outside the interference region it might well grow owing to purely hadronic effects, but of course we cannot calculate it. Indeed it is somewhat miraculous that we can estimate A for small t by lumping all our ignorance of the strong interactions into a few qualitative features plus the value of σ_{tot}.

At high energies the range of t where $A(t)$ is a few per-cent corresponds to extremely small laboratory scattering angles, so that it is immensely difficult to carry out the asymmetry measurement. Nonetheless work has progressed at Brookhaven on a CNI (Coulomb nuclear interference) polarimeter for use with RHIC and the method was tested at Fermilab (Grosnick *et al.*, 1990).

It should be noted that though the analysing power is small, it would be totally negligible if the proton had no anomalous magnetic moment. For then the helicity-flip amplitude ϕ_5^{EM} in (8.1.5) would have arisen from γ^μ

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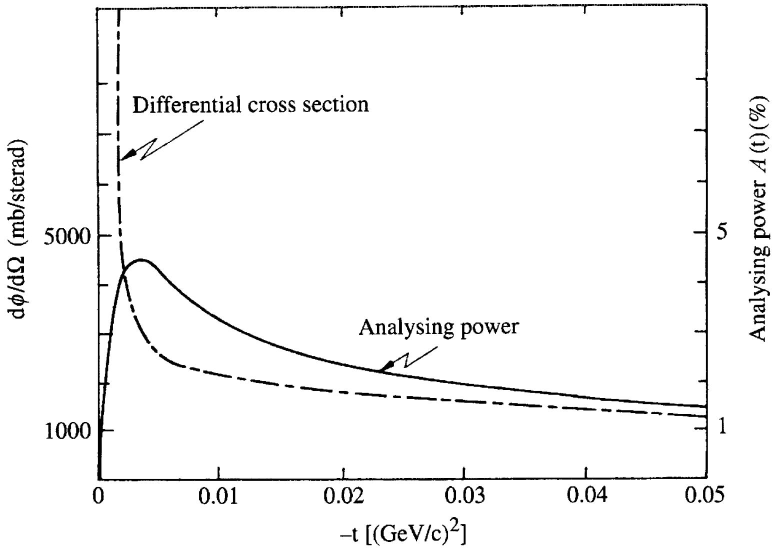

Fig. 8.2 Differential cross-section and analysing power $A(t)$ for $pp \to pp$ at $p_{\mathrm{L}} = 200$ GeV/c.

coupling, which, at high energies, conserves helicity (see subsection 4.6.2), so that we would have found an extra factor of $m/\sqrt{s}$ in ϕ_5^{EM}.

At the time of writing there is great interest in being able to measure beam polarizations at high energies to an accuracy of about 5%. On the theoretical side, attempts have been made to test or improve the accuracy of (8.1.13) by inclusion of hadronic helicity-flip amplitudes (Jakob and Kroll, 1992; Trueman, 1996). On the experimental side attempts have been made to measure A in the Coulomb interference region in pp elastic scattering at 200 GeV/c (Akchurin *et al.*, 1993).

The experiment is exceedingly difficult and the data points, with large errors, are compatible with the result (8.1.13) but do not really test it to any significant degree of accuracy.

It is not clear at present whether one will be able to calculate A to an accuracy of 5%, though a somewhat optimistic conclusion was reached at the RHIC–Brookhaven workshop on CNI polarimetry (Leader, 1997; Leader and Trueman, 1997). This was based upon a new analysis of the magnitude and phase expected for the part of the hadronic-flip amplitudes that might survive at asymptotically high energies. It was suggested that a more accurate expression than (8.1.13), valid for $|t| \lesssim 0.01(\mathrm{GeV}/c)^2$, is

$$A\frac{d\sigma}{dt} = \frac{\alpha\sigma_{\mathrm{tot}}}{m\sqrt{-t}}\left(\frac{\kappa}{2} - \mathrm{Im}\ r_5\right) \qquad (8.1.16)$$

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where r_5 is defined by

$$\phi_5 = r_5 \left(\frac{\sqrt{-t}}{m}\right) \text{ Im } \left(\frac{\phi_1 + \phi_3}{2}\right).$$

The unknown parameter r_5 is in principle a function of both energy and t, but it is argued that Im r_5 in (8.1.16) can be taken as a constant in the RHIC energy range and for $|t|$ as specified above.

Very recently it has been discovered that pp elastic scattering in the CNI region is self-calibrating, in the sense that if enough spin-dependent observables are measured then one can determine not only the values of the various helicity amplitudes but also, most surprisingly, the value of the polarizations of the initial protons (Buttimore *et al.*, 1999). Thus in effect one has an absolute polarimeter for which the theoretical error in the expression for the analysing power is of the order of the fine structure constant α. This will be discussed in Chapter 14.

(*ii*) *Primakoff-type reactions*

In this variant of the original Primakoff effect a π^0 is diffractively produced in the interaction of a proton with the Coulomb field of a heavy nucleus Z:

$$p + Z \to p + \pi^0 + Z.$$

When the final state $p\pi^0$ is moving almost forwards, i.e. at very small momentum transfer to the nucleus, the reaction is dominated by one-photon exchange, as shown in Fig. 8.3.

The Feynman diagram involves the amplitude for the 'reaction'

$$\text{virtual photon } + p \to \pi + p$$

and for very small momentum transfers in the Primakoff process, say $|k^2| \approx 10^{-3}(\text{ GeV}/c)^2$, the virtual photon is almost on mass shell. Thus to a very good approximation we should be able to consider the amplitude involved as the physical amplitude for genuine photoproduction $\gamma + p \to \pi + p$, a reaction which has been well studied at low and medium energies.

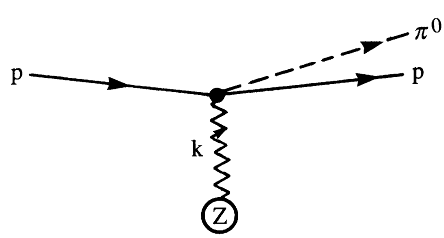

Fig. 8.3. Feynman diagram for Primakoff effect.

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The beautiful and subtle point is that, even for a high energy initial proton, the CM energy of the photoproduction reaction (let us call it $M_{\pi p}$) is small, as we shall show, and at low energies it is known empirically that the photoproduction analysing power is large.

Consider a high energy proton, mass m and with magnitude of momentum p_L incident along OZ upon a fixed target in the Lab made up of heavy nuclei of mass $M \gg m$. If we focus only on reactions in which $t \equiv k^2$ is very small in modulus, $|t| \lesssim 10^{-3}(\text{ GeV}/c)^2$, then one can show that the maximum value of $M_{\pi p}$ is given by

$$\left(M_{\pi p}^{\max}\right)^2 = m^2 + 2p_L\sqrt{-t}. \tag{8.1.17}$$

Thus even for $p_L = 300$ GeV$/c$, $M_{\pi p} \lesssim 4.5$ GeV$/c^2$ and we are dealing with a relatively low energy reaction, which has been well studied experimentally.

The relevant analysing power, usually denoted $T(\theta)$, of the $\gamma p \to \pi^0 p$ reaction varies both with energy and CM scattering angle θ. It is large in the region $1.36 \leq M_{\pi p} \leq 1.52$ GeV$/c^2$ and has a maximum magnitude of about 90%.

The realization that a measurement of the proton polarization at high energies can be linked to low energy photoproduction is due to Underwood (1979). The basic theory was developed by Margolis and Thomas (1978) and a practical feasibility analysis was presented by Kuroda (1982). The experimental possibilities of the approach were finally demonstrated at Fermilab in 1989 by *measuring* the analysing power of the Primakoff reaction using a 185 GeV$/c$ proton beam of *known* polarization and demonstrating that it is in accord with the theoretical expectation. (Carey *et al.*, 1990).

Referring to Fig. 8.4, let p^μ be the 4-momentum of the initial proton and let

$$P^\mu = q^\mu + p'^\mu \qquad (P^2 = M_{\pi p}^2) \tag{8.1.18}$$

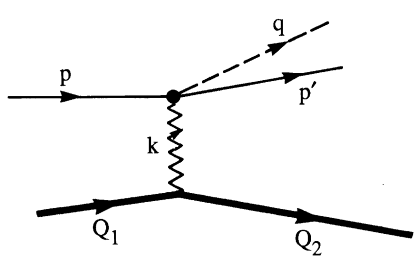

Fig. 8.4. Kinematics for Primakoff effect. Q_1, Q_2 are the initial and final momenta of the heavy nucleus Z.

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be the total 4-momentum of the final πp pair. Define the invariant momentum transfer t for the following reactions:

$$p + Z \to (\pi p) + Z: \qquad t = (P - p)^2 = k^2 \tag{8.1.19}$$

$$\gamma + p \to \pi + p: \qquad t_0 = (p' - p)^2, \tag{8.1.20}$$

Then, assuming that only the diagram in Fig. 8.3 contributes, with a spinless target nucleus of charge Ze, one can show (Margolis and Thomas, 1978) that

$$\frac{d\sigma}{dM_{\pi p}^2 dt dt_0 d\phi_\pi} = \frac{\alpha Z^2}{\pi} \frac{|F(t)|^2}{M_{\pi p}^2 - m^2} \left(\frac{P_\perp^2}{t^2}\right) \frac{d\sigma}{dt_0 d\phi_\pi}(\vec{\gamma}\vec{p} \to \pi^0 p) \tag{8.1.21}$$

where the arrow overbars indicate a polarized particle, $F(t)$ is an unknown nuclear electromagnetic form factor and $\mathbf{P}_\perp$ is the transverse momentum vector of the πp system in the Lab:

$$P_\perp^2 = -t - \frac{\left(M_{\pi p}^2 - m^2\right)^2}{4p_{\mathrm{L}}^2}. \tag{8.1.22}$$

The cross-section $d\sigma/dt_0 d\phi_\pi$ is the differential cross-section for $\gamma p \to \pi p$ with polarized photon and polarized initial proton. The angle ϕ_π is, strictly speaking, the azimuthal angle of the π in the πp rest system, i.e. in the $\gamma p \to \pi p$ CM, with Z-axis along $\mathbf{P}$ in the Lab and some fixed Y-axis. Because the direction of $\mathbf{P}$ differs only infinitesimally from the direction of $\mathbf{p}_{\mathrm{L}}$, for the kinematic region under study, ϕ_π is then also simply the azimuthal angle of the produced pion in the Lab reference frame.

Margolis and Thomas (1978) showed that the almost-real photon is linearly polarized. Then if ϕ_γ is the angle between the polarization vector $\boldsymbol{\varepsilon}$ and the reaction plane, and if $\boldsymbol{\mathcal{P}}$ is the spin-polarization vector for the initial proton in the CM, one has (see Storrow, 1978)

$$\frac{d\sigma}{dt_0 d\phi_\pi} = \frac{1}{2\pi} \left.\frac{d\sigma}{dt_0}\right|_{\text{unpol.}} \Big\{1 - \mathscr{P}_{\text{lin}} \left[\Sigma(\theta)\cos 2\phi_\gamma + \mathscr{P}_x H(\theta)\sin 2\phi_\gamma \right.$$
$$\left. + \mathscr{P}_y P(\theta)\cos 2\phi_\gamma - \mathscr{P}_z G(\theta)\sin 2\phi_\gamma\right] + \mathscr{P}_y T(\theta)\Big\} \tag{8.1.23}$$

where θ is the CM scattering angle, $\mathscr{P}_{\text{lin}}$ is the linear polarization and here the direction OY is along the normal to the reaction plane, i.e. along $\mathbf{k} \times \mathbf{q}$ in the CM. The various functions $\Sigma(\theta)$, $H(\theta)$, $P(\theta)$, $G(\theta)$ and $T(\theta)$ are dynamics-dependent reaction parameters that also depend upon the energy of the $\gamma p \to \pi p$ reaction.

Margolis and Thomas (1978) also showed that $\boldsymbol{\varepsilon}$ lies along the direction of the vector $\mathbf{P}_\perp$ as seen in the $\gamma p \to \pi p$ CM. Moreover the cross-section (8.1.21) is independent of the azimuthal angle Φ of $\mathbf{P}$ in the Lab. Hence if for fixed ϕ_π we average over the direction of $\mathbf{P}_\perp$ we are in effect averaging over ϕ_γ. In this case almost all terms in (8.1.23) average to zero and we are left with

$$\left\langle \frac{d\sigma}{dM_{\pi p}^2 dt dt_0 d\phi_\pi} \right\rangle = \frac{\alpha Z^2}{\pi} \frac{|F(t)|^2}{M_{\pi p}^2 - m^2} \left(\frac{P_\perp^2}{t^2} \right) \times \frac{1}{2\pi} \left. \frac{d\sigma}{dt_0} \right|_{\text{unpol.}} [1 + \boldsymbol{\mathcal{P}} \cdot \mathbf{n} T(\theta)] \tag{8.1.24}$$

where $\mathbf{n}$ is the unit normal to the reaction plane and the angle brackets imply an average over Φ.

Since we are using the reaction as an analyser and we do not know the direction of $\boldsymbol{\mathcal{P}}$ it is perhaps simplest to discuss (8.1.24) in the CM reference frame with fixed X- and Y-axes such that the pion has azimuthal angle ϕ_π. Then the polarization-dependent term in (8.1.24) is just

$$1 + \left(\mathscr{P}_y \cos\phi_\pi - \mathscr{P}_x \sin\phi_\pi \right) T(\theta)$$

where $T(\theta)$ is supposed known.

A study of the ϕ_π-dependence of the cross-section thus gives information on $\mathscr{P}_x$ and $\mathscr{P}_y$. Ideally this could be done for values of θ where $T(\theta)$ is large, but it may be necessary in practice to integrate over θ to increase the statistics.

Unfortunately the very beautiful result (8.1.24) cannot be used directly for polarimetry, because we have ignored, in the above, all contributions arising from the purely hadronic diffraction production of the πp system. It is usually assumed that the hadronic amplitude is due to Pomeron exchange, does not depend on helicity and is essentially imaginary. Typically it is taken to be of the form $iC \exp(-bP_\perp^2)$ for very small $P_\perp^2$, with C real. The slope b should reflect the 'size' R of the nucleus $Z: b \propto 1/R^2$. For Pb one estimates $b \sim 250\ (\text{GeV}/c)^2$, so that the hadronic differential cross-section has a slope of about 500 $(\text{GeV}/c)^2$. However, the cross-section in (8.1.24) has a sharp peak at

$$P_\perp^2 = \frac{\left(M_{\pi p}^2 - m^2 \right)^2}{2p_\text{L}^2},$$

which, for $M_{\pi p} \simeq 1.23\ \text{GeV}/c^2$ and $p_\text{L} = 200\ \text{GeV}/c$, corresponds to the tiny value $P_\perp^2 = 1.5 \times 10^{-5}\ (\text{GeV}/c)^2$. Thus a fit to the $P_\perp^2$ distribution can help to estimate the hadronic part of the cross-section.

For the region of such small values of $P_\perp^2$ and $|t|$, the form factor $F(t)$ can safely taken to be $F(0) = 1$. The observed $P_\perp^2$ distribution can then be fitted by

$$\left.\frac{d\sigma}{dP_\perp^2}\right|_{\text{Expt}} = \left.\frac{d\sigma}{dP_\perp^2}\right|_{\text{Primakoff}} + C^2 e^{-bP_\perp^2} \tag{8.1.25}$$

from which C and b could be determined, in principle.

The net effect would then be that the ϕ_π-dependent part of (8.1.24) becomes

$$1 + \boldsymbol{\mathcal{P}} \cdot \mathbf{n} T(\theta) f(P_\perp^2) \tag{8.1.26}$$

where

$$f(P_\perp^2) = \left.\frac{d\sigma}{dP_\perp^2}\right|_{\text{Primakoff}} \Big/ \left.\frac{d\sigma}{dP_\perp^2}\right|_{\text{expt}} \tag{8.1.27}$$

is a dilution factor such that the effective analysing power is $T(\theta)f(P_\perp^2)$.

In the experiment of Carey *et al.* (1990) mentioned earlier, where a proton beam of *known* polarization was used, a somewhat similar approach was taken to the analysis, and f, averaged over the $P_\perp^2$ of the experiment, was measured to be 0.55 with an error of about ± 0.18.

Clearly there is a significant influence from the hadronic amplitude and (8.1.24) cannot be used as an absolute polarimeter as it stands. But the dilution factor is not catastrophic and (8.1.26) seems to offer a practicable approach to high energy polarimetry provided f can be measured accurately.

The argument that led to (8.1.26) is actually flawed. The photoproduction analysing power $T(\theta)$ would be zero if the photoproduction amplitudes were all real! (see Appendix section A10.4).

Thus there could be important interference effects between the Primakoff and hadronic amplitudes. However, as we shall now explain, this cannot change the *form* of (8.1.26) – only the physical interpretation of $f(P_\perp^2)$ changes.

Using methods based on (5.4.31) and on the analysis of resonance decay to be dealt with in Section 8.2, in which parity conservation is assumed, it is possible to show that the cross-section for

$$p + Z \to \pi + p + Z,$$

averaged over $\mathbf{P}_\perp$, depends on the initial proton polarization only via a factor

$$1 + A(p_{\text{L}}, P_\perp^2, M_{\pi p}^2, \theta) \boldsymbol{\mathcal{P}} \cdot \mathbf{n} \tag{8.1.28}$$

where $A(p_{\text{L}}, P_\perp^2, M_{\pi p}^2, \theta)$ is the proton analysing power of the reaction

$$p + Z \to \pi + p + Z,$$

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irrespective of the dynamical mechanism. It has to be determined experimentally.

This can be seen in a simpler way. The cross-section must be invariant under space inversion. Thus the pseudovector $\boldsymbol{\mathcal{P}}$ must occur in a scalar product with some other pseudovectors. *A priori* the latter could be $\mathbf{k} \times \mathbf{q}$ and $\mathbf{k} \times \mathbf{p}'$. However, at fixed $\mathbf{q}$, it follows that $\mathbf{p}' = \mathbf{P} - \mathbf{q}$ and so averaging over $\mathbf{P}_\perp$ causes $\boldsymbol{\mathcal{P}} \cdot (\mathbf{k} \times \mathbf{p}')$ to reduce to $-\boldsymbol{\mathcal{P}} \cdot (\mathbf{k} \times \mathbf{q})$. Thus the only possibility is $\boldsymbol{\mathcal{P}} \cdot \hat{\mathbf{n}}$ and (8.1.28) is the most general form possible. From this more general point of view the Primakoff analysis simply tells us in what kinematic regime we can expect to find a significant analysing power for the reaction

$$p + Z \to \pi + p + Z.$$

Despite these rather disappointing conclusions, it could still be possible to have an absolute Primakoff polarimeter if one had enough events to be able to restrict oneself to really very small $P_\perp^2$ i.e. of order 10^{-5} $(\text{GeV}/c)^2$. For, from the study of Carey *et al.* (1990), one can deduce that in this kinematic region $d\sigma|_{\text{Primakoff}} \gtrsim 100 d\sigma|_{\text{Hadronic}}$.

(*iii*) *Møller scattering*

The reaction

$$e + e \to e + e$$

is, from a spin point of view, formally identical to elastic proton–proton scattering, so that all the formula relating CM reaction parameters to helicity amplitudes may be taken over from Table A10.3.

Being an electromagnetic interaction we treat it in the Born, i.e. the one-photon-exchange, approximation, in which case all the helicity amplitudes are real. From the first two entries of Table A10.5 we see that the standard analysing power is then zero. However, the initial state spin correlation parameters $A_{\alpha\beta}$ will be non-zero and the reaction can be used to measure the polarization of the beam provided that we use a polarized target with known polarization. This can be achieved by using very thin magnetized ferromagnetic foils, in which degrees of polarization of about 8% are attained. The direction of the spin-polarization is easily reversed.

To begin with we work in the CM of the reaction. We choose our Y-axis such that the known target spin-polarization vector lies in the YZ-plane. Let the beam and target spin-polarization vectors be specified in the CM frame by

$$\boldsymbol{\mathcal{P}} = (\mathscr{P}_x, \mathscr{P}_y, \mathscr{P}_z) \qquad \text{and} \qquad \boldsymbol{\mathcal{P}}^{\mathrm{T}} = (0, \mathscr{P}_y^{\mathrm{T}}, \mathscr{P}_z^{\mathrm{T}}). \tag{8.1.29}$$

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The general form of the differential cross-section $d^2\sigma/dtd\phi$ is given in (5.6.12), in which particle B is the target. We must now calculate the reaction parameters A_{ij} for our process.

We are particularly interested in high energy electrons, so we may greatly simplify the calculation by going to the CM, where all the electrons are highly relativistic, and making use of the result (see subsection 4.6.2) that helicity is conserved for vector coupling in this situation. Thus neglecting terms of order m^2/s, where m is the electron mass, in our normalization, the only non-negligible, correctly symmetrized (see (4.2.16)) helicity amplitudes are (Buttimore, Gotsman and Leader, 1978, Appendix A):

$$\begin{aligned}\phi_1 &= \sqrt{4\pi}\alpha\left(\frac{1}{t}+\frac{1}{u}\right)\\ \phi_3 &= \sqrt{4\pi}\alpha\frac{1}{t}\left(1+\frac{t}{s}\right)e^{i\phi}\\ \phi_4 &= -\sqrt{4\pi}\alpha\frac{1}{u}\left(1+\frac{u}{s}\right)e^{i\phi}\end{aligned} \tag{8.1.30}$$

where we have used (4.1.7) to generalize the Buttimore *et al.* results to $\phi \neq 0$. Using these in the expressions given in Table A10.5 we find

$$\begin{aligned}A^{(A)} = A^{(B)} = A_{XZ} &= A_{ZX} = 0\\ A_{XX} = A_{YY} &= -\frac{\sin^4\theta}{(4-\sin^2\theta)^2}\\ A_{ZZ} &= \frac{\sin^2\theta(8-\sin^2\theta)}{(4-\sin^2\theta)^2}.\end{aligned} \tag{8.1.31}$$

Then (5.6.12) becomes

$$\frac{d^2\sigma}{dtd\phi} = \frac{1}{2\pi}\frac{d\sigma}{dt}\left[1+\frac{\sin^4\theta}{(4-\sin^2\theta)^2}\mathscr{P}_y\mathscr{P}_y^{\mathrm{T}} - \frac{\sin^2\theta(8-\sin^2\theta)}{(4-\sin^2\theta)^2}\mathscr{P}_z\mathscr{P}_z^{\mathrm{T}}\right] \tag{8.1.32}$$

Note that there is no azimuthal dependence. Under reversal of $\boldsymbol{\mathcal{P}}^{\mathrm{T}}$ we then have the asymmetry:

$$\frac{\frac{d\sigma}{dt}(\boldsymbol{\mathcal{P}},\boldsymbol{\mathcal{P}}^{\mathrm{T}}) - \frac{d\sigma}{dt}(\boldsymbol{\mathcal{P}},-\boldsymbol{\mathcal{P}}^{\mathrm{T}})}{\frac{d\sigma}{dt}(\boldsymbol{\mathcal{P}},\boldsymbol{\mathcal{P}}^{\mathrm{T}}) + \frac{d\sigma}{dt}(\boldsymbol{\mathcal{P}},-\boldsymbol{\mathcal{P}}^{\mathrm{T}})} = \frac{\sin^2\theta}{(4-\sin^2\theta)^2} \times\left[\sin^2\theta\,\mathscr{P}_y\mathscr{P}_y^{\mathrm{T}} - (8-\sin^2\theta)\mathscr{P}_z\mathscr{P}_z^{\mathrm{T}}\right] \tag{8.1.33}$$

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so that $\mathscr{P}_y$ and $\mathscr{P}_z$ can be determined by suitably varying $\mathscr{P}_y^{\mathrm{T}}$ and $\mathscr{P}_z^{\mathrm{T}}$. Note that the 'analysing powers' vary strongly with CM angle θ and are large near $\theta = \pi/2$. Also the cross-section in this region is relatively large so that an efficient polarimeter is feasible.

The technique of Møller polarimetry has been used successfully in a range of experiments, most recently in the SLAC E142, E143, E154 and E155 experiments (Feltham and Steiner, 1997; Band, 1997) on polarized deep inelastic scattering. A statistical precision of 1%–2% is achieved in typically 15 minutes!

For access to the experimental aspects of Møller polarimetry the reader is referred to the *Proceedings of the 12th International Symposium on High-Energy Spin Physics, Amsterdam, 1996* (World Scientific, Singapore, 1997).

(*iv*) *Compton scattering*

The reaction

$$\gamma + e \rightarrow \gamma + e$$

was studied in great detail in lowest-order QED by Lipps and Tolhoek (1954). For an unpolarized initial photon the cross-section is independent of the polarization of the electron. However, it does depend upon any linear polarization of the photon and for a circularly polarized photon depends also upon the polarization of the electron. Hence by scattering a laser beam of *known* circular polarization off the electrons and measuring the differential cross-section, one can learn about the spin-polarization vector of the electrons.

This technique has been used with great success at several e^+e^- storage rings, most recently at SLAC and HERA.

At the SLC/SLD at SLAC (see Fig. 8.5), a 532 nm YAG laser beam, corresponding to circularly polarized photons of energy 2.33 eV in the Lab, collides almost head-on with a high energy (45.6 GeV) polarized electron beam. The electrons are scattered into a narrow forward cone and are detected in a Čerenkov detector. The photons are backscattered but are not detected. The polarization measurement involves a comparison of the detection rates when the circular polarization of the laser beam is reversed. The Compton polarimeter is capable of achieving a statistical accuracy of 1% in 3 minutes and polarizations are now quoted with a total accuracy of $\pm 0.67\%$!

At HERA there are two Compton polarimeters in operation, one using a pulsed Nd:YAG laser, the other using a continuous argon-ion laser. In both it is the backscattered photons that are detected. Statistical errors of 0.4% are achieved in 10 minutes and an overall accuracy of about 3% is expected.

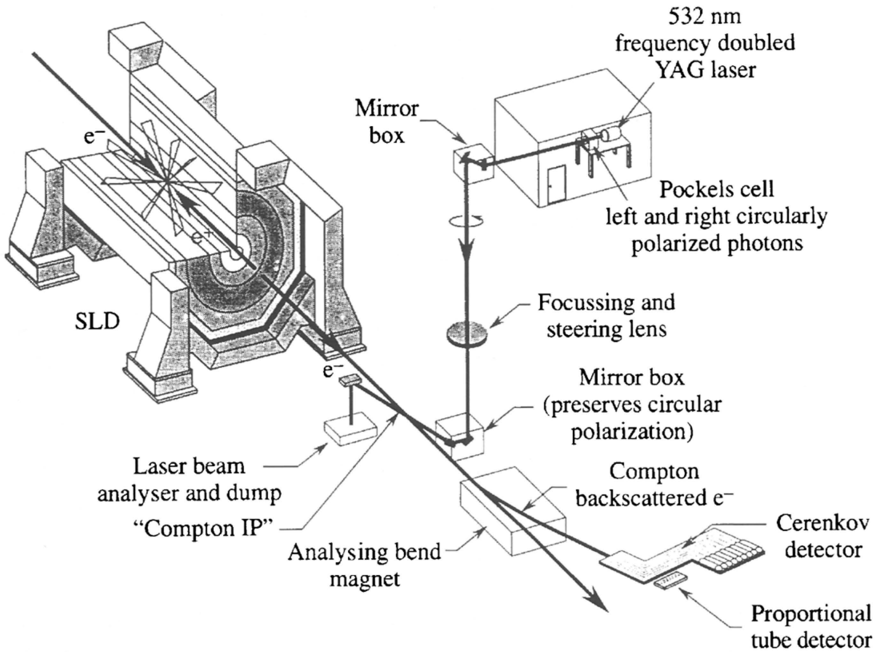

Fig. 8.5 Compton polarimeter at SLC/SLD (courtesy of J. Clendenin, L. Piemontese and M. Swartz).

Consider now the kinematics of Compton scattering. Usually, in the literature, the reaction is discussed in what is called the laboratory frame, meaning the frame where the target electron is at rest. Here, the collision in the laboratory frame is between the laser beam and the fast-moving electron beam. So we shall, for clarity, refer to these frames as the *electron rest frame* and the *Lab collision frame*. The kinematics in these frames is shown in Fig. 8.6. The angle between the incoming laser beam and the electron beam is so small that it can be ignored, so that we have, in effect, a head-on collision and the two frames are related by a simple boost along the Z-axis. We assume some fixed axis system with the electron beam incoming along OZ. The photon is scattered into polar angles $\theta = \pi - \theta_\gamma,\ \phi = \phi_\gamma$ in the electron rest frame, as shown.

Note that in the Lab collision frame a very high energy electron collides with a very low energy photon, so that the final state particles are largely swept forward along OZ. On the contrary, in the electron rest frame a very high energy photon collides with the stationary electron and the final state particles are largely swept along the negative OZ direction.

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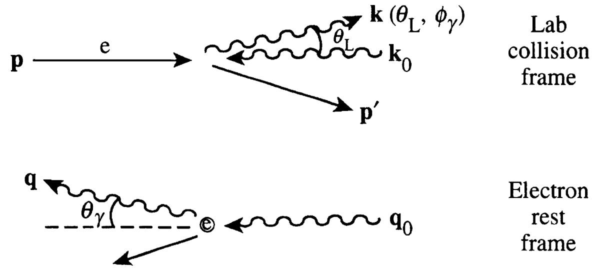

Fig. 8.6 Kinematics of Compton scattering. Initially the electron moves along OZ in the Lab collision frame; in both parts of the figure the Z-axis is to the right.

In the electron rest frame we have the famous Compton relation

$$\frac{m_e}{q} - \frac{m_e}{q_0} = 1 - \cos\theta_\gamma. \tag{8.1.34}$$

The connection between the electron rest frame variables and the given Lab collision frame variables, the photon energy k_0, the electron energy E_e and the angle θ_L as shown, are (neglecting m_e in comparison with E_e),

$$q_0 = \frac{2E_e k_0}{m_e} \qquad \tan\theta_\mathrm{L} = \left(\frac{m_e}{E_e}\right)\frac{\sin\theta_\gamma}{1-\cos\theta_\gamma}. \tag{8.1.35}$$

Some care must be exercised in taking over the result of the Lipps–Tolhoek papers. Firstly, their results are presented in an electron rest frame with the Y-axis perpendicular to the reaction plane whereas we wish to analyse the electron's spin-polarization vector with respect to some fixed-axis system. Secondly, their paper was written prior to the invention of helicity states, so their photon density matrix is given in a basis that utilizes the states of linear polarization $|\mathbf{e}_{(x)}\rangle$ and $|\mathbf{e}_{(y)}\rangle$, given in (3.1.75), rather than in the helicity basis. The necessary alterations to the Lipps–Tolhoek results can easily be made by use of the results in subsection 3.1.12 and Section 5.4.

We suppose that a laser beam moving along the negative OZ direction contains a fraction $1-f$ of its photons with linear polarization $\mathscr{P}_\mathrm{lin}$ along an axis at angle γ to OX and a fraction f of its photons with circular polarization $\mathscr{P}_\mathrm{circ}$, where $\mathscr{P}_\mathrm{circ} > 0$ corresponds to positive helicity and to left-circular polarization in classical optics. If the beam collides with an electron whose spin-polarization vector is $\boldsymbol{\mathcal{P}} = (\mathscr{P}_x, \mathscr{P}_y, \mathscr{P}_z)$ in the fixed reference frame then, the invariant differential cross-section is given, in the electron rest frame, by

$$\frac{d\sigma}{dtd\phi} = \frac{m_e^2 r_0^2}{s^2}\left(\frac{q}{q_0}\right)\left\{1+\cos^2\theta_\gamma + \frac{1}{m_e}(q_0-q)(1-\cos\theta_\gamma)\right.$$
$$+(1-f)\mathscr{P}_{\rm lin}\cos(2\gamma-2\phi_\gamma)\sin^2\theta_\gamma + f\frac{\mathscr{P}_{\rm circ}}{m_e}\left[\mathscr{P}_z(q_0+q)\cos\theta_\gamma\right.$$
$$\left. -\ (\mathscr{P}_x\cos\phi_\gamma + \mathscr{P}_y\sin\phi_\gamma)q\sin\theta_\gamma\right](1-\cos\theta_\gamma)\Big\} \tag{8.1.36}$$

where r_0 is the 'classical electron radius' $e^2/(m_e c^2)$.

If we consider a purely circularly polarized laser beam ($f = 1$) and we analyse the data in the electron rest frame then a study of the ϕ_γ-dependence will, in principle, yield all the components of the electron's spin-polarization vector $\boldsymbol{\mathcal{P}}$.

In practice the polarization of the laser beam is known with great precision, whereas the cross-section measurement suffers from significant normalization errors. It is therefore better to study the asymmetry under reversal of the circularity of the photon polarization, i.e. to study

$$\frac{d\sigma(\mathscr{P}_{\rm circ}) - d\sigma(-\mathscr{P}_{\rm circ})}{d\sigma(\mathscr{P}_{\rm circ}) + d\sigma(-\mathscr{P}_{\rm circ})}$$
$$= \mathscr{P}_{\rm circ}\frac{\left[\mathscr{P}_z(q_0+q)\cos\theta_\gamma - (\mathscr{P}_x\cos\phi_\gamma + \mathscr{P}_y\sin\phi_\gamma)q\sin\theta_\gamma\right](1-\cos\theta_\gamma)}{m_e(1+\cos^2\theta_\gamma)+(q_0-q)(1-\cos\theta_\gamma)}. \tag{8.1.37}$$

For longitudinally polarized electrons, where we attempt to measure $\mathscr{P}_z$, one uses cross-sections integrated over the azimuthal angle ϕ_γ, in which case the measured asymmetry becomes

$$\frac{\langle d\sigma(\mathscr{P}_{\rm circ}) - d\sigma(-\mathscr{P}_{\rm circ})\rangle}{\langle d\sigma(\mathscr{P}_{\rm circ}) + d\sigma(-\mathscr{P}_{\rm circ})\rangle} = -\mathscr{P}_{\rm circ}\mathscr{P}_z A_{\rm C}(\theta_\gamma) \tag{8.1.38}$$

where the underlying Compton asymmetry is

$$A_{\rm C}(\theta_\gamma) = \frac{-(q_0+q)\cos\theta_\gamma(1-\cos\theta_\gamma)}{m_e(1+\cos^2\theta_\gamma)+(q_0-q)(1-\cos\theta_\gamma)}. \tag{8.1.39}$$

For 100% right (R) or left (L) circularly polarized light the measured asymmetry is thus

$$\frac{\langle d\sigma({\rm R}) - d\sigma({\rm L})\rangle}{\langle d\sigma({\rm R}) + d\sigma({\rm L})\rangle} = \mathscr{P}_z A_{\rm C}(\theta_\gamma). \tag{8.1.40}$$

The basic asymmetry $A_{\rm C}(\theta_\gamma)$ is shown as a function of $\cos\theta_\gamma$ in Fig. 8.7 for $E_e = 47$ GeV.

Note that (8.1.39) can be written as a function of E'_e/E_e, where E'_e is the electron recoil energy in the Lab collision frame. Figure 8.8 shows the

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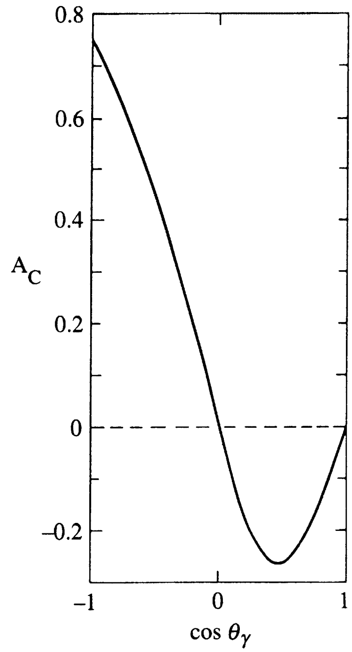

Fig. 8.7 The underlying Compton asymmetry $A_C(\theta_\gamma)$ as a function of $\cos\theta_\gamma$, for $E_e = 47$ GeV (from Alexander *et al.*, 1996).

measured and theoretical asymmetry (8.1.40) at SLAC, for $\mathscr{P}_z = 77.5\%$, as a function of E'_e.

It can be seen from these figures that the asymmetries are very large for θ_γ near π, which corresponds to electrons with smaller recoil energy E'_e.

(*v*) *Polarimetry via decay distributions*

A seminal series of experiments on polarized deep inelastic lepton hadron scattering has been in progress at CERN for more than a decade, utilizing a high energy polarized muon beam (the European Muon Collaboration, Ashman *et al.* 1989; the Spin Muon Collaboration, Adams *et al.*, 1997).

The muons, say μ^+, are produced by the decay in flight of high energy pions, $\pi^+ \to \mu^+ + \nu_\mu$. In the Standard Model the neutrinos are entirely left-handed i.e. $\lambda_\nu = -1/2$. Consequentially, in the rest frame of the pion, where the muon and neutrino are produced back to back, the μ^+ can only be produced with $\lambda_\mu = -1/2$, because the π is spinless. Thus in the pion rest system (more correctly in the helicity rest frame of the muon reached from the pion rest frame) the muons are -100% longitudinally polarized.

In the Lab frame (more correctly in the muon helicity rest frame reached from the Lab frame) the muon spin-polarization vector will be different on account of the Wick helicity rotation discussed in subsection 2.2.2. The

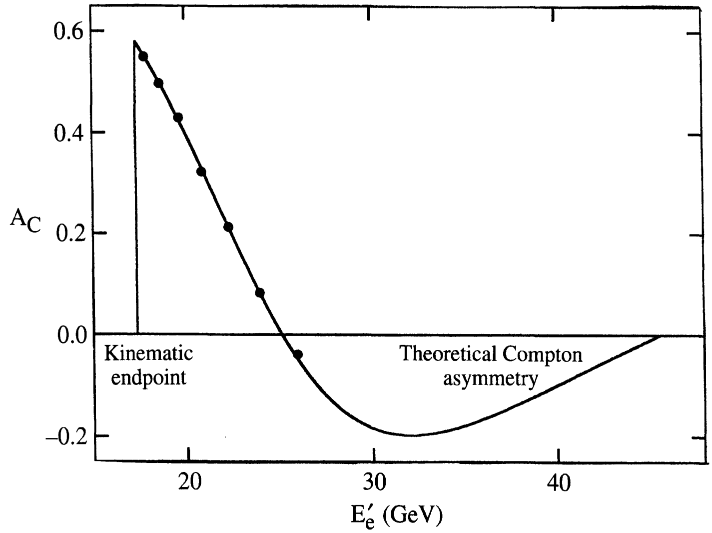

Fig. 8.8 The measured and theoretical Compton asymmetry $\mathscr{P}_z A_C(E'_e)$ at SLAC for $E_e = 45.6$ GeV and for $\mathscr{P}_z = 77.5\%$, as a function of electron recoil energy E'_e. The solid line gives the theory; the points are the measurements, made using a Čerenkov detector.

Wick angle θ_{Wick} will be given by (2.2.6), in which $\delta = \theta \equiv \theta^*$ is the angle between the muon and pion directions of flight, in the pion rest system and $\beta = -\beta_\pi \approx -1$. The rotation is about an axis perpendicular to the plane containing the muon and pion momenta, the latter taken to be along OZ in the Lab.

As a consequence the purely longitudinal mean spin vector of a given muon in the pion rest frame will appear in the Lab frame to have z-component

$$\mathscr{P}_z^{(\mu)}(\theta^*) = -\frac{E^* \cos\theta^* + p^*}{E^* + p^* \cos\theta^*}, \tag{8.1.41}$$

where E^* and p^* are variables in the pion rest system, and will also have a component of $\boldsymbol{\mathcal{P}}$ perpendicular to OZ. Averaging over the azimuthal angles of all muons with a given value of θ^* yields $\mathscr{P}_\perp^{(\mu)}(\theta^*) = 0$ for the ensemble.

Only for strictly forward-going muons $\theta^* = 0$, i.e. the most energetic ones in the Lab, will $\mathscr{P}_z = -1$. (Indeed for $\theta^* = \pi$, $\mathscr{P}_z = +1$!) But the muon beam, of necessity, contains a cone of particles with a range of momenta. It will thus be an ensemble that is longitudinally polarized, with $\boldsymbol{\mathcal{P}}^{(\mu)} = \mathscr{P}_z^{(\mu)}\mathbf{e}_{(z)}$, where $|\mathscr{P}_z^{(\mu)}| < 1$. For accurate work it is therefore necessary to measure $\mathscr{P}_z^{(\mu)}$ for the beam.

Consider the concrete case of positively charged muons. The μ^+ eventually undergo β-decay in flight, $\mu^+ \to e^+ \nu_e \bar{\nu}_\mu$, and the shape of the positron energy spectrum is sensitive to the polarization of the muon and is controlled by the so-called Michel parameter (see Commins and Bucksbaum, 1983, Chapter 3.2; for the transformation to the Lab see Combley and Picasso, 1974). One has

$$\frac{dN}{dy} = N\left[\frac{5}{3} - 3y^2 + \frac{4y^3}{3} - \mathscr{P}_z^{(\mu)}\left(\frac{1}{3} - 3y^2 + \frac{8y^3}{3}\right)\right] \tag{8.1.42}$$

where $y = E_e/E_\mu$ is the ratio of the Lab energies of the e^+ and μ^+ and N is the total number of muon decays.

The spectrum dN/dy is shown as a function of y in Fig. 8.9. In practice QED corrections, which have a small but non-negligible effect on the spectrum, are also taken into account (Adeva *et al.*, 1994).

Using this approach the SMC were able to determine that $\mathscr{P}_z = -0.82$ with an error of $\pm 3\%$ for their 100–200 GeV muon beam.

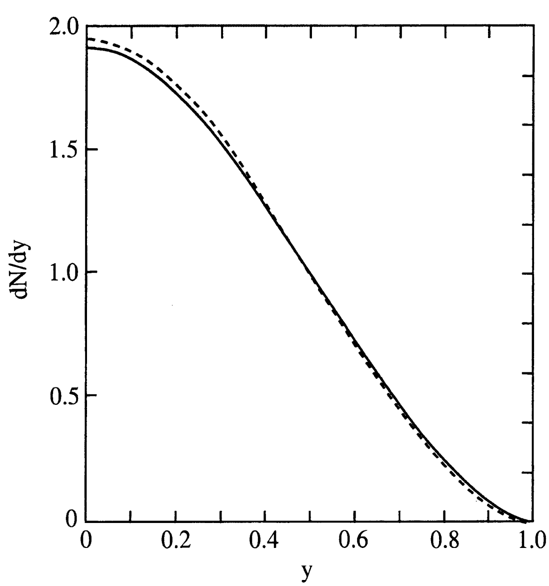

Fig. 8.9 Positron energy spectrum in $\mu^+ \to e^+ \nu_e \bar{\nu}_\mu$ as a function of $y = E_e/E_\mu$ with (- - -) and without (—) QED corrections (from Adeva *et al.*, 1994.)

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8.1.2 Reaction mechanism not known

It is sometimes possible to use a reaction as a polarization analyser even though its analysing powers are not calculable from first principles. Often, some ingenuity is required in order to do so.

(i) Beam and/or target polarization adjustable

If the target and/or the beam can be put into a state of known polarization, either by electromagnetic means or by utilizing a parity-violating decay to produce the beam (see Section 6.4), so that the initial t^l_m or t^L_M are known, then by measuring the ϕ-dependence of $d^2\sigma/dtd\phi$ one can measure the generalized analysing powers $(l,m;0,0|0,0;0,0)$ and/or $(0,0;L,M|0,0;0,0)$. This is discussed at length in Section 5.4.

(ii) Double-scattering experiments

Another approach is to rely on time-reversal invariance and to do a double-scattering experiment $A+B \to C+D$ followed by $C+D \to A+B$. The initial beam is scattered through an angle θ off one target and the scattered beam is re-scattered off an identical target through the same angle θ. Suppose that both targets are unpolarized and that the initial beam is also unpolarized. The scattered beam C will be polarized, in general, so an analysis of the ϕ-dependent cross-section asymmetries for the *second reaction*, in the natural Lab analysing frame S_{LC}, will yield the combination, see (5.4.14),

$$\begin{aligned} C_m(\theta) \equiv \sum_{l\geq m}(2l+1)\left[t^l_m(C;\theta)\right]_{S_{LC}} \\ \times (l,m;0,0|0,0;0,0)^{CD\to AB}_{\theta;\phi=0}. \end{aligned} \tag{8.1.43}$$

In deriving (8.1.43) we have used a form of (5.4.11) in which C is now the beam particle and we have stressed the θ-dependence. C is produced with multipole parameters $t^l_m(C;\theta)$ in the CM of the first reaction, but, as discussed in subsection 3.3.2(ii), it is the multipole parameters in the appropriate frame S_{LC}, where C is the incoming beam particle, that must be used in (8.1.43). From eqn (3.3.14),

$$\left[t^l_m(C;\theta)\right]_{S_{LC}} = \sum_{m'} d^l_{mm'}(\alpha_C)t^l_{m'}(C;\theta). \tag{8.1.44}$$

Now from (5.4.2), for an unpolarized initial state,

$$t^l_m(C;\theta) = (0,0;0,0|l,m;0,0)^{AB\to CD}_{\theta;\phi=0} \tag{8.1.45}$$

and assuming time-reversal invariance we have, from (5.3.9) and (5.3.3),

$$\begin{aligned} (0,0;0,0|l,m;0,0)^{AB\to CD}_{\theta;\phi=0} &= (l,m;0,0|0,0;0,0)^{CD\to AB}_{\theta;\phi=\pi} \\ &= e^{-im\pi}(l,m;0,0|0,0;0,0)^{CD\to AB}_{\theta;\phi=0} \end{aligned} \tag{8.1.46}$$

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Thus

$$\left[t^l_m(C;\theta)\right]_{S_{LC}} = \sum_{m'} d^l_{mm'}(\alpha_C)e^{-im'\pi} \times (l,m';0,0|0,0;0,0)^{CD\to AB}. \qquad (8.1.47)$$

We see from (8.1.43) and (8.1.47) that what is measured is a bilinear combination of the analysing powers of the reaction $CD \to AB$. If the beam can be passed through a magnetic field between the two scatterings (see subsection 5.4.2(ii)) then for the same angle θ several combinations $C_m(\theta,\delta_i)$ can be measured, involving different bilinear combinations, with known coefficients, of the analysing powers. A sufficient number of measurements with various values of the magnetic field may allow one to solve (8.1.42) and (8.1.47) for the $(l,m;0,0|0,0;0,0)$, though sign ambiguities may remain that must be settled by other means. A beautiful example of this technique can be found in Button and Mermod's experiment on non-relativistic deuteron–deuteron scattering (Button and Mermod, 1960).

A word of caution is necessary, however. What is hidden in the usual discussion (and in the above) is the fact that the reaction parameters one is trying to measure also depend, in principle, on the CM energy of each collision. Since the scattered particle from the first reaction loses energy in the Lab frame, especially if it scatters through a large angle, the CM energy of the second reaction will be less than for the first. Care should be taken to assess in any given situation whether this is a relevant consideration.

We consider some interesting practical examples.

(a) $p+p \to p+p$ *followed by* $p+p \to p+p$

After the first reaction, taken to lie in the XZ-plane, proton C emerges with spin-polarization vector $\boldsymbol{\mathcal{P}} = (0, A(\theta), 0)$ where $A(\theta)$ is the analysing power. There is no Wick helicity rotation in this case (as can be seen using (8.1.44) or from the fact that the rotation is about OY), so that the same vector $\boldsymbol{\mathcal{P}}$ specifies the spin polarization for the incident nucleon in the second reaction. For the second reaction the Y-axis of the S_{LC} is in the same direction as the Y-axis of the Lab frame so we may use (5.6.12) to get

$$\frac{d^2\sigma}{dt\,d\phi} = \frac{1}{2\pi}\frac{d\sigma}{dt}\left[1+\cos\phi\ A(\theta)\mathscr{P}_y\right] = \frac{1}{2\pi}\frac{d\sigma}{dt}\left[1+\cos\phi\ A^2(\theta)\right] \qquad (8.1.48)$$

so that the ϕ-dependence yields $A(\theta)$ up to a sign. The method is relatively simple for spin-1/2 particles. The practical problem is that $A(\theta)$ gets very small at high energies, at least for moderate t-values.

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(b) $A + B \to \gamma + D$ *followed by* $\gamma + D \to A + B$
Consider the state of polarization of the γ produced in the first reaction from an unpolarized initial state. From (5.4.2)

$$t^l_m(\gamma, \theta) = (0, 0; 0, 0|l, m; 0, 0)^{A+B\to\gamma+D}. \tag{8.1.49}$$

The absence of helicity zero for photons implies, from the properties of the matrices $T^l_m(s = 1)$, see (3.1.26), that the only non-zero reaction parameters, see (5.3.2), have $m = 0$ or $m = 2$. Also, from (5.3.7), $(0, 0; 0, 0|1, 0; 0, 0) = 0$. Thus (3.1.84) and (3.1.87) imply that the photon can only be linearly polarized. Moreover, from (5.3.8), for a parity-conserving reaction $(0, 0; 0, 0|2, 2; 0, 0)$ is real. Thus in (3.1.87) $\gamma = 0$ and from (3.1.86) the photon is linearly polarized along OX in its standard helicity frame.

Because we are dealing with a photon there is no Wick helicity rotation and the $t^l_m(\gamma; \theta)$ given by (8.1.49) are the correct multipole parameters for the photon as the beam particle in the second reaction. (The second reaction is described in a frame whose Y-axis is in the same direction as the Lab Y-axis of the first reaction.)

For the time-reversed reaction $\gamma + D \to A + B$, with linearly polarized γ, we have from (5.4.11)

$$\frac{d^2\sigma}{dtd\phi} = \left(\frac{2}{3}\right)\frac{1}{2\pi}\frac{d\sigma}{dt}\sum_{l,m}(2l + 1)t^l_m(\gamma; \theta) \times (l, m; 0, 0|0, 0; 0, 0)^{\gamma D\to AB}_{\theta,\phi} \tag{8.1.50}$$

If conventionally we define the photon tensor analysing power of the above reaction $\gamma D \to AB$ by

$$\Sigma(\theta) \equiv \frac{2\sqrt{5}}{\sqrt{3}}(2, 2; 0, 0|0, 0; 0, 0)^{\gamma D\to AB}_{\theta,\phi=0} \tag{8.1.51}$$

and use the fact that for photons (as can be deduced from (5.3.2) and (3.1.26) or from (5.4.2) and (3.1.87))

$$(2, 0; 0, 0|0, 0; 0, 0) = \frac{1}{\sqrt{10}} \tag{8.1.52}$$

and that

$$t^2_2(\gamma; \theta) = (0, 0; 0, 0|2, 2; 0, 0)^{AB\to\gamma D} = (2, 2; 0, 0|0, 0; 0, 0)^{\gamma D\to AB}_{\phi=\pi}$$

by time reversal, and so by (5.3.3)

$$t^2_2(\gamma; \theta) = (2, 2; 0, 0|0, 0; 0, 0)^{\gamma D\to AB}_{\phi=0} = \frac{\sqrt{3}}{2\sqrt{5}}\Sigma(\theta),$$

then we eventually end up, for the second reaction, with

$$\frac{d^2\sigma}{dtd\phi} = \frac{1}{2\pi}\frac{d\sigma}{dt}\left\{1 + [\Sigma(\theta)]^2 \cos 2\phi\right\} \tag{8.1.53}$$

so that $\Sigma(\theta)$ can be found, up to a sign, from the azimuthal dependence.

Of course if one has a photon beam of *known* linear polarization, $\Sigma(\theta)$ can be measured in a much simpler fashion: see Appendix 10.4.

(*iii*) *Asymmetries in inclusive reactions*

It is an empirical fact that many high energy inclusive hadronic reactions have a significant polarizing power (e.g. $pp \to \Lambda X$) and/or a large analysing power (e.g. $\vec{p}p \to \pi^{\pm}X$). These polarizing and analysing powers appear to remain big even for large-momentum-transfer reactions, in contradiction to naive perturbative QCD expectations. Some possible mechanisms for these are discussed in Chapter 13.

There is, to date, no generally agreed dynamical explanation for the observed behaviour. However, the event rates are high and provided the analysing power can be *measured empirically* to sufficient accuracy they can be utilized as polarimeters.

Perhaps the most promising reaction for polarization analysis of proton beams is

$$\vec{p} + p \to \pi^{+,-,0} + X. \tag{8.1.54}$$

For an unpolarized target, let the unknown spin-polarization vector for the incoming proton beam be $\boldsymbol{\mathcal{P}} = (\mathscr{P}_x, \mathscr{P}_y, \mathscr{P}_z)$ with respect to some fixed Lab or CM reference frame. Then, according to the arguments given in Section 5.8, the differential cross-section will be given by eqn (5.6.12) with the target spin-polarization vector $\boldsymbol{\mathcal{P}}^B$ put to zero and the various analysing powers now referring to the reaction (8.1.54). One has then

$$\frac{d^2\sigma}{dtd\phi} = \frac{1}{2\pi}\frac{d\sigma}{dt}\left[1 + A_N\left(\mathscr{P}_y \cos\phi - \mathscr{P}_x \sin\phi\right)\right] \tag{8.1.55}$$

where we have followed convention and utilized (see (5.6.13) and discussion thereafter)

$$A_N \equiv A^{(A)} \tag{8.1.56}$$

for the proton analysing power of the reaction.

For an unpolarized beam and polarized target exactly the same formula holds with $A_N \to -A_N$ (since $A^{(B)} = -A^{(A)}$ in this case) provided the spin-polarization vector of the target is specified in its natural helicity rest frame reached from the CM, as explained in subsection 3.3.1.

The azimuthal dependence in (8.1.55) then allows us to determine the component of $\boldsymbol{\mathcal{P}}$ perpendicular to the collision axis, provided $A_N(\theta)$ is known.

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Measurements of $A_N(\theta)$ were originally carried out at Argonne, Brookhaven and CERN, but the experiments of most relevance to high energy polarimetry have been those carried out at Fermilab using the 200 GeV/c tertiary polarized proton beam described in Section 6.4.

8.2 Unstable particles

There is a vast literature concerning the derivation of the properties, especially the spin and parity, of a resonance from an analysis of its decay, stemming from the heyday of the 1960s when vast numbers of new particles were discovered (Jackson, 1965). Recently this kind of analysis has again come into vogue with the study of the charm and bottom families and the search for glueballs. Our concern is with cases where the resonance or unstable particle is well known and we wish to use its decay purely to learn about its density matrix as it emerges from the production reaction.

The resonance typically decays into two or three particles and their angular distribution is presented in their CM, i.e. in a rest frame of the decaying particle. As mentioned in subsection 3.3.2 several choices of frame are popular – helicity, Gottfried–Jackson, Adair, transversity. The density matrix in one of these frames then differs from the density matrix in the CM of the production reaction by at most a rotation.

We shall discuss decay distributions almost entirely in the helicity rest frame of the decaying particle. Details about other frames can be found in the review article by Bourrely, Leader and Soffer (1980). The treatment we shall give is both general and straightforward, and the use of multipole parameters is far simpler and clearer than density matrix elements.

8.2.1 Two-particle decay of spin-J resonance

We consider the decay of particle C of arbitrary spin J, $C \to E + F$, where E, F are also of arbitrary spin. We consider this decay in the *helicity rest frame* S_C of C, where E emerges with momentum $\mathbf{p}_E = (p_E, \theta_E, \phi_E)$. The initial state of C is then described by the CM helicity density matrix $\rho(C)$ or the CM multipole parameters $t^l_m(C)$ of the *production reaction.*

The decay amplitude is a special case of (4.1.8) in which the initial state has a unique value of J:

$$H_{ef;c}(\theta_E, \phi_E) \propto \langle ef|T^J|c\rangle e^{ic\phi_E} d^J_{c\mu}(\theta_E)$$

where e, f, c refer to the helicities of E, F and C and $\mu = e - f$. However, for a single particle C we must have $c \equiv \lambda_C = J_z$. Then by rotational invariance the matrix element cannot depend on c.

We thus write

$$H_{ef;c}(\theta_E, \phi_E) \propto M_C(e, f) e^{ic\phi_E} d^J_{c\mu}(\theta_E) \tag{8.2.1}$$

where the $M_C(e, f)$, the *reduced helicity amplitudes*, are dynamics dependent parameters describing the decay.

We shall normalize the $M_C(e, f)$ so that

$$\sum_{e,f} |M_C(e, f)|^2 = 1. \tag{8.2.2}$$

(*i*) *The decay angular distribution*

From (5.2.1) and (5.2.3) the *normalized* angular distribution of E is given by

$$W(\theta_E, \phi_E) = \frac{2J+1}{4\pi} \sum_{\substack{e,f\\c,c'}} |M_C(e, f)|^2 \times e^{i\phi_E(c-c')} d^J_{c\mu}(\theta_E) \rho_{cc'}(C) d^J_{c'\mu}(\theta_E). \tag{8.2.3}$$

Since ρ is hermitian (8.2.3) can be rewritten as

$$W(\theta_E, \phi_E) = \frac{2J+1}{4\pi} \sum_{\substack{e,f\\c,c'}} |M_C(e, f)|^2 \left\{ \cos\left[\phi_E(c-c')\right] \operatorname{Re} \rho_{cc'} - \sin\left[\phi_E(c-c')\right] \operatorname{Im} \rho_{cc'} \right\} d^J_{c\mu}(\theta_E) d^J_{c'\mu}(\theta_E). \tag{8.2.4}$$

If parity is conserved in the *decay* then from (4.2.3)

$$M_C(e, f) = \frac{\eta_E \eta_F}{\eta_C} (-1)^{J-s_E-s_F} M_C(-e, -f) \tag{8.2.5}$$

and one obtains, for any production process,

$$W(\theta_E, \phi_E) = W(\pi - \theta_E, \pi + \phi_E), \tag{8.2.6}$$

for a parity-conserving decay, W is symmetric under reflection through the origin of S_C.

If parity is conserved in the *production* reaction and if either

(1) the initial state in that reaction is unpolarized, or
(2) it is polarized, the state of polarization satisfies the experimental conditions (s_1)–(s_3) of subsection 5.4.2(iv) or (s_4) of subsection 5.4.4 and C emerges in the beam-containing plane perpendicular to the quantization plane

then $\rho(C)$ will satisfy (5.4.8) and for *any* decay mechanism, one has

$$W(\theta_E, \phi_E) = W(\pi - \theta_E, \pi - \phi_E), \tag{8.2.7}$$

i.e. W is symmetric under reflection through the Y-axis of S_C.

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If the above holds for the production reaction *and* the decay conserves parity then (8.2.6) and (8.2.7) give

$$W(\theta_E, \phi_E) = W(\theta_E, -\phi_E), \tag{8.2.8}$$

which implies that the part of (8.2.4) that depends on Im $\rho_{cc'}$ must vanish in this case, and one has

$$W(\theta_E, \phi_E) = \frac{2J+1}{4\pi} \sum_{\substack{e,f \\ c,c'}} |M_C(e,f)|^2 \cos\left[\phi_E(c-c')\right] \times \mathrm{Re}\ \rho_{cc'}\ d^J_{c\mu}(\theta_E) d^J_{c'\mu}(\theta_E). \tag{8.2.9}$$

Hence the decay distribution in this case yields information only on Re $\rho_{cc'}$. Since for this case (5.4.8) holds, this is equivalent to obtaining information only on the *even-polarization* part of ρ. In fact, this is a special case of a completely general result (see the next section) that in any parity-conserving decay, no matter what the production reaction is, $W(\theta_E, \phi_E)$ depends only on the *even* multipole parameters $t^l_m(C)$ of C.

(ii) Distribution of the final state multipole parameters

The reaction formalism developed in Section 5.3 simplifies enormously when the initial state consists of a single particle. With the multipole parameters all specified in the helicity rest frame of C, the final state multipole parameters are controlled by reaction parameters $(l, m|l', m'; L', M')_{\phi_E}$ in which the dependence on θ_E is now explicit:

$$(l, m|l', m'; L', M')_{\phi_E} = \frac{1}{4\pi} \frac{1}{\sqrt{2l+1}} C_l(l', m'; L', M') \times \mathscr{D}^{(l)}_{m,m'-M'}(\phi_E, \theta_E, 0) \tag{8.2.10}$$

where the C_l, the *decay parameters* dependent on the decay amplitude, are constants and the angular functions are the well-known representation functions of the rotation group (Rose, 1957). In fact

$$\mathscr{D}^{(l)}_{m,m'-M'}(\phi, \theta, 0) = e^{-im\phi} d^l_{m,m'-M'}(\theta). \tag{8.2.11}$$

Explicitly, with our normalization convention (8.2.2) and when the spin of C is J,

$$\begin{aligned} C_l(l', m'; L', M') = &\left[\frac{(2J+1)(2s_E+1)(2s_F+1)}{(2l'+1)(2L'+1)}\right]^{1/2} (-1)^{J+s_F-s_E} \\ &\times \sum_{e,f} M^*_C(e,f) M_C(e-m', f-M') \langle l', m'|s_E, e; s_E, m'-e\rangle \\ &\times \langle L', M'|s_F, f; s_F, M'-f\rangle \\ &\times \langle l, m'-M'|J, e-f; J, m'-M'-e+f\rangle. \end{aligned} \tag{8.2.12}$$

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The relation between the initial and final state multipole parameters is then

$$W(\theta_E, \phi_E) t^{l'L'}_{m'M'}(E, F) = \frac{1}{4\pi} \sum_{l \geq |m'-M'|}^{2J} \sqrt{2l+1}\, C_l(l', m'; L', M') \times \sum_m t^l_m(C) \mathscr{D}^{(l)}_{m, m'-M'}(\phi_E, \theta_E, 0). \qquad (8.2.13)$$

The decay parameters enjoy the following properties:

(α) Normalization:

$$C_0(0,0;0,0) = \sum_{e,f} |M_C(e,f)|^2 = 1. \qquad (8.2.14)$$

(β) l-value constraint:

$$C_l(l', m'; L', M') = 0 \qquad \text{if } |m' - M'| > l. \qquad (8.2.15)$$

(γ) Reality:

$$C_l^*(l', m'; L', M') = C_l(l', -m'; L', -M'). \qquad (8.2.16)$$

(δ) Parity: if parity is conserved in the decay then

$$C_l(l', -m'; L', -M') = (-1)^{l+l'+L'} C_l(l', m'; L', M'). \qquad (8.2.17)$$

Thus

$$C_l(l', 0; L', 0) = 0 \quad \text{if } l + l' + L' \text{ is odd}. \qquad (8.2.18)$$

When (8.2.18) is combined with (8.2.16) we have, for a parity conserving decay

$$C_l(l', m'; L', M') \text{ is } \left\{ \begin{array}{c} \text{real} \\ \text{imaginary} \end{array} \right\} \text{ if } l + l' + L' \text{ is } \left\{ \begin{array}{c} \text{even} \\ \text{odd} \end{array} \right\}. \qquad (8.2.19)$$

It is clear from (8.2.12) that there will be linear relationships amongst the different $C_l(l', m'; L', M')$ for a *fixed* set of values of m' and M', since they are all expressed in terms of the same product $M_C^*(e, f) M_C(e - m', f - M')$ of matrix elements. An example will be given in subsection 8.2.1(ix).

As a special case of (8.2.13) the decay distribution is

$$W(\theta_E, \phi_E) = \frac{1}{4\pi} \sum_{l \geq 0}^{2J} C_l(0,0;0,0) \sum_m t^{l*}_m(C) Y_{lm}(\theta_E, \phi_E) \qquad (8.2.20)$$

where we have used the relation for the spherical harmonics

$$\mathscr{D}^{(l)}_{m,0}(\phi, \theta, 0) = \sqrt{\frac{4\pi}{2l+1}} Y^*_{lm}(\theta, \phi) \qquad (8.2.21)$$

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as well as

$$t^{l*}_m = (-1)^m t^l_{-m} \qquad \text{and} \qquad Y^*_{lm} = (-1)^m Y_{l-m}.$$

A crucial feature is that for a parity-conserving decay only *even* values of l appear on the right-hand side of (8.2.20).

It is important to note that the $t^{l'L'}_{m'M'}(E,F)$ are the multipole parameters in the helicity rest frame S_C of C. They are also the correct multipole parameters in the helicity rest frames S_E, S_F of E and F reached from S_C.

The generalization of (8.2.20) to the two-body decays of unstable particles C_1, C_2 of spin J_1 and J_2 created in some reaction, i.e.

$$\begin{aligned} A + B &\to C_1 + C_2 \\ C_1 &\to E_1 + F_1 \\ C_2 &\to E_2 + F_2, \end{aligned}$$

where the decays are *independent* of each other, is straightforward:

$$\begin{aligned} W(\theta_1, \phi_1; \theta_2, \phi_2) = \frac{1}{4\pi} \sum_{l_1 \ge 0}^{2J_1} \sum_{l_2 \ge 0}^{2J_2} & C_{l_1}(0,0;0,0) C_{l_2}(0,0;0,0) \\ \times & \sum_{m_1, m_2} t^{l_1 l_2}_{m_1 m_2}(C_1, C_2) Y_{l_1 m_1}(\theta_1, \phi_1) Y_{l_2 m_2}(\theta_2, \phi_2) \end{aligned} \tag{8.2.22}$$

where the $t^{l_1 l_2}_{m_1 m_2}(C_1, C_2)$ are the joint multipole parameters for the produced C_1 and C_2 particles. Of course all angles in (8.2.22), i.e. θ_1, ϕ_1 for E_1 and θ_2, ϕ_2 for E_2, refer to the helicity rest frames S_{C_1} and S_{C_2} of C_1 and C_2. (There is also an obvious generalization of (8.2.13).) A nice application of this formalism to $e^- e^+ \to \tau^- \tau^+$ followed by $\tau^- \to \pi^- \nu_\tau$ and $\tau^+ \to \pi^+ \bar{\nu}_\tau$ will be given in subsection 9.2.1(iii).

In the following we shall deal just with the simpler case (8.2.20).

We note the following general properties that follow from (8.2.20) or (8.2.13).

(α) If parity is conserved in the decay then (8.2.18) implies that $W(\theta_E, \phi_E)$ depends only on those $t^l_m(C)$ with l even.

(β) For an unpolarized initial state, i.e. $t^l_m(C) = 0$ for $l \ge 1$,

$$t^{l'L'}_{m'M'}(E,F) = 0 \qquad \text{if } m' \ne M'. \tag{8.2.23}$$

In particular, for the effective multipole parameters of E (or F)

$$t^{l'}_{m'}(E) = 0 \qquad \text{if } m' \ne 0. \tag{8.2.24}$$

Note that (8.2.23) and (8.2.24) are much stronger results than in the $2 \to 2$ reaction case.

(γ) If parity is conserved in the decay then

$$t^{l'L'}_{m'M'}(E,F;\theta_E,\phi_E) = (-1)^{l'+L'}t^{l'L'}_{-m'-M'}(E,F;\pi-\theta_E,\pi+\phi_E). \tag{8.2.25}$$

For an unpolarized initial state the only non-zero effective multipole parameters for E (or F) are the $t^{l'}_0$ with l' even.

(δ) If the production reaction gives a $\rho(C)$ that satisfies (5.4.8) (see the discussion after eqn (8.2.6)) and if the decay conserves parity then

$$t^{l'L'}_{m'M'}(\theta_E,\phi_E) = (-1)^{l'+L'+m'+M'}t^{l'L'}_{-m'-M'}(\theta_E,-\phi_E). \tag{8.2.26}$$

Equation (8.2.8) for $W(\theta_E,\phi_E)$ is just a special case of this general relation.

(*iii*) *Moments of the experimental distributions*

Let $G(\theta_E,\phi_E)$ be any function of the polar angles of E in S_C. We denote by $\langle G\rangle$ the average, over all angles, of G weighted by the normalized decay distribution $W(\theta_E,\phi_E)$, i.e.

$$\langle G\rangle \equiv \int_0^{2\pi} d\phi_E \int_0^{\pi} \sin\theta_E\; d\theta_E\, W(\theta_E,\phi_E)G(\theta_E,\phi_E). \tag{8.2.27}$$

By taking moments of suitable multipole parameters of the decay products one can isolate individual multipoles $t^l_m(C)$ of the initial state. In complete generality one has, from (8.2.13),

$$\left\langle t^{l'L'}_{m'M'}\mathscr{D}^{(l)^*}_{m,m'-M'}\right\rangle = \frac{1}{\sqrt{2l+1}}C_l(l',m';L',M')t^l_m(C) \qquad (l\le 2J). \tag{8.2.28}$$

The simplest example, where no spin properties of E or F are measured, is (see (8.2.13) and (8.2.21))

$$\langle Y_{lm}\rangle = \frac{1}{\sqrt{4\pi}}C_l(0,0;0,0)t^l_m(C) \qquad (l\le 2J). \tag{8.2.29}$$

To use (8.2.28) in practice one requires a table of $\mathscr{D}$-functions. These can be obtained from (8.2.11) and the explicit table of $d^l_{\lambda\mu}$ in Appendix 1. For $l\ge 3$ one must resort to recursion relations. A new and simplified form of these is given in Appendix 1, where also the detailed symmetry properties of the $d^l_{\lambda\mu}$ are stated.

Note that:

(α) In principle, a particular $t^l_m(C)$ can be found from many different moments, i.e. from all those with $|m'-M'|\le l$. In practice, however, one will not know the $C_l(l',m';L',M')$ needed on the right-hand side of (8.2.28) for arbitrary values of its arguments and one will not, in any case, be able to measure joint multipole parameters very easily. The simplest way to obtain the $t^l_m(C)$ is discussed below.

(β) For a parity-conserving decay the right-hand side of (8.2.29) is zero for l odd. The equation is nevertheless useful, and the left-hand side should be calculated from the data also for odd l as a check on experimental biases.

(γ) Moments with $l > 2J$ must give zero. (This is a general result on the maximal angular complexity in the decay of a spin-J particle.) If they do not there are either experimental biases or C is not what it was thought to be!

We see in general that the moments of the angular distribution of the particles, or of their multipole parameters, give us information only about the *product* of $t^l_m(C)$ and certain decay parameters whose value depends upon the decay mechanism.

For a *parity-violating* decay, the decay distribution functions as a complete analyser of the polarization state of C, so all we require to know are the decay parameters $C_l(0,0;0,0)$. Of these,

$$N \equiv \min\{(2s_E+1)(2s_F+1)-1;2J\}$$

are independent.

For a *parity-conserving* decay we need the decay distribution and the distribution of any one of the odd multipole parameters of either E or F in order to get a complete analysis of the state of polarization of C. We thus require to know the $C_l(0,0;0,0)$ and, say, the $C_l(1,0;0,0)$. In total these depend upon

$$N_{\mathscr{P}} \equiv \begin{cases} \min\left\{\dfrac{(2s_E+1)(2s_F+1)}{2}-1;\ 2J\right\} \\ \text{or} \\ \min\left\{\dfrac{(2s_E+1)(2s_F+1)}{2}-\dfrac{1}{2};\ 2J\right\} \end{cases}$$

real parameters, for $(2s_E+1)(2s_F+1)$ even or odd, respectively.

If we cannot calculate the C_l from first principles, then in order to use a decay as an analyser we must first carry out N or $N_{\mathscr{P}}$ measurements on the decay products in such a way that the polarization state of C is irrelevant, i.e. by measuring moments with $l = m = 0$ (see eqn (8.2.28)). This gives us the minimum required number of C_l and thereafter the decay can function as a complete analyser for arbitrary states of polarization of C.

Fortunately many decays in Nature have $N_{\mathscr{P}} = 0$ or $N = 1$ so the whole problem of finding the C_l disappears or at least becomes relatively

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simple. We shall refer to decays with $N_{\mathscr{P}} = 0$ and which thus automatically function as analysers as *magic* decays.

Moreover, many high-spin resonances undergo a 'decay chain', i.e. a sequence of two-body decays of which the last, at least, has $N_{\mathscr{P}} = 0$ or $N = 1$, and we shall show later how this fact can be used to bypass the problem of finding the C_l for the intermediate links of the chain.

(*iv*) *Magic decays*

We now list the magic decays. The decay parameters are given in terms of vector addition coefficients, which are fully tabulated in Appel (1968) or can be extracted from the table of Clebsch–Gordan coefficients in any biennial 'Review of particle properties' in *Phys Rev D*, for example Barnett *et al.* (1996). The following results are for the case where the particle with non-zero spin has decay angles θ_E, ϕ_E.

(a) $J \to 0+0$ (e.g. $\rho \to \pi\pi$, $f \to \pi\pi$, $D \to K\pi$):

$$C_l(0,0;0,0) = (-1)^J\sqrt{2J+1}\,\langle l,0|J,0;J,0\rangle \qquad l \text{ even.} \tag{8.2.30}$$

(b) $J \to 1/2+0$ with parity conserved and $J \to \nu+0$ with maximal parity violation as in the Standard Model (e.g. $\Delta \to N\pi$, $\tau^- \to \nu\pi$):

$$\begin{aligned} &C_l(0,0;0,0) = (-1)^{J-1/2}\sqrt{2J+1}\,\langle l,0|J,1/2;J,-1/2\rangle \qquad l \text{ even.}\\ &C_l(1,0;0,0) = \tfrac{1}{\sqrt{3}}(-1)^{J-1/2}\sqrt{2J+1}\,\langle l,0|J,1/2;J,-1/2\rangle \qquad l \text{ odd.}\\ &C_l(1,1;0,0) = 0. \end{aligned} \tag{8.2.31}$$

(c) $J \to$ photon $+0$, parity conserved (e.g. $\omega \to \gamma\pi$, $D^{*+} \to \gamma D$):

$$\begin{aligned} &C_l(0,0;0,0) = (-1)^{J-1}\sqrt{2J+1}\,\langle l,0|J,1;J,-1\rangle \qquad l \text{ even}\\ &C_l(1,0;0,0) = (-1)^{J}\sqrt{2J+1}\,\langle l,0|J,1;J,-1\rangle \qquad l \text{ odd}\\ &C_l(2,0;0,0) = (-1)^{J-1}\sqrt{\tfrac{1}{2}}\sqrt{2J+1}\,\langle l,0|J,1;J,-1\rangle \qquad l \text{ even}\\ &C_l(2,2;0,0) = (-1)^{J-1}\sqrt{\tfrac{3}{5}}\sqrt{2J+1}\,\langle l,2|J,1;J,1\rangle \qquad l \text{ even}\\ &C_l(1,1;0,0) = C_l(2,1;0,0) = 0. \end{aligned} \tag{8.2.32}$$

(d) $J \to 1+0$, parity conserved *and* intrinsic parities satisfy $\eta_0\eta_1 = (-1)^J\eta_J$ (e.g. $\psi \to \rho\pi$, $a_2 \to \rho\pi$, $\omega(1670) \to \rho\pi$): same as (c).

(*v*) *The decay* $C(\text{spin } J) \to E(\text{spin } 1/2) + F(\text{spin } 0)$

The decay $J \to 1/2+0$ with or without parity conservation is of particular interest, so we study it in some detail. We split each decay amplitude into a parity-conserving (PC) and a parity-violating (PV) piece,

$$M(\lambda) = M_{\mathrm{PC}}(\lambda) + M_{\mathrm{PV}}(\lambda) \qquad \lambda = \pm 1/2, \tag{8.2.33}$$

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such that

$$M_{\mathrm{PC/PV}}(-\lambda) = \pm\eta M_{\mathrm{PC/PV}}(\lambda)$$

where the plus and minus apply to M_{PC} and M_{PV} respectively and where η is the phase factor in eqn (8.2.5).

There are three real linearly independent parameters controlling the decay,

$$\alpha = \frac{2\ \mathrm{Re}\ (M_{\mathrm{PC}}M^*_{\mathrm{PV}})}{|M_{\mathrm{PC}}|^2 + |M_{\mathrm{PV}}|^2}, \qquad \beta = \frac{2\ \mathrm{Im}\ (M_{\mathrm{PC}}M^*_{\mathrm{PV}})}{|M_{\mathrm{PC}}|^2 + |M_{\mathrm{PV}}|^2},$$
$$\gamma = -\frac{|M_{\mathrm{PC}}|^2 - |M_{\mathrm{PV}}|^2}{|M_{\mathrm{PC}}|^2 + |M_{\mathrm{PV}}|^2} \tag{8.2.34}$$

which satisfy $\alpha^2+\beta^2+\gamma^2 = 1$. Our definitions of α, β, γ have been chosen so as to agree with those used in the *Review of Particle Properties* (Particle Data Group, 1978) for the case $\Lambda \to N\pi$. As a consequence our γ is opposite in sign to the γ defined in Jackson (1965).

If parity is conserved one has $\alpha = \beta = 0$, $\gamma = -1$.

The decay parameters are given in terms of these by

$$\begin{aligned}
C_l(0,0;0,0) &= (-1)^{J-1/2}\sqrt{2J+1}\,\langle l,0|J,1/2;J,-1/2\rangle \\
&\quad \times \begin{Bmatrix} 1 \\ \alpha \end{Bmatrix} \qquad \text{for } l \begin{Bmatrix} \text{even} \\ \text{odd} \end{Bmatrix} \\
C_l(1,0;0,0) &= (-1)^{J-1/2}\sqrt{\tfrac{1}{3}}\sqrt{2J+1}\,\langle l,0|J,1/2;J,-1/2\rangle \\
&\quad \times \begin{Bmatrix} \alpha \\ 1 \end{Bmatrix} \qquad \text{for } l \begin{Bmatrix} \text{even} \\ \text{odd} \end{Bmatrix} \\
C_l(1,1;0,0) &= \bar{\eta}\sqrt{\tfrac{1}{6}}\sqrt{2J+1}\,\langle l,1|J,1/2;J,1/2\rangle\,(-\gamma+i\beta) \qquad \text{for } l \text{ odd} \\
&= 0 \qquad \text{for } l \text{ even}
\end{aligned} \tag{8.2.35}$$

with $\bar{\eta} = \eta_C\eta_E\eta_F$ in terms of intrinsic parities.

From these and eqns (8.2.13), (8.2.20) and (8.2.28) we learn the following.

(a) If parity is conserved we have a magic decay; thus a measurement of $W(\theta_E, \phi_E)$ and the distribution of any one component of the polarization of E will yield a complete analysis of the $t^l_m(C)$. (Recall that $\mathscr{P}_z = \sqrt{3}t^1_0(E)$, $\mathscr{P}_x = \sqrt{3/2}(t^1_{-1} - t^1_1)$ and $\mathscr{P}_y = i\sqrt{3/2}(t^1_{-1} + t^1_1)$, the components referring to the axes in the helicity rest frame of E.)

(b) If parity is not conserved a measurement of the decay distribution $W(\theta_E, \phi_E)$ yields the $t^l_m(C)$ for l even and $\alpha t^l_m(C)$ for l odd. The asymmetry parameter α can be found most simply from the average longitudinal

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polarization of E:

$$\langle \mathscr{P}_z(E) \rangle = \sqrt{3} C_0(1,0;0,0) = \alpha. \tag{8.2.36}$$

The most common decays of this type are e.g. $\Lambda \to N\pi$, $\Sigma \to N\pi$.

Let us assume C is produced in a parity-conserving reaction in a situation such that eqn (5.4.8) is satisfied. Then $t^1_{-1}(C) = t^1_1(C)$, $t^1_0(C) = 0$ and the polarization vector of C is normal to the production plane, i.e. along the Y-axis in the helicity rest frame of C. Explicitly then

$$W(\theta_E, \phi_E) = \frac{1}{4\pi}\left(1 + \alpha \mathscr{P}^C_y \sin\theta_E \sin\phi_E\right) \tag{8.2.37}$$

giving rise to the well-known up–down symmetry with respect to the production plane.

If $(\mathscr{P}^E_{x'}, \mathscr{P}^E_{y'}, \mathscr{P}^E_{z'})$ are the components of the polarization vector for E in its helicity rest frame S_E, one has

$$\begin{aligned}
W(\theta_E, \phi_E)\mathscr{P}^E_{z'} &= \frac{1}{4\pi}\left(\alpha + \mathscr{P}^C_y \sin\theta_E \sin\phi_E\right) \\
W(\theta_E, \phi_E)\mathscr{P}^E_{x'} &= -\frac{\eta \mathscr{P}^C_y}{4\pi}\left(\gamma \cos\theta_E \sin\phi_E + \beta \cos\phi_E\right) \\
W(\theta_E, \phi_E)\mathscr{P}^E_{y'} &= \frac{\eta \mathscr{P}^C_y}{4\pi}\left(\beta \cos\theta_E \sin\phi_E - \gamma \cos\phi_E\right)
\end{aligned} \tag{8.2.38}$$

where η is the phase factor in eqn (8.2.5).

If we define a 3-vector $\bar{\mathscr{P}}^E$ in the rest frame S_C of C, such that its components $(\bar{\mathscr{P}}^E_x, \bar{\mathscr{P}}^E_y, \bar{\mathscr{P}}^E_z)$ are numerically equal to the components of $\mathscr{P}^E$ along the axes of the Adair rest frame of E (the Adair rest frame is reached from S_C by a *pure* Lorentz transformation along the direction of motion of E), then (8.2.38) can be written as a 3-vector equation in S_C:

$$\begin{aligned}
4\pi W(\theta_E, \phi_E)\bar{\mathscr{P}}^E &= (\alpha + \mathbf{e}\cdot\boldsymbol{\mathcal{P}}^C)\mathbf{e} + \eta\gamma\left[\mathbf{e}\times(\mathbf{e}\times\boldsymbol{\mathcal{P}}^C)\right] \\
&\quad + \eta\beta(\mathbf{e}\times\boldsymbol{\mathcal{P}}^C)
\end{aligned} \tag{8.2.39}$$

where $\mathbf{e}$ is a unit vector along the momentum of E. Note that our formula has explicit reference to the relative parities of the particles in it (through η) and also to the spin J of C. It will thus only agree with the formula given by the Particle Data Group in reactions like $\Lambda \to N\pi$ where $\eta = -1$.

(vi) Fermionic decay $C(J) \to E(J_1) + 0$, $J \geq 3/2$

It is now necessary to measure the average values for each of the $t^{l'}_0(E)$, which, from (8.2.28) yield

$$\left\langle t^{l'}_0(E) \right\rangle = C_0(l', 0; 0, 0). \tag{8.2.40}$$

(If parity is conserved in the decay, the right-hand side $= 0$ for l' odd.)

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Then from (8.2.12) we obtain

$$C_l(0,0;0,0) = \sqrt{2l+1}C_0(l,0;0,0), \tag{8.2.41}$$

which is another example of a reaction's power to analyse being related to its power to polarize.

There are then two possibilities.

(a) If parity is not conserved, all the $C_l(0,0;0,0)$ being now known the distribution of E functions as a complete analyser for the state of polarization of C.
(b) If parity is conserved the $C_l(0,0;0,0)$ with odd l are zero. The decay distribution yields the $t^l_m(C)$ only for l even.

To find the $t^l_m(C)$ with odd l, one studies the distribution of $t^{l'}_0(E)$ for any odd value of l'. One thus requires the decay parameters $C_l(l',0;0,0)$, which are calculated as follows.

One solves (8.2.12) for the moduli squared of the decay matrix elements

$$\begin{aligned} |M_C(\lambda_1)|^2 = \frac{1}{2J_1+1}\sum_l (2l+1)\,\langle J_1,\lambda_1|J_1,\lambda_1;l,0\rangle \\ \times C_0(l,0;0,0) \end{aligned} \tag{8.2.42}$$

(note that the left-hand side is just $\rho_{\lambda_1\lambda_1}(E_1)$ if C is *unpolarized*) and then substitutes in

$$\begin{aligned} C_l(l',0;0,0) = \sqrt{2l+1}\sum_{\lambda_1}\langle J_1,\lambda_1|J_1,\lambda_1;l',0\rangle \\ \times \langle J,\lambda_1|J,\lambda_1;l,0\rangle\,|M_C(\lambda_1)|^2. \end{aligned} \tag{8.2.43}$$

In electroweak interactions, which violate parity, it often happens that the decaying particle C is produced with longitudinal polarization $\mathscr{P}^C_z$. In this case the angular distribution of E is given by

$$W(\theta_E,\phi_E) = \frac{1}{4\pi}\left(1+\alpha\mathscr{P}^C_z\cos\theta_E\right). \tag{8.2.44}$$

For many resonance decays the values of the decay parameter α are quite well known, as shown in Table 8.1. Equation (8.2.44) will be of use in discussing reactions like $e^+e^- \to \Lambda + X$ where, in the Standard Model, one expects the Λs to be highly polarized longitudinally (see subsection 9.2.2(ii)).

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Table 8.1. Values of the decay parameter α for various hyperon decays

Decay	Value of α
$\Lambda \to p\pi^-$	0.642 ± 0.013
$\Sigma^+ \to p\pi^0$	-0.980 ± 0.016
$\Sigma^+ \to n\pi^+$	0.068 ± 0.013
$\Sigma^- \to n\pi^-$	-0.068 ± 0.008
$\Xi^0 \to \Lambda\pi^0$	-0.411 ± 0.022
$\Xi^- \to \Lambda\pi^-$	-0.456 ± 0.014
$\Lambda_c^+ \to \Lambda\pi^+$	-0.98 ± 0.19
$\Lambda_c^+ \to \Sigma^+\pi^0$	-0.45 ± 0.32

(*vii*) *Fermionic decay chains*

A common situation is that the produced fermion C decays sequentially, emitting a spin-0 meson at each step. We thus consider

$$\begin{array}{l} C(J) \to E_1(J_1) + 0 \\ \qquad\qquad \searrow E_2(J_2) + 0 \\ \qquad\qquad\qquad \searrow \cdots \\ \qquad\qquad\qquad\quad \searrow E_n(J_n) + 0 \\ \qquad\qquad\qquad\qquad\quad \searrow E(1/2) + 0. \end{array} \tag{8.2.45}$$

We denote by $t^l_m(E_k)$ the multipole parameters of E_k in its helicity rest system as reached from the helicity rest frame of its parent E_{k-1}.

From the discussion in (iv) above, a study of the distribution of E in the frame S_{E_n}, and if necessary of the distribution of one of its polarized components, will always yield all the $t^l_m(E_n)$ whether or not parity is conserved. From (iv) or (v) the distribution of E_n and one of its odd multipoles $t^{l_n}_0(E_n)$ is enough to give all the $t^l_m(E_{n-1})$ etc.

In principle, therefore, one can work one's way back up the chain until one obtains all the $t^l_m(C)$.

(*viii*) *Bosonic decay chains*

We consider the decay of a heavy boson and suppose that it is dominated by a sequence of two-body decay as in (8.2.45), but with various possibilities for the final decay. We assume that parity is conserved in each decay.

There are three cases of interest, depending on the form of the last decay in the chain.

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(a) If the last decay is of the form $E_n(J_n) \to E(0)+0$ then the distribution of E in S_{E_n} gives the *even-l* $t^l_m(E_n)$, but there is no way to find the $t^l_m(E_n)$ with odd l. Proceeding up the chain one ends up with only the even-multipole parameters of the original resonance C.

(b) If the last reaction is of the form $E_n(J_n) \to$ photon $+ 0$ and if the state of polarization of the photon can be measured (see subsection 3.1.12) then since the decay is magic all the required decay parameters are known (see eqn (8.2.32)) and one can obtain all the $t^l_m(E_n)$. One can then work back up the chain to obtain all $t^l_m(C)$ of the original resonance.

(c) If it is possible to detect decay into a stable fermion–antifermion pair, e.g. $\rho \to \mu^-\mu^+$, and if the longitudinal polarization of the fermions can be measured then all $t^l_m(E_n)$ can be found and, proceeding up the chain, eventually all $t^l_m(C)$.

Suppose the final decay is $E_n(J_n) \to E(1/2)+\bar{E}(1/2)$. The decay parameters that appear in the angular distribution of E and of its longitudinal polarization are, from (8.2.12),

$$\begin{aligned} C_l(0,0;0,0) &= (-1)^{J_n}\sqrt{2J_n+1}\,\big[(1-\epsilon)\,\langle l,0|J_n,0;J_n,0\rangle \\ &\qquad -\epsilon\,\langle l,0|J_n,1;J_n,-1\rangle\big] \qquad \text{for } l \text{ even} \\ C_l(1,0;0,0) &= \epsilon(-1)^{J_n+1}\sqrt{\frac{2J_n+1}{3}} \\ &\qquad \times\,\langle l,0|J_n,1;J_n,-1\rangle \qquad \text{for } l \text{ odd}, \end{aligned} \tag{8.2.46}$$

where ϵ is a measure of the relative decay probabilities into helicities $++$ or $+-$. Specifically,

$$|M_{E_n}(+-)|^2/|M_{E_n}(++)|^2 = \epsilon/(1-\epsilon). \tag{8.2.47}$$

Thus a measurement of the moments of the distribution of E and of its longitudinal polarization will give all the $t^l_m(E_n)$ as functions of one parameter ϵ. To find ϵ requires the measurement of a correlation between the spins of E and $\bar{E}$. If the spin projections for E and $\bar{E}$ are referred to the axes of their respective helicity rest frames, then

$$\int_0^{2\pi} d\phi_E \int_0^{\pi} \sin\theta_E\, d\theta_E\, \langle \sigma_z(E)\sigma_z(\bar{E})\rangle_{\theta_E,\phi_E}\, W(\theta_E,\phi_E) = 1-2\epsilon. \tag{8.2.48}$$

For $J_n = 1$, and where the decay is into a lepton–anti-lepton pair coupled purely through a minimal electromagnetic-type γ^μ coupling (e.g. $\rho \to \mu^-\mu^+$) one finds that

$$\epsilon = \left(1+2m_l^2/M_{E_n}^2\right)^{-1}$$

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where m_l and M_{E_n} are the lepton and resonance masses. Thus $\epsilon \sim 1$ and the decay functions as a very efficient analyser. In this case, clearly one need not measure ϵ.

(ix) Decay of W-boson

As discussed in Chapter 9 one of the most remarkable developments in particle physics has been the prediction and discovery of the massive vector bosons $W^{\pm}$, Z^0 associated with the unification of the weak and electromagnetic forces. Polarization phenomena in the decay of the Z^0 are treated in detail in subsection 9.2.1. Here we describe a fundamental test to show that the W is really a spin-1 boson.

Suppose that the W had spin J, and consider its decay

$$W \to E + F$$

where E, F have spins S_E, S_F respectively.

Using the properties of the vector addition coefficients (Rose, 1957), one can show that

$$\sqrt{\frac{J(J+1)}{3}}C_1(0,0;0,0) = \sqrt{S_E(S_E+1)}C_0(1,0;0,0) - \sqrt{S_F(S_F+1)}C_0(0,0;1,0). \qquad (8.2.49)$$

To turn this into a relation amongst measured quantities we note that from (8.2.29)

$$\begin{aligned}\langle \cos\theta_E \rangle \equiv \langle P_1(\theta_E) \rangle &= \sqrt{\frac{4\pi}{3}}\,\langle Y_{10} \rangle \\ &= \sqrt{\frac{1}{3}}\sqrt{\frac{J}{(J+1)}}C_1(0,0;0,0)\mathscr{P}_z(W) \\ &= \sqrt{\frac{1}{3}}\frac{1}{\sqrt{J(J+1)}}C_1(0,0;0,0)\,\langle \hat{s}_z \rangle_W . \end{aligned} \qquad (8.2.50)$$

Also, from (8.2.28),

$$\left\langle t^1_0(E) \right\rangle = C_0(1,0;0,0) \qquad \left\langle t^1_0(F) \right\rangle = C_0(0,0;1,0). \qquad (8.2.51)$$

Thus, substituting in (8.2.49) and using (3.1.35) and (1.1.27)

$$\langle \cos\theta_E \rangle = \frac{1}{J(J+1)}\left[\langle \lambda_E \rangle - \langle \lambda_F \rangle\right]\langle \hat{s}_z \rangle_W \qquad (8.2.52)$$

where $\langle \lambda_i \rangle$ is the mean helicity of particle i. This result, clearly, is quite general and does not depend specifically on the initial particle's being a W. Indeed (8.2.52) was first derived by Jacob (1958) in the context of strange particle decay, but its most dramatic success was in connection with the original discovery of $W^+ \to e^+\nu_e$, where the value $\langle \cos\theta_{e^+}\rangle = 0.5$ was found. This 'killed three birds with one stone'! It required $J_W = 1$, $\langle\lambda_{e^+}\rangle = -\langle\lambda_{\nu_e}\rangle = 1/2$ and $\langle\hat{s}_z\rangle_W = 1$, all in perfect agreement with the predictions of the standard model of electroweak interactions.

8.2.2 Three-particle decay of a spin-J resonance

We consider the decay of a particle C of arbitrary spin J and mass m_C,

$$C \to E_1 + E_2 + E_3,$$

where the particles E_i have arbitrary spin. The most common decays, in practice, are those in which all E_i have spin zero, e.g. 3π, or in which one particle, say E_1, has spin 1/2 and the others are pions. We shall thus not discuss the most general case but limit ourselves to the situation where the polarization, or the multipole parameters of, at most *one* of the decay products is observed. We shall always refer to this particle as E_1.

(*i*) Decay amplitudes

There are many ways to specify the configuration of a three-particle state. Werle (1963) used the polar angles θ_1, ϕ_1 of E_1 and one further angle to specify the orientation of the *decay triangle*, i.e. the triangle formed by the momenta $\mathbf{p}_i$ of the final particles in the rest system of C. Berman and Jacob (1965) characterized the state by the polar angles of the *normal* to the decay triangle and an angle specifying the orientation of the triangle once the normal is fixed. We will utilize the latter only, since we have found that it leads to simpler results.

Let S_C with axes X, Y, Z be the helicity rest frame of C as reached from the CM of the production reaction.[1] Let ω_i, $\mathbf{p}_i$ be the energies and momenta of the particles E_i, with

$$\omega_1 + \omega_2 + \omega_3 = m_C. \tag{8.2.53}$$

An arbitrary state in which the E_i have helicities λ_i is written as

$$|\omega_1\lambda_1, \omega_2\lambda_2, \omega_3\lambda_3; \phi_n, \theta_n, \gamma\rangle,$$

where $\theta = \theta_n$, $\phi = \phi_n$ are the polar angles of the normal $\mathbf{n}$ to the decay triangle, the direction of $\mathbf{n}$ being along $\mathbf{p}_1 \times \mathbf{p}_2$.

The significance of γ is best seen by noting that the above state can be obtained from a 'standard' one by a rotation through Euler angles ϕ_n, θ_n, γ:

$$|\omega_i\lambda_i; \phi_n, \theta_n, \gamma\rangle = U[r(\phi_n, \theta_n, \gamma)]|\omega_i\lambda_i; 0, 0, 0\rangle \tag{8.2.54}$$

[1] Any other rest frame of C is equally good provided the correct density matrix for C is used in the formulae that follow.

where, in $|\omega_i\lambda_i; 0, 0, 0\rangle$, $\mathbf{p}_1$ lies along OX and $\mathbf{p}_2$ lies in the XY-plane with $p_2 > 0$, so that $\mathbf{n}$ lies along OZ.

For a given event, once the polar angles θ_n, ϕ_n of the normal are determined, the angle γ can be found from the polar angles θ_1, ϕ_1 of E_1 as follows:

$$\begin{aligned} \cos\gamma &= \cos\theta_1 / \sin\theta_n \\ \sin\gamma &= \sin\theta_1(\sin\phi_1\cos\phi_n - \cos\phi_1\sin\phi_n). \end{aligned} \tag{8.2.55}$$

The decay amplitude is of the form

$$\begin{aligned} &H_{\lambda_1\lambda_2\lambda_3;\lambda_C}(\omega_1, \omega_2, \omega_3; \phi_n, \theta_n, \gamma) \\ &\qquad \propto \sum_{\mathscr{M}=-J}^{J} F_{\mathscr{M}}(\omega_1\lambda_1, \omega_2\lambda_2, \omega_3\lambda_3)\mathscr{D}^{(J)^*}_{\lambda_C\mathscr{M}}(\phi_n, \theta_n, \gamma) \end{aligned}$$

where, because $\lambda_C = J_z$, $F_{\mathscr{M}}$ is independent of λ_C by rotational invariance. The physical significance of $\mathscr{M}$ is that it is the projection of $\mathbf{J}$ along the normal to the decay plane; the dependence on $\mathscr{M}$ is a new feature compared with the two-body decay situation.

We normalize $F_{\mathscr{M}}$ in such a way that

$$\int\!\!\int d\omega_1 d\omega_2 \sum_{\mathscr{M}} \sum_{\lambda_i} |F_{\mathscr{M}}(\omega_i\lambda_i)|^2 = 1 \tag{8.2.56}$$

where the integration over ω_1, ω_2 corresponds to summing over the Dalitz plot.

The most detailed distribution we consider involves measurement of the helicity multipole moments of particle E_1 and of their dependence on ϕ_n, θ_n, γ. (These are the multipole moments in the helicity rest frame S_{E_1} of E_1 reached from S_C. S_{E_1} has its Z-axis along $\mathbf{p}_1$ and OX opposite to the normal to the decay plane.) It is assumed that a summation over the Dalitz plot is performed.[1] Then

$$W(\phi_n, \theta_n, \gamma)t^l_m(E_1) = \frac{1}{8\pi^2}\sum_{L,M}(2L+1)t^L_M(C)[LM|lm]_{\phi_n\theta_n\gamma} \tag{8.2.57}$$

where $t^L_M(C)$ are the helicity multipole parameters of C, $W(\phi_n, \theta_n, \gamma)$ is the normalized decay distribution and

$$\begin{aligned} [LM|lm]_{\phi_n\theta_n\gamma} \equiv &\sum_{\mathscr{M}\mathscr{M}'} R_{\mathscr{M}\mathscr{M}'}(l, m)\langle J, \mathscr{M}'; L, \mathscr{M}-\mathscr{M}'|J, \mathscr{M}\rangle \\ &\times \mathscr{D}^{(L)}_{M,\mathscr{M}-\mathscr{M}'}(\phi_n, \theta_n, \gamma) \end{aligned} \tag{8.2.58}$$

[1] When particles E_2 and E_3 are identical we relax this assumption, as discussed later.

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in which the *decay parameters* are

$$R_{\mathcal{M}\mathcal{M}'}(l,m) = \iint d\omega_2 d\omega_3 \sum_{\lambda_i} \langle s_1, \lambda_1 | s_1, \lambda_1 - m; l, m\rangle \\ \times F^*_{\mathcal{M}}(\omega_2, \omega_3; \lambda_1, \lambda_2, \lambda_3) F_{\mathcal{M}'}(\omega_2, \omega_3; \lambda_1 - m, \lambda_2, \lambda_3). \tag{8.2.59}$$

Note that

$$R^*_{\mathcal{M}\mathcal{M}'}(l,m) = (-1)^m R_{\mathcal{M}'\mathcal{M}}(l,-m). \tag{8.2.60}$$

For simplicity we shall write $R_{\mathcal{M}'\mathcal{M}}$ for $R_{\mathcal{M}'\mathcal{M}}(0,0)$.

(*ii*) *The angular distribution of the normal to the decay plane*

If we take $l = m = 0$ in (8.2.57) and integrate over γ we are left with the normalized angular distribution for the normal to the decay triangle:

$$\begin{aligned} \bar{W}(\phi_n, \theta_n) &\equiv \int_0^{2\pi} d\gamma\, W(\phi_n, \theta_n, \gamma) \\ &= \frac{1}{\sqrt{4\pi}} \sum_{L,M} \sqrt{2L+1}\, t^{L\,*}_M(C) R_L Y_{LM}(\theta_n, \phi_n) \end{aligned} \tag{8.2.61}$$

where the real constants R_L are given by

$$R_L = \sum_{\mathcal{M}} R_{\mathcal{M}\mathcal{M}} \langle J, \mathcal{M}; L, 0 | J, \mathcal{M}\rangle\,. \tag{8.2.62}$$

The normalization condition (8.2.56) implies that

$$R_0 = \sum_{\mathcal{M}} R_{\mathcal{M}\mathcal{M}} = 1.$$

Note that (8.2.61) is exactly analogous to the two-particle decay distribution (8.2.20), with the R_L, which depend on the dynamics of the process, playing the rôle of the decay parameters $C_L(0,0;0,0)$.

(a) *Consequences of parity conservation*

If parity is conserved in the decay process, one finds that

$$F_{\mathcal{M}}(\omega_2, \omega_3; \lambda_1, \lambda_2, \lambda_3) = e^{i\pi\mathcal{M}} \eta_C \eta_1 \eta_2 \eta_3 (-1)^{s_1 - \lambda_1 + s_2 - \lambda_2 + s_3 - \lambda_3} \\ \times F_{\mathcal{M}}(\omega_2, \omega_3; -\lambda_1, -\lambda_2, -\lambda_3) \tag{8.2.63}$$

where the η are intrinsic parities. If the E_i are spinless particles then (8.2.63) will cause the vanishing of either the even-$\mathcal{M}$ or the odd-$\mathcal{M}$ amplitudes, depending on the intrinsic parities. In the general spin case one has, from (8.2.59),

$$R_{\mathcal{M}\mathcal{M}'}(l,-m) = (-1)^{l+\mathcal{M}-\mathcal{M}'} R_{\mathcal{M}\mathcal{M}'}(l,m). \tag{8.2.64}$$

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Thus

$$R_{\mathscr{M}\mathscr{M}'}(l,0) = 0 \qquad \text{if } l + \mathscr{M} - \mathscr{M}' \text{ is odd.} \tag{8.2.65}$$

In contrast to the two-body case, (8.2.18), parity invariance here does not make R_L vanish for odd L in general, and $\bar{W}(\phi_n, \theta_n)$ serves as an analyser of both the even and odd parts of the polarization of C.[1]

We see from (8.2.61) that the ratios $t^L_M(C)/t^L_{M'}(C)$ for any L and arbitrary M, M' can be obtained directly from $\bar{W}(\phi_n, \theta_n)$ without requiring any knowledge of the R_L. To obtain the dependence on L does require a knowledge of the R_L.

(b) *Identical particles*

If particles E_2 and E_3 are identical, or if they are in an eigenstate of isotopic spin, the decay amplitude must be made either symmetric or antisymmetric under the exchange of the space and spin labels of E_2 and E_3. The correctly symmetrized amplitude $F^{\mathscr{S}}$ is then found to satisfy

$$\begin{aligned} F^{\mathscr{S}}_{\mathscr{M}}(\omega_1\lambda_1, \omega_2\lambda_2, \omega_3\lambda_3) = &\pm (-1)^{J+\lambda_1+\lambda_2+\lambda_3} \\ &\times F^{\mathscr{S}}_{-\mathscr{M}}(\omega_1\lambda_1, \omega_3\lambda_3, \omega_2\lambda_2). \end{aligned} \tag{8.2.66}$$

Introducing the quantity $R_{\mathscr{M}\mathscr{M}'}(l, m; \omega_2, \omega_3)$ as the unintegrated analogue of $R_{\mathscr{M}\mathscr{M}'}(l, m)$ (see eqn (8.2.59)), one obtains, for the symmetrized form,

$$R^{\mathscr{S}}_{\mathscr{M}\mathscr{M}'}(l, m; \omega_2, \omega_3) = e^{-i\pi m} R^{\mathscr{S}}_{-\mathscr{M}-\mathscr{M}'}(l, m; \omega_3, \omega_2). \tag{8.2.67}$$

If we also introduce the unintegrated version of R_L, i.e.

$$R_L(\omega_2, \omega_3) = \sum_{\mathscr{M}} R_{\mathscr{M}\mathscr{M}}(0, 0; \omega_2, \omega_3) \langle J, \mathscr{M}; L, 0 | J, \mathscr{M} \rangle \tag{8.2.68}$$

then we obtain

$$\begin{aligned} R^{\mathscr{S}}_L(\omega_2, \omega_3) = &\tfrac{1}{2} \sum_{\mathscr{M}} \langle J, \mathscr{M}; L, 0 | J, \mathscr{M} \rangle \\ &\times \left[R^{\mathscr{S}}_{\mathscr{M}\mathscr{M}}(0, 0; \omega_2, \omega_3) + (-1)^L R^{\mathscr{S}}_{\mathscr{M}\mathscr{M}}(0, 0; \omega_3, \omega_2) \right\} \end{aligned} \tag{8.2.69}$$

and we see that after integration over the whole Dalitz plot we will find

$$R^{\mathscr{S}}_L = 0 \qquad \text{for } L \text{ odd.}$$

To avoid this loss of analysing efficiency when $E_2 = E_3$ or when E_2 and E_3 are in eigenstates of isotopic spin, one should restrict the Dalitz-plot integration region to, say, $\omega_2 > \omega_3$.

[1] The one exception to the above occurs when C has spin 1, all the decay particles have spin zero and $\eta_C = \eta_1\eta_2\eta_3$. In this case $\mathscr{M} = \pm 1, 0$ only and, by (8.2.63), $F_{\pm 1} = 0$. Then $R_L = 0$ for odd L by (8.2.62) and only the even polarization of C is analysed.

Several examples for $J \leq 2$ are worked out in detail in Berman and Jacob (1965) and in the following section.

(c) *Moments of the experimental distributions*
Let $G(\phi_n, \theta_n, \gamma)$ be any function of the angles ϕ_n, θ_n, γ that specify the configuration of the three-particle final state in the rest frame S_C. We denote by $\langle G \rangle$ the angle average $(\phi_n, \theta_n, \gamma)$ of G weighted by the normalized decay distribution:

$$\langle G \rangle \equiv \int_0^{2\pi} d\phi_n \int_0^{\pi} \sin\theta_n \, d\theta_n \int_0^{2\pi} d\gamma \, W(\phi_n, \theta_n, \gamma) G(\phi_n, \theta_n, \gamma). \tag{8.2.70}$$

The most general case we consider involves the taking of moments of the angular distribution, or of the distribution of helicity multipole parameters of E_1. From (8.2.57) and (8.2.58) one has immediately:

$$\left\langle t^l_m(E_1) \mathscr{D}^{(L)^*}_{M\mu} \right\rangle = t^L_M(C) \sum_{\mathscr{M}} \langle J, \mathscr{M} - \mu; L, \mu | J, \mathscr{M} \rangle \times R_{\mathscr{M}, \mathscr{M}-\mu}(l, m). \tag{8.2.71}$$

Note that moments with $\mu = 0$ just correspond to integrating over the angle γ.

We see that information about a specific $t^L_M(C)$ can be obtained from many different moments, by choosing different values of μ, or from studying the distribution of the different $t^l_m(E_1)$.

In general one may need to know some of the decay parameters in order to extract the t^L_M. Just as in the two-body decay case, some reactions are *magic* and yield a complete analysis of the $t^L_M(C)$. Other reactions may yield only ratios of t^L_M at fixed M as L varies, and these reactions require the knowledge of certain of the dynamic-dependent decay parameters in order to get actual values of the $t^L_M(C)$. These parameters can be obtained in a model-independent fashion only if the resonance C can be prepared in a definite state of polarization, and that seems to be impossible if $J \geq 1$.

We shall look at several cases of interest.

(c.1) *Decay into spinless particles.* We are here limited to moments of the angular distribution only, i.e. to $l = m = 0$ in (8.2.71). From (8.2.63), if parity is conserved in the decay then $F_{\mathscr{M}} = 0$ for $\mathscr{M}$ odd or even according to whether $\eta_C \eta_1 \eta_2 \eta_3 = \pm 1$. In both cases $R_{\mathscr{M}, \mathscr{M}-\mu}$ will vanish for μ odd. Thus only moments with μ even are non-zero. There are then several possibilities.

• $J = 1$ and $\eta_C \eta_1 \eta_2 \eta_3 = +1$. In this case only $F_0 \neq 0$ and we are left with just R_{00}, which is equal to unity by the normalization condition (8.2.56). Then from (8.2.71)

$$\left\langle \mathscr{D}^{(1)^*}_{M0} \right\rangle = 0 \qquad \left\langle \mathscr{D}^{(2)^*}_{M0} \right\rangle = \langle 1, 0; 2, 0 | 1, 0 \rangle \, t^2_M(C) = -\sqrt{\tfrac{2}{5}} t^2_M(C) \tag{8.2.72}$$

so no information is obtained about the $t^1_M(C)$ but the $t^2_M(C)$ are fully determined.

• $J = 2$ and $\eta_C\eta_1\eta_2\eta_3 = +1$. Now we have R_{22}, R_{00}, $R_{-2-2} = 1 - R_{22} - R_{00}$, $R_{20} = R^*_{02}$, $R_{-20} = R^*_{0-2}$, R_{2-2} and from (8.2.71)

$$\left\langle \mathscr{D}^{(L)^*}_{M0} \right\rangle = t^L_M(C)\,\{\langle 2,2;L,0|2,2\rangle + R_{00}\,[\langle 2,0;L,0|2,0\rangle - \langle 2,2;L,0|2,2\rangle]\} \qquad \text{for } L = 0,2,4 \quad (8.2.73)$$

$$\left\langle \mathscr{D}^{(L)^*}_{M0} \right\rangle = t^L_M(C)\,\langle 2,2;L,0|2,2\rangle \times (R_{00} + 2R_{22} - 1) \qquad \text{for } L = 1,3 \quad (8.2.74)$$

$$\left\langle \mathscr{D}^{(L)^*}_{M2} \right\rangle = t^L_M(C)\,\langle 2,0;L,2|2,2\rangle \times \left[R_{20} + (-1)^L R_{0-2}\right] \qquad \text{for } L = 2,3,4 \quad (8.2.75)$$

$$\left\langle \mathscr{D}^{(L)^*}_{M4} \right\rangle = t^L_M(C)\,\langle 2,-2;L,4|2,2\rangle\, R_{2-2} \qquad \text{for } L = 4. \quad (8.2.76)$$

From (8.2.75) we can obtain the ratio $t^2_M(C)/t^4_M(C)$. The same ratio is also obtained from (8.2.73), as a function of R_{00}, which can thus be determined. Hence we obtain explicitly the even-rank multipoles. From (8.2.74) we can obtain the ratio $t^1_M(C)/t^3_M(C)$ but not the explicit values of the odd multipoles.

• Arbitrary integer J and $\eta_C\eta_1\eta_2\eta_3 = +1$. The previous method can be generalized, and one obtains the even multipoles explicitly and the ratios only of the odd multipoles. Briefly, choosing $\mu = 2J - 2$ yields $t^{2J-2}_M(C)/t^{2J}_M(C)$. Using this in the moment with $\mu = 2J - 4$ allows the elimination of $R_{J,-J+4} + R_{J-4,-J}$, after which $t^{2J-4}_M(C)/t^{2J-2}_M(C)$ can be evaluated. Proceeding thus, one ends up with explicit values for all the even multipoles. For the odd multipoles, $\mu = 2J - 4$ yields $t^{2J-3}_M(C)/t^{2J-1}_M(C)$, which, used in the moment with $\mu = 2J - 6$, yields $t^{2J-5}_M(C)/t^{2J-3}_M(C)$. Proceeding thus, one ends up with the ratios of all the odd multipoles but not their explicit value.

• $J = 1$ and $\eta_C\eta_1\eta_2\eta_3 = -1$. The non-zero parameters are R_{11} (real); $R_{-1-1} = 1 - R_{11}$ and $R_{1-1} = R^*_{-11}$. Then from (8.2.71)

$$\left\langle \mathscr{D}^{(1)^*}_{M0} \right\rangle = \tfrac{1}{\sqrt{2}}\,[2R_{11} - 1]\, t^1_M(C)$$
$$\left\langle \mathscr{D}^{(2)^*}_{M0} \right\rangle = \tfrac{1}{\sqrt{10}} t^2_M(C) \qquad \left\langle \mathscr{D}^{(2)^*}_{M2} \right\rangle = \sqrt{\tfrac{3}{5}}\, R_{1-1} t^2_M(C). \quad (8.2.77)$$

Again the $t^2_M(C)$ are fully determined, and now the $t^1_M(C)$ are also determined, up to one overall factor.

• $J = 2$ and $\eta_C\eta_1\eta_2\eta_3 = -1$. We have R_{11}, $R_{-1-1} = 1 - R_{11}$, $R_{1-1} = R^*_{-11}$ and from (8.2.71)

$$\begin{aligned}
\left\langle \mathscr{D}^{(L)^*}_{M0} \right\rangle &= t^L_M(C)\,\langle 2,1;L,0|2,1\rangle \qquad \text{for } L = 0,2,4 \\
\left\langle \mathscr{D}^{(L)^*}_{M0} \right\rangle &= t^L_M(C)\,\langle 2,1;L,0|2,1\rangle\,(2R_{11} - 1) \qquad \text{for } L = 1,3 \\
\left\langle \mathscr{D}^{(L)^*}_{M2} \right\rangle &= t^L_M(C)\,\langle 2,-1;L,2|2,1\rangle\,R_{1-1} \\
\left\langle \mathscr{D}^{(L)^*}_{M3} \right\rangle &= \left\langle \mathscr{D}^{(L)^*}_{M4} \right\rangle = 0.
\end{aligned} \tag{8.2.78}$$

Thus the even-L multipoles are fully determined but for the odd-L ones only their ratio is determined.

• Arbitrary integer J and $\eta_C\eta_1\eta_2\eta_3 = -1$. The method described above for the case arbitrary J and $\eta_C\eta_1\eta_2\eta_3 = +1$ is applicable and yields the explicit value of the even multipoles and the ratios of the odd multipoles.

In summary, for arbitrary J and $\eta_C\eta_1\eta_2\eta_3 = \pm 1$ all even multipoles are determined but only the ratios of the odd multipoles. It should be noted that moments of the γ distribution are essential for this to be possible, once $J \geq 2$.

(c.2) *Decay into one spin-*$1/2$ *and two spinless particles.* Here, in principle, we can utilize moments of the angular distribution and of the distribution of the spin components of E_1. The decay then functions as a *magic* analyser and the *complete* density matrix of the decaying particle can be determined.

We consider the decay of a spin-$3/2$ resonance in detail and outline the approach for the case of arbitrary half-integer spin.

• $C(3/2) \to E_1(1/2) + E_2(0) + E_3(0)$. Because of parity invariance and because E_1 can have only two values for its helicity, it turns out that all $R_{\mathscr{M}\mathscr{M}'}(l,m)$ can be written in terms of one combination of the decay amplitudes, namely

$$Q_{\mathscr{M}\mathscr{M}'} \equiv \int d\omega_2 d\omega_3 F^*_{\mathscr{M}}(\omega_2,\omega_3;1/2)F_{\mathscr{M}'}(\omega_2,\omega_3;1/2). \tag{8.2.79}$$

One has then from (8.2.59)

$$\begin{aligned}
R_{\mathscr{M}\mathscr{M}'} \equiv R_{\mathscr{M}\mathscr{M}'}(0,0) &= \left[1 + (-1)^{\mathscr{M}-\mathscr{M}'}\right] Q_{\mathscr{M}\mathscr{M}'} \\
R_{\mathscr{M}\mathscr{M}'}(1,0) &= \tfrac{1}{\sqrt{3}} \left[1 - (-1)^{\mathscr{M}-\mathscr{M}'}\right] Q_{\mathscr{M}\mathscr{M}'} \\
R_{\mathscr{M}\mathscr{M}'}(1,1) &= -(-1)^{\mathscr{M}-\mathscr{M}'} R_{\mathscr{M}\mathscr{M}'}(1,-1) \\
&= \eta_C\eta_1\eta_2\eta_3 e^{i\pi\mathscr{M}'} \sqrt{\tfrac{2}{3}} Q_{\mathscr{M}\mathscr{M}'}.
\end{aligned} \tag{8.2.80}$$

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Finally, it will turn out convenient to define the combinations

$$Q^{\pm}_{\mathcal{M}\mathcal{M}'} = Q_{\mathcal{M}\mathcal{M}'} \pm Q_{-\mathcal{M}'-\mathcal{M}}. \tag{8.2.81}$$

For the $\mu = 0$ moments of the angular distribution one then finds

$$\begin{aligned}
\left\langle \mathscr{D}^{(L)^*}_{M0} \right\rangle &= t^L_M(C) \Big\{ \langle 3/2, 3/2; L, 0 | 3/2, 3/2 \rangle \\
&\qquad + 2Q^{+}_{1/2\ 1/2} \left[\langle 3/2, 1/2; L, 0 | 3/2, 1/2 \rangle \right. \\
&\qquad \left. - \langle 3/2, 3/2; L, 0 | 3/2, 3/2 \rangle \right] \Big\} \qquad \text{for } L = 0, 2 \\
\left\langle \mathscr{D}^{(L)^*}_{M0} \right\rangle &= 2t^L_M(C) \Big\{ \langle 3/2, 3/2; L, 0 | 3/2, 3/2 \rangle \, Q^{-}_{3/2\ 3/2} \\
&\qquad + \langle 3/2, 1/2; L, 0 | 3/2, 1/2 \rangle \, Q^{-}_{1/2\ 1/2} \Big\} \\
&\qquad\qquad \text{for } L = 1, 3.
\end{aligned} \tag{8.2.82}$$

For the $\mu = 0$ moments of the distribution of the longitudinal component of the spin of E_1 one finds

$$\left\langle \mathscr{P}_z(E_1) \mathscr{D}^{(L)^*}_{M0} \right\rangle = \sqrt{3} \left\langle t^1_0(E_1) \mathscr{D}^{(L)^*}_{M0} \right\rangle = 0 \tag{8.2.83}$$

where the subscript refers to the axis OZ of the helicity rest frame S_{E_1}.

In particular the mean longitudinal polarization is zero after integration over γ.

For the $\mu = 0$ moments of the transverse polarization, along OY of S_{E_1}, one finds similarly

$$\left\langle \mathscr{P}_y(E_1) \mathscr{D}^{(L)^*}_{M0} \right\rangle = i\sqrt{\tfrac{3}{2}} \left\langle \left[t^1_1(E_1) + t^1_{-1}(E_1) \right] \mathscr{D}^{(L)^*}_{M0} \right\rangle = 0. \tag{8.2.84}$$

Again the mean transverse polarization, in the decay plane, is zero.

For the component of spin $\mathscr{P}_n$ along the normal to the decay plane, since OX in the helicity rest frame S_{E_1} is opposite to $\mathbf{n}$ we have (with $\eta = \eta_C \eta_1 \eta_2 \eta_3$)

$$\begin{aligned}
\left\langle \mathscr{P}_n(E_1) \mathscr{D}^{(L)^*}_{M0} \right\rangle &= \sqrt{\tfrac{3}{2}} \left\langle \left[t^1_1(E_1) - t^1_{-1}(E_1) \right] \mathscr{D}^{(L)^*}_{M0} \right\rangle \\
&= -2i\eta t^L_M(C) \Big\{ \langle 3/2, 3/2; L, 0 | 3/2, 3/2 \rangle \, Q^{\mp}_{3/2,3/2} \\
&\qquad - \langle 3/2, 1/2; L, 0 | 3/2, 1/2 \rangle \, Q^{\mp}_{1/2\ 1/2} \Big\} \\
&\qquad\qquad \text{for } L \text{ even/odd}
\end{aligned} \tag{8.2.85}$$

We note in particular that from the moment with $L = M = 0$ we get

$$\langle \mathscr{P}_n(E_1) \rangle = -2i\eta \left(Q^{-}_{3/2\ 3/2} - Q^{-}_{1/2\ 1/2} \right). \tag{8.2.86}$$

We now show how all the multipole moments $t^L_M(C)$ can be obtained explicitly from just the $\mu = 0$ moments of the angular distribution and those of the polarization along the normal to the decay plane.

Use of (8.2.86) allows us to eliminate $Q^-_{3/2\ 3/2}$ in eqns (8.2.85) and (8.2.82). Then for $L = 1$ and 2 in (8.2.82) and (8.2.85) we have, schematically,

$$\begin{aligned}\left\langle \mathscr{D}^{(2)^*}_{M0} \right\rangle &= t^2_M(C)\left(a + bQ^+_{1/2\ 1/2}\right)\\ \left\langle \mathscr{D}^{(1)^*}_{M0} \right\rangle &= t^1_M(C)\left(c + dQ^-_{1/2\ 1/2}\right)\\ \left\langle \mathscr{P}_n(E_1)\mathscr{D}^{(2)^*}_{M0} \right\rangle &= t^2_M(C)\left(e + fQ^-_{1/2\ 1/2}\right)\\ \left\langle \mathscr{P}_n(E_1)\mathscr{D}^{(1)^*}_{M0} \right\rangle &= t^1_M(C)\left(g + hQ^+_{1/2\ 1/2}\right)\end{aligned} \tag{8.2.87}$$

where a, b, c, d, e, f, g, h are *known* constants, combination of vector addition coefficients and the measured value of $\langle \mathscr{P}_n(E_1)\rangle$.

If the four quantities (for fixed M) on the left-hand side of (8.2.87) are measured then we have four equations for the four unknowns, t^1_M, t^2_M, $Q^+_{1/2\ 1/2}$ and $Q^-_{1/2\ 1/2}$. Thus we can get the explicit values of $t^1_M(C)$ and $t^2_M(C)$. With $Q^-_{1/2\ 1/2}$ now known, a measurement of $\left\langle \mathscr{D}^{(3)^*}_{M0} \right\rangle$ will yield the value of $t^3_M(C)$ via (8.2.82).

If particles E_2 and E_3 are identical, care should be taken to integrate over the region $\omega_2 > \omega_3$ only in the Dalitz plot, as discussed in subsection (b) above.

The moments described above have $\mu = 0$, so that γ is simply integrated over, and they suffice to determine all the $t^L_M(C)$. Moments with $\mu \neq 0$ are also interesting if one wishes to study the dynamics of the decay mechanisms. The following general rule holds for moments with arbitrary μ:

$$\left\langle \left[t^l_m(E_1) - (-1)^{l+\mu} t^l_{-m}(E_1)\right] \mathscr{D}^{(L)^*}_{M\mu} \right\rangle = 0. \tag{8.2.88}$$

It follows that $\left\langle t^1_0(E_1)\mathscr{D}^{(L)^*}_{M\mu} \right\rangle = 0$ if $l + \mu$ is odd. The result (8.2.88) can be used as a check on experimental biases. The vanishing of the $\mu = 0$ moments of $\mathscr{P}_z$ and $\mathscr{P}_y$ mentioned above is a special case of (8.2.88).

• C (arbitrary half-integer J) $\to E_1(1/2) + E_2(0) + E_3(0)$. We outline briefly how the $t^L_M(C)$ can be obtained from the moments of the various distributions. For $J \geq 5/2$ it is necessary to utilize moments with $\mu > 0$ as well.

For fixed L, the dependence of the $t^L_M(C)$ upon M is trivially obtained from the ratio of moments with the same L and various M. As can be seen from (8.2.71) all the decay-dependent parameters will cancel out.

We thus concentrate on moments with one value of M, namely $M = 0$.

For simplicity let us denote by M^L_μ a moment of the type $\left\langle \mathscr{D}^{(L)^*}_{0\mu} \right\rangle$ and by N^L_μ the type $\left\langle \mathscr{P}_n(E_1)\mathscr{D}^{(L)^*}_{0\mu} \right\rangle$, and let us put t^L for $t^L_0(C)$.

For $\mu = 2J - 1$, M^{2J}_{2J-1} and N^{2J-1}_{2J-1} will be proportional to $t^{2J} Q^-_{J,-J+1}$ and $t^{2J-1} Q^-_{J,-J+1}$ respectively. Their ratio gives t^{2J-1}/t^{2J}.

For $\mu = 2J - 3$ there are four unknown decay constants, $Q^{\pm}_{J,-J+3}$ and $Q^{\pm}_{J-1,-J+2}$. By taking ratios of M^{2J}_{2J-3}, M^{2J-1}_{2J-3}, N^{2J}_{2J-3} and N^{2J-1}_{2J-3} and using the known value of t^{2J-1}/t^{2J} one is able to obtain, say, $Q^{\pm}_{J-1,-J+2}$ and $Q^-_{J,-J+3}$ in the form (known constant) $\times\, Q^+_{J,-J+3}$. Making these substitutions everywhere, $Q^+_{J,-J+3}$ becomes a common factor in the equations relating moments to t^L. Thus we can obtain t^{2J-2}/t^{2J-1} and t^{2J-3}/t^{2J-2}.

Proceeding in this way one finds for each choice of μ information from the measured moments and from the previously determined multipole ratios that is more than enough, to express all $Q^{\pm}_{\mathcal{M},\mathcal{M}-\mu}$ in the form (known constant) $\times Q^+_{J,J-\mu}$. Substitution of this into the equations yields the new ratios $t^{\mu}/t^{\mu+1}$, $t^{\mu+1}/t^{\mu+2}$.

At the last stage, $\mu = 0$, we know the ratios $t^2/t^3, t^3/t^4, \ldots, t^{2J-1}/t^{2J}$ and we can obtain all $Q^{\pm}_{\mathcal{M},\mathcal{M}}$ in the form (known constant) $\times\, Q^+_{J,J}$, whereafter the ratio t^1/t^2 can be found. Now, however, we have also the normalization condition $\sum_{\mathcal{M}>0} Q^+_{\mathcal{M},\mathcal{M}} = 1/2$, which follows from (8.2.56), (8.2.79) and (8.2.81), and this determines the actual value of $Q^+_{J,J}$. Then the value of t^1, say, can be found explicitly, from which follow the values of all the other t^L.

Note that we end up with not just the desired t^L_M but also the whole set of decay parameters $Q^{\pm}_{\mathcal{M},\mathcal{M}-\mu}$ (μ even).

(c.3) *Two-body resonance domination of three-body state.* If the three-body final state is dominated by resonance formation between two of the particles then we regard the decay as a two-step process

$$\begin{array}{l} C \to C' + E_3 \\ \qquad\ \ \searrow \\ \qquad\quad E_1 + E_2 \end{array}$$

and this is discussed fully in subsections 8.2.1(vii), (viii) above.

(c.4) *Decay into photon and two spinless particles, C(J integer) $\to$ Photon$+$ $E_2(0) + E_3(0)$.* Because $\lambda = 0$ is forbidden to a photon, this case is very similar to that where E_1 has spin 1/2. Parity conservation allows all decay parameters to be related to the combination

$$Q_{\mathcal{M}\mathcal{M}'} \equiv \int d\omega_2 d\omega_3 F^*_{\mathcal{M}}(\omega_2, \omega_3; 1) F_{\mathcal{M}'}(\omega_2, \omega_3; 1). \tag{8.2.89}$$

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We have

$$\begin{aligned}
R_{\mathscr{M}\mathscr{M}'}(0,0) &= \left[1+(-1)^{\mathscr{M}-\mathscr{M}'}\right] Q_{\mathscr{M}\mathscr{M}'} \\
R_{\mathscr{M}\mathscr{M}'}(2,0) &= \tfrac{1}{\sqrt{10}} R_{\mathscr{M}\mathscr{M}'}(00) \\
R_{\mathscr{M}\mathscr{M}'}(2,2) &= \eta_C\eta_2\eta_3(-1)^{\mathscr{M}'+1}\sqrt{\tfrac{3}{5}}\, Q_{\mathscr{M}\mathscr{M}'} \\
R_{\mathscr{M}\mathscr{M}'}(1,0) &= \tfrac{1}{\sqrt{2}}\left[1-(-1)^{\mathscr{M}-\mathscr{M}'}\right] Q_{\mathscr{M}\mathscr{M}'}.
\end{aligned} \tag{8.2.90}$$

If the state of polarization of the photon can be determined then the measured moments will yield the $t^L_M(C)$ via (8.2.71) and (8.2.90) in similar manner to case (c.2) above.

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