FOURTH-ORDER NUMERICAL METHODS FOR THE COUPLED KORTEWEG–DE VRIES EQUATIONS

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Abstract

We compare six fixed-stepsize fourth-order numerical methods for a number of test problems described by a system of coupled Korteweg–de Vries equations. Particular attention is paid to the ability of these methods to preserve fixed points (solitary waves) and the invariants of the system, and establishing to what extent the conservation of integral invariants is indicative of the solution error for these methods.

1. Introduction

In this paper we consider the numerical integration of a Hamiltonian system of two coupled Korteweg–de Vries (KdV) equations

$$\alpha_1 A_t + (\delta_1 A + \kappa B + \lambda_1 A_{xx} + \sigma B_{xx} + \mu_1 A^2 + \nu_1 B^2 + 2\nu_2 AB)_x = 0,$$
$$\alpha_2 B_t + (\kappa A + \delta_2 B + \sigma A_{xx} + \lambda_2 B_{xx} + \nu_2 A^2 + \mu_2 B^2 + 2\nu_1 AB)_x = 0, \tag{1.1}$$

where $\alpha_{1,2}, \delta_{1,2}, \lambda_{1,2}, \mu_{1,2}, \nu_{1,2}, \kappa, \sigma \in \mathbb{R}, x \in \mathbb{R}$ and for $t \geq 0$ both $A(x,t)$ and $B(x,t)$ are real-valued functions. This system has been derived in the context of wave interactions in two- or three-layer models of fluids [9]. Such models began to attract interest in the 1960s, following the publication of a paper by Eckart [5] on internal waves in the ocean. Variations of system (1.1) were obtained for Eady models of baroclinic instability [16], and quasigeostrophic baroclinic waves in the context of atmospheric blocking [8]. Theoretical properties of (1.1) have also been extensively studied (see, for example, [22]).

In general, system (1.1) is not solvable by the inverse scattering transform [9], therefore it is of considerable interest to determine optimal numerical methods for

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finding its solutions. Two systematic comparisons of numerical methods applied to the KdV equation (the underlying uncoupled equation for (1.1)) have been previously published. Taha and Ablowitz [21] considered seven finite-difference and two pseudospectral methods, all of first- or second-order accuracy in time-stepping. Later, Nouri and Sloan [17] compared six pseudospectral second-order methods. Various numerical methods (including the fourth-order ones) have been tested on the KdV equation, transformed to the so-called semilinear form

$$u_t = \mathcal{L}u + \mathcal{N}(u) \equiv F(u), \quad u(t_0) = u_0, \tag{1.2}$$

where $\mathcal{L}$ and $\mathcal{N}$ are linear and nonlinear operators, respectively. This has usually been done in the context of exponential integrators [15] which we will discuss later. While the results of these comparisons can provide indications regarding the optimal methods for (1.1), they are not directly translatable to the system we consider, because the linear operator obtained in (1.2) is represented by a sparse nondiagonal matrix, whereas for the uncoupled KdV equation the linear operator is a diagonal matrix. In particular, the exponential integrators require computation of certain special functions which depend on the structure of this matrix, which suggests that, should this involve a substantial computational effort, exponential integrators may be more expensive than other methods. Furthermore, unlike the regular KdV equation, the system we consider is potentially unstable due to coupling and transverse variations (see, for example, [20]), which poses an additional challenge to numerical methods. At the same time, it is Hamiltonian, which is an advantage that we can make use of to identify numerical methods best suited to solve system (1.1).

In this paper we compare six popular fixed-stepsize fourth-order methods of different types, which include the classical Runge–Kutta (CRK4) method as a benchmark. All methods except CRK4 treat the linear part of the problem exactly, hence they are well suited to stiff problems. The previously widespread, computationally cheap, second-order methods cannot provide the improved accuracy and stability, often desirable for solving practical problems. In Section 2 we formulate the problem and then briefly describe the numerical methods included in the comparison in Section 3. The results of the comparison of four test problems are presented and discussed in Section 4. Finally, we summarize our findings in Section 5.

2. Problem formulation

We assume that the functions A and B are periodic on a spatial interval L_x. Let us discretize L_x by $N + 1$ equidistant points, $\{X_0, X_1, \ldots, X_N\}$ with $X_j = j\,dx$, $j = 0, 1, \ldots, N - 1$, $dx = L_x/N$. Taking the discrete Fourier transform of (1.1), and dividing the first equation by α_1 and the second one by α_2, we obtain the semilinear system (1.2) for $u = [\mathcal{F}(A), \mathcal{F}(B)]^T \in \mathbb{C}^{2N}$, where

$$\mathcal{L}u \equiv iLu, \quad L \equiv \begin{bmatrix} \text{diag}(\lambda_1 \mathbf{k}^3 - \delta_1 \mathbf{k})/\alpha_1 & \text{diag}(\sigma \mathbf{k}^3 - \kappa \mathbf{k})/\alpha_1 \\ \text{diag}(\sigma \mathbf{k}^3 - \kappa \mathbf{k})/\alpha_2 & \text{diag}(\lambda_2 \mathbf{k}^3 - \delta_2 \mathbf{k})/\alpha_2 \end{bmatrix}, \tag{2.1}$$

with $i^2 = -1$, and

$$\mathcal{N}(u) \equiv \begin{bmatrix} N_1(u) \\ N_2(u) \end{bmatrix} \equiv -i \begin{bmatrix} \mathbf{k} \circ \mathcal{F}(\mu_1 A^2 + \nu_1 B^2 + 2\nu_2 AB)/\alpha_1 \\ \mathbf{k} \circ \mathcal{F}(\nu_2 A^2 + \mu_2 B^2 + 2\nu_1 AB)/\alpha_2 \end{bmatrix}. \tag{2.2}$$

Here $\mathcal{F}(\cdot)$ denotes the coefficients of the discrete Fourier transform, $\mathbf{k} = [k_1, k_2, \ldots, k_N]^T = [-N/2, \ldots, -1, 0, 1, \ldots, N/2 - 1]^T (2\pi)/L_x$, and $\circ$ denotes the elementwise product of two vectors. Note that in (2.1) Lu represents an ordinary matrix multiplication of matrix $L \in \mathbb{R}^{2N \times 2N}$ by vector $u \in \mathbb{C}^{2N}$.

3. Numerical methods

In this section we provide a brief description of the numerical methods that will be applied to (1.2), detailed in (2.1) and (2.2). Throughout, the size of a fixed time step will be denoted by h, and u_n will denote the value of a function u at the nth time instant.

As a benchmark, we compare the methods outlined below with the classical, fourth-order Runge–Kutta method.

3.1. The φ-functions The exact solutions of (1.2) on $[t_n, t_{n+1}]$ with $t_{n+1} = t_n + h$ can be represented by the expansion

$$u_{n+1} = \exp(\mathcal{L}h)u_n + \sum_{k=1}^{\infty} \varphi_k(\mathcal{L}h)h^k \frac{d^{k-1}}{dt^{k-1}} \mathcal{N}(u)|_{t=t_n},$$

where

$$\varphi_k(\mathcal{L}h) = \frac{1}{h^k} \int_0^h \frac{\tau^{k-1}}{(k-1)!} \exp(\mathcal{L}(h - \tau)) \, d\tau, \quad k = 1, \ldots, m. \tag{3.1}$$

This result is discussed in detail by Minchev and Wright [15]. Functions of the form (3.1) are known as φ-*functions*. By definition, $\varphi_0(\mathcal{L}h) \equiv \exp(\mathcal{L}h)$, and other φ-functions can be calculated using the recurrence relation

$$\varphi_s(\mathcal{L}h) = (\mathcal{L}h)^{-1}(\varphi_{s-1}(\mathcal{L}h) - I/(s-1)!), \quad s = 1, 2, \ldots,$$

where I is the identity matrix. The question of how to evaluate these functions, including $\varphi_0(\mathcal{L}h)$, is very important when it comes to implementation of stiff integration methods which we will discuss later. Evaluation of the φ-functions (3.1) is a well-known and rather difficult problem because of the cancellation errors occurring if the eigenvalues of $\mathcal{L}h$ are close to zero. A number of approaches to solving this problem have been suggested. Here we use the scaling and squaring approach based on Padé approximations as implemented in the EXPINT package [19]. There is sufficient evidence [1] to suggest that it may represent the most efficient approach to evaluating φ-functions to date.

3.2. Split-step Fourier methods The split-step Fourier methods are based on the representation of a formal exact solution to (1.2) on a time interval $[t_0, t_0 + h]$, $u(t_0 + h) = \exp(h(\mathcal{L} + \mathcal{N}))u_0$, using sequential operator compositions to approximate

$$\exp(h(\mathcal{L} + \mathcal{N})) \approx \exp(c_1 h\mathcal{L}) \circ \exp(d_1 h\mathcal{N}) \circ \cdots \circ \exp(c_p h\mathcal{L}) \circ \exp(d_p h\mathcal{N}), \quad (3.2)$$

where $\{c_1, c_2, \ldots, c_p\}$ and $\{d_1, d_2, \ldots, d_p\}$ are real numbers. Practically, this involves solving the linear part of (1.2) exactly, and the nonlinear part numerically. Here the nonlinear problem is solved using CRK4. We consider here SSF4, a numerical method corresponding to (3.2) with $p = 4$ and

$$c_1 = c_4 = \frac{1}{2(2 - 2^{1/3})}, \quad c_2 = c_3 = \frac{1 - 2^{1/3}}{2(2 - 2^{1/3})},$$
$$d_1 = d_3 = \frac{1}{2 - 2^{1/3}}, \quad d_2 = -\frac{2^{1/3}}{2 - 2^{1/3}}, \quad d_4 = 0.$$

3.3. Linearly implicit Runge–Kutta methods Linearly implicit Runge–Kutta (RK) methods belong to the class of implicit–explicit (IMEX) numerical methods whose key feature is that the linear part of an equation is advanced with an implicit RK method, while an explicit RK method is applied to the nonlinear part. In particular, we consider LIRK4 as proposed by Calvo et al. [2]. The algorithm for this method is as follows:

$$K_1 = u_n,$$
$$K_2 = u_n + h\{(1/4)\mathcal{L}K_2 + (1/4)\mathcal{N}(K_1)\},$$
$$K_3 = u_n + h\{(1/2)\mathcal{L}K_2 + (1/4)\mathcal{L}K_3 - (1/4)\mathcal{N}(K_1) + \mathcal{N}(K_2)\},$$
$$K_4 = u_n + h\{(17/50)\mathcal{L}(K_2) - (1/25)\mathcal{L}K_3 + (1/4)\mathcal{L}K_4$$
$$\quad - (13/100)\mathcal{N}(K_1) + (43/75)\mathcal{N}(K_2) + (8/75)\mathcal{N}(K_3)\},$$
$$K_5 = u_n + h\{(371/1360)\mathcal{L}K_2 - (137/2720)\mathcal{L}K_3 + (15/544)\mathcal{L}K_4$$
$$\quad + (1/4)\mathcal{L}K_5 - (6/85)\mathcal{N}(K_1) + (42/85)\mathcal{N}(K_2)$$
$$\quad + (179/1360)\mathcal{N}(K_3) - (15/272)\mathcal{N}(K_4)\},$$
$$K_6 = u_n + h\{(25/24)\mathcal{L}K_2 - (49/48)\mathcal{L}K_3 + (125/16)\mathcal{L}K_4 - (85/12)\mathcal{L}K_5$$
$$\quad + (1/4)\mathcal{L}K_6 + (79/24)\mathcal{N}(K_2) - (5/8)\mathcal{N}(K_3) + (25/2)\mathcal{N}(K_4)$$
$$\quad - (85/6)\mathcal{N}(K_5)\},$$
$$u_{n+1} = u_n + h\{(25/24)\mathcal{L}K_2 - (49/48)\mathcal{L}K_3 + (125/16)\mathcal{L}K_4 - (85/12)\mathcal{L}K_5$$
$$\quad + (1/4)\mathcal{L}K_6 + (25/24)\mathcal{N}(K_2) - (49/48)\mathcal{N}(K_3) + (125/16)\mathcal{N}(K_4)$$
$$\quad - (85/12)\mathcal{N}(K_5) + (1/4)\mathcal{N}(K_6)\}.$$

3.4. Exponential time differencing methods Exponential time differencing (ETD) methods have been developed specifically for semilinear systems of the form (1.2). They solve the linear part of the system exactly and the nonlinear part

numerically, and aim to efficiently handle the issue of stiffness introduced by the linear part.

The fourth-order scheme ETDRK4CM, which solves the nonlinear part of an equation with CRK4, was suggested by Cox and Matthews [4]. It is specified as follows:

$$
\begin{aligned}
K_1 &= u_n, \\
K_2 &= h\{(1/2)\varphi_1(\mathcal{L}h/2)\mathcal{N}(K_1)\} + \varphi_0(\mathcal{L}h/2)u_n, \\
K_3 &= h\{(1/2)\varphi_1(\mathcal{L}h/2)\mathcal{N}(K_2)\} + \varphi_0(\mathcal{L}h/2)u_n, \\
K_4 &= h\{(1/2)\varphi_1(\mathcal{L}h/2)(\varphi_0(\mathcal{L}h/2) - I)\mathcal{N}(K_1) \\
     &\quad + \varphi_1(\mathcal{L}h/2)\mathcal{N}(K_3)\} + \varphi_0(\mathcal{L}h)u_n, \\
u_{n+1} &= h\{(\varphi_1(\mathcal{L}h) - 3\varphi_2(\mathcal{L}h) + 4\varphi_3(\mathcal{L}h))\mathcal{N}(K_1) \\
     &\quad + (2\varphi_2(\mathcal{L}h) - 4\varphi_3(\mathcal{L}h))\mathcal{N}(K_2) + (2\varphi_2(\mathcal{L}h) - 4\varphi_3(\mathcal{L}h))\mathcal{N}(K_3) \\
     &\quad + (4\varphi_3(\mathcal{L}h) - \varphi_2(\mathcal{L}h))\mathcal{N}(K_4)\} + \varphi_0(\mathcal{L}h)u_n.
\end{aligned}
$$

We also consider a representative of the Lie group methods which can be viewed as an ETD method. The Lie group methods aim to ensure that the numerical solution of an equation evolves on the same manifold as that on which the exact solution evolves. The method we include in our comparison is known as CF4 [3]. This differs from other Lie group methods in that it does not use commutators (hence the name commutator-free (CF)) and concentrates on minimizing the number of φ-function calculations. It can be described by the same intermediate stages as ETDRK4CM, with the only difference being in the solution update performed as

$$
\begin{aligned}
u_{n+1} &= h\{((1/2)\varphi_1(\mathcal{L}h) - (1/3)\varphi_1(\mathcal{L}h/2))\mathcal{N}(K_1) \\
     &\quad + (1/3)\varphi_1(\mathcal{L}h)\mathcal{N}(K_2) + (1/3)\varphi_1(\mathcal{L}h)\mathcal{N}(K_3) \\
     &\quad + ((1/3)\varphi_1(\mathcal{L}h/2) - (1/6)\varphi_1(\mathcal{L}h))\mathcal{N}(K_4)\} + \varphi_0(\mathcal{L}h)u_n.
\end{aligned}
$$

3.5. Integrating factor methods Integrating factor (IF) methods were first developed by Lawson [13]. They aim to reduce the stiffness of an equation via a change of variables such that explicit dependence on $\mathcal{L}$ in the transformed equation remains only inside the exponential. This equation is then solved by any numerical method (in this scheme it is CRK4), and the solution obtained is transformed back to the original variable. It is pertinent to note that the IF methods can be viewed as a subclass of exponential integrators (see [14] for discussion).

Since all IF methods deal with the fast time scale introduced into the nonlinear term, their error coefficient is relatively large [6]. Furthermore, unlike ETD methods, they do not preserve fixed points of the original equation (see [12]). Rather, we consider one of the generalized integrating factor (GIF) methods derived by Krogstad [12]. This method appeared in the work of Ostermann et al. [18] as a modification of one of Krogstad's methods with improved stability.

Let us introduce a vector $y^{[n-1]} = [y_1^{[n-1]}, y_2^{[n-1]}] = [u_{n-1}, h\mathcal{N}(u_{n-2})]$ which will be passed between successive levels. Then the scheme of interest here, known as a GIFRK4M, is a second type GIF method using RK4:

$$K_1 = y_1^{[n-1]},$$
$$\begin{aligned} K_2 &= h((1/2)\varphi_1(\mathcal{L}h/2) + (1/4)\varphi_2(\mathcal{L}h/2))\mathcal{N}(K_1) \\ &\quad + \varphi_0(\mathcal{L}h/2)y_1^{[n-1]} - (1/4)\varphi_2(\mathcal{L}h/2)y_2^{[n-1]}, \end{aligned}$$
$$\begin{aligned} K_3 &= h\{((1/2)\varphi_1(\mathcal{L}h/2) + (1/4)\varphi_2(\mathcal{L}h/2) - (3/4)I)\mathcal{N}(K_1) + (1/2)I\mathcal{N}(K_2)\} \\ &\quad + (\varphi_0(\mathcal{L}h/2)y_1^{[n-1]} + (-1/4)\varphi_2(\mathcal{L}h/2) + (1/4)I)y_2^{[n-1]}, \end{aligned}$$
$$\begin{aligned} K_4 &= h\{(\varphi_1(\mathcal{L}h) + \varphi_2(\mathcal{L}h) - (3/2)\varphi_0(\mathcal{L}h/2))\mathcal{N}(K_1) + \varphi_0(\mathcal{L}h/2)\mathcal{N}(K_3)\} \\ &\quad + \varphi_0(\mathcal{L}h)y_1^{[n-1]} + (-\varphi_2(\mathcal{L}h) + (1/2)\varphi_0(\mathcal{L}h/2))y_2^{[n-1]}, \end{aligned}$$
$$\begin{aligned} y_1^{[n]} &= h\{(\varphi_1(\mathcal{L}h) - 2\varphi_3(\mathcal{L}h) - (1/2)\varphi_0(\mathcal{L}h/2))\mathcal{N}(K_1) + (1/3)\varphi_0(\mathcal{L}h/2)\mathcal{N}(K_2) \\ &\quad + (1/3)\varphi_0(\mathcal{L}h/2)\mathcal{N}(K_3) + ((1/2)\varphi_2(\mathcal{L}h) + \varphi_3(\mathcal{L}h) - (1/4)\varphi_0(\mathcal{L}h/2))\mathcal{N}(K_4)\} \\ &\quad + \varphi_0(\mathcal{L}h)y_1^{[n-1]} + ((1/2)\varphi_2(\mathcal{L}h) + \varphi_3(\mathcal{L}h) - (1/4)\varphi_0(\mathcal{L}h/2))y_2^{[n-1]}, \end{aligned}$$
$$y_2^{[n]} = h\mathcal{N}(K_1).$$

An important property of this multilevel method is that if $\mathcal{L} = 0$, it reduces to CRK4. To start the method, we use one step of the CF4 method.

4. Comparison of methods

As we mentioned before, our system has Hamiltonian structure such that the energy is specified as

$$\begin{aligned} \mathcal{H} = \int_{-\infty}^{\infty} \Big\{ &\frac{1}{2}\delta_1 A^2 + \kappa AB + \frac{1}{2}\delta_2 B^2 - \frac{1}{2}\lambda_1(A_x)^2 - \sigma A_x B_x - \frac{1}{2}\lambda_2(B_x)^2 \\ &+ \frac{1}{3}\mu_1 A^3 + \nu_2 A^2 B + \nu_1 AB^2 + \frac{1}{3}\mu_2 B^3 \Big\} \, dx \end{aligned}$$

and momentum as

$$\mathcal{P} = \int_{-\infty}^{\infty} \{\alpha_1 A^2 + \alpha_2 B^2\} \, dx. \tag{4.1}$$

Both $\mathcal{H}$ and $\mathcal{P}$ are invariants of motion [9]. We use them to evaluate the numerical methods described in the previous section. First, we consider first two test problems such that their exact solutions are known (let us denote them as $u^* = [A^*, B^*]^T$); we calculate a numerical solution at a specified time instant (denoted by u) and then, using the L_2-norm, the relative error

$$\|u - u^*\|_{L_2}/\|u\|_{L_2}.$$

We also calculate the relative change of the invariants:

$$|\mathcal{P} - \mathcal{P}^*|/|\mathcal{P}|, \quad |\mathcal{H} - \mathcal{H}^*|/|\mathcal{H}|,$$

where $\mathcal{P}^*$ and $\mathcal{H}^*$ are the values of invariants at the initial time instant ($t = 0$), while $\mathcal{P}$ and $\mathcal{H}$ are their values at the end of the calculations. In the first two test problems we compare these changes with the solution error to justify using invariants in the case
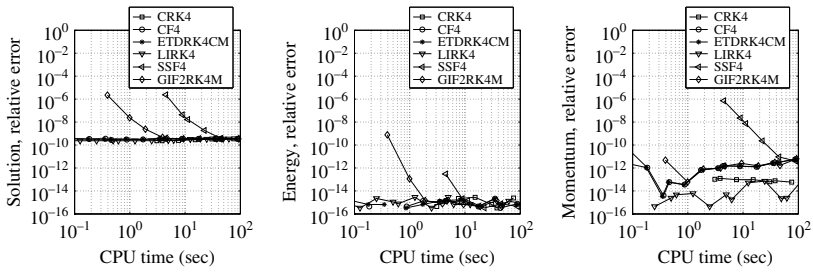
FIGURE 1. Relative error for solution and invariants for Problem 4.1.

where no exact solutions are available. Then we look at further test problems with interacting solitons where we can only compare the numerical methods using these integral invariants.

PROBLEM 4.1 (Linear coupling, exact stationary solution). The first example represents a simple situation when the equations of the system (1.2) are only coupled linearly. Consider (1.1) with the following coefficients:

$$\alpha_1 = \alpha_2 = 1, \quad \delta_1 = \delta_2 = 1.5, \quad \kappa = -1,$$
$$\lambda_1 = 0.1, \quad \lambda_2 = -0.2, \quad \mu_1 = 1, \quad \mu_2 = -1, \quad \sigma = \nu_1 = \nu_2 = 0.$$

Let $x \in [-24, 24], t \in [0, 20]$ and the number of points on the spatial interval be $N = 256$. This system has an exact stationary solution

$$A = 0.75 \; \text{sech}^2(x \sqrt{5}/2), \quad B = 1.5 \; \text{sech}^2(x \sqrt{5}/2).$$

Relative error and conservation of invariants for this problem are shown in Figure 1.

Interestingly, the relative error remains saturated, $O(10^{-10})$, for all tested time steps by all methods except SSF4 and GIF2RK4M. The latter demonstrated the expected fourth-order slope, and of the two, SSF4 produced the largest error for a given CPU time. While similar results were observed for the conservation of energy, we note that the saturation level of the error for the energy was much smaller, $O(10^{-15})$. As for the conservation of momentum, we note that SSF4 again exhibits fourth-order errors. The exponential integrator methods are all clustered together, which suggests that the saturation error is dominated by errors due to the calculation of the φ-functions. Figure 1 suggests that SSF4 and GIF2FRK4 are unable to preserve fixed points. For the first method this is expected, but for the latter it suggests that the claim that it preserves fixed points may only be valid for relatively small time steps. We also looked at the maximal stepsize supported by numerical methods for the current problem: CF4, ETDRK4CM and LIRK4 appeared unconditionally stable, while the maximal time step of SSF4, GIF2RK4M and CRK4 decreased as $O(N^{-3})$.

PROBLEM 4.2 (Nonlinear coupling, exact solution). We now consider an example of (1.1) with only nonlinear coupling. Given

$$\alpha_1 = \alpha_2 = 1, \delta_1 = \delta_2 = -1, \kappa = \sigma = 0, \lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 1, \nu_1 = \nu_2 = 0.25,$$
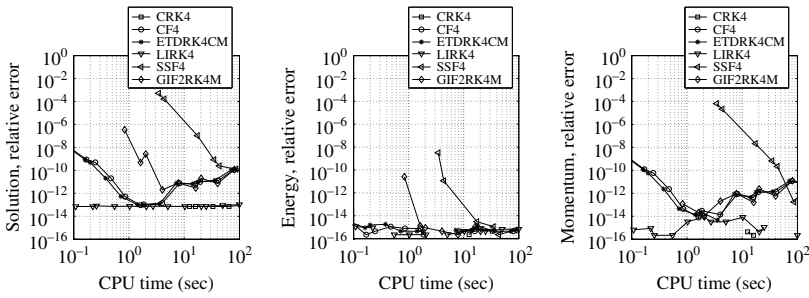
FIGURE 2. Relative error for solution and invariants for Problem 4.2.

the system has an exact stationary solution

$$A(x, t) = B(x, t) = \tfrac{6}{7} \operatorname{sech}^2(0.5x).$$

Let $x \in [-80, 80]$, $t \in [0, 60]$, $N = 512$. In Figure 2 we observe that the exponential integrator methods, ETDRK4CM and CF4, produced an increasing solution error for larger time steps (small CPU time) as well as for small time steps (large CPU time), which we would again attribute to the cancellation problem, inherent in calculation of φ-functions. This, together with a similar error behaviour for the conservation of momentum, indicates the existence of an optimal time step to employ in the calculation of φ-functions. As in Problem 4.1, the error of SSF4 and GIF2RK4M increases with stepsize, hence their ability to preserve fixed points is again doubtful.

PROBLEM 4.3 (Linear and nonlinear coupling). Consider an example of (1.1) discussed by Gear and Grimshaw [7], where

$$\alpha_1 = 21.513\,946, \quad \alpha_2 = 2.267\,029, \quad \delta_1 = 0, \quad \delta_2 = -1, \quad \kappa = 0, \quad \lambda_1 = 21.513\,946,$$
$$\lambda_2 = 1, \quad \sigma = 2.998\,816, \quad \mu_1 = 10.756\,973, \quad \mu_2 = 0.5, \quad \nu_1 = 0.275\,139,$$
$$\nu_2 = 7.989\,041,$$

and the initial conditions

$$A(x, 0) = \frac{1}{2} \operatorname{sech}^2\Big(\frac{\sqrt{6}}{12}(x + 50)\Big) + \frac{1}{4} \operatorname{sech}^2\Big(\frac{\sqrt{3}}{12}(x + 10)\Big),$$
$$B(x, 0) = \operatorname{sech}^2\Big(\frac{\sqrt{3}}{6}(x + 50)\Big) + \frac{1}{2} \operatorname{sech}^2\Big(\frac{\sqrt{3}}{12}(x + 10)\Big).$$

These are two-soliton profiles showing soliton-like interactions as they evolve. In particular, each of the two components of the profiles remains phase-locked with the corresponding component of the other profile throughout the simulation, and at a time point around 90, the components undergo a phase shift while preserving their identity. We assume that $-100 \leq x \leq 100$, $0 \leq t \leq 100$, and choose $N = 512$. Conservation of invariants for this problem (Figure 3) reveals a very slight advantage for GIF2RK4M, but this method, SSF4 and CRK4, which showed the best accuracy, were far less stable
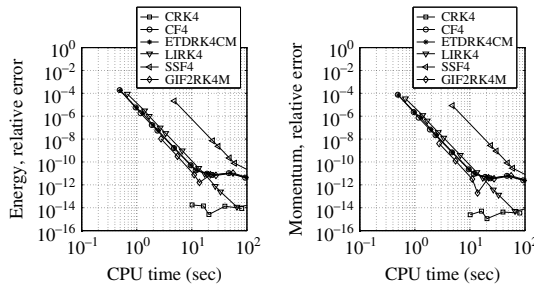
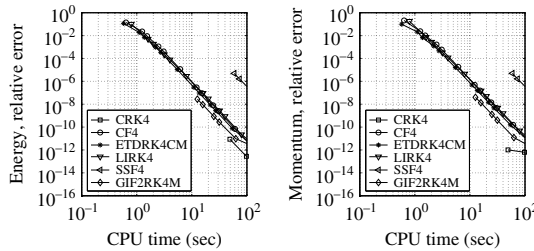FIGURE 3. Conservation of invariants for Problem 4.3.

FIGURE 4. Conservation of invariants for Problem 4.4.

than the other methods, and as mentioned before, the need for improved stability was the reason why methods different from CRK4 were investigated. In this case all the more stable methods performed almost equally well.

PROBLEM 4.4 (Repulsing solitons). For this problem [8], the coefficients of (1.1) are chosen as

$$\alpha_1 = 0.3, \quad \alpha_2 = -0.2, \quad \delta_1 = -0.675, \quad \delta_2 = -0.45, \quad \kappa = 0.06, \quad \lambda_1 = 0.3,$$
$$\lambda_2 = 0.2, \quad \mu_1 = 0.9, \quad \mu_2 = 0.6, \quad \sigma = \nu_1 = \nu_2 = 0,$$

and the initial conditions are

$$A(x, 0) = \text{sech}^2\left(\frac{1}{\sqrt{2}}(x - 5)\right), \quad B(x, 0) = \text{sech}^2\left(\frac{1}{\sqrt{2}}(x + 5)\right).$$

In this example, the two solitons collide and then repulse and then collide again—this process continues throughout the simulation. Here we take $-60 \le x \le 60$, $0 \le t \le 35$, $N = 512$.

As shown in Figure 4, the performance of the various methods seen in the first three problems is reinforced. GIF2RK4M and CRK4 are clearly the most accurate methods, albeit for significantly smaller time steps than can be achieved with CF4, ETDRK4CM and LIRK4, while SSF4 remains relatively inaccurate. Both Problems 4.3 and 4.4 suggest that in general there is a strong correlation between the conservation of the energy and momentum, which contrasts with the results for the solitary wave examples.

5. Conclusions

We have compared, using integral invariants, six numerical methods of broadly different types applied to the coupled KdV equations (1.1). The widely used CRK4 was able to calculate invariants very accurately in all test problems. However, this scheme normally worked only with relatively small time steps. We observed a disappointing performance by SSF4, which appeared to demonstrate only mediocre stability while being very resource-intensive. This is consistent with the results obtained by Kassam and Trefethen [10]. The accuracy and stability of ETDRK4CM, CF4 and LIRK4 were remarkably close in all test problems. Moreover, LIRK4 was on a par with the other two methods, and it did not require calculation of φ-functions, which is a potential advantage. Since these three methods preserved fixed points, they could potentially be applied to develop efficient schemes for the computation of solitary waves, similar to the Petviashvili or squared-operator methods [23]. The GIF2RK4M method consistently produced very small errors, compared to those by CRK4, with the ability to work with significantly larger time steps than CRK4, although not with time steps as large as those successfully handled by ETDRK4CM, LIRK4 and CF4. While GIF2RK4M was supposed to conserve fixed points [12], we observed in Problems 4.1 and 4.2 that its error increased for larger time steps.

We were also interested in determining whether the conservation of invariants could be a reliable indicator of the performance of a method. Based on our findings, we conclude that one can indeed evaluate the relative performance of numerical methods by looking at how they treat invariants. In Problems 4.1 and 4.2, the methods appeared to have an increased accuracy for conservation of energy. A similar situation was observed for ETDRK4CM by Klein [11]. On the other hand, the order is the same as for the solution error for conservation of momentum, which raises the question which invariant is preferable, since as we can see, their conservation is not necessarily identical. In view of this and the fact that due to the form of the expression for momentum (4.1), its conservation is assured provided the L_2-norm of both variables is conserved, we are inclined to favour conservation of energy, $\mathcal{H}$, because it is simply more informative. Monitoring this invariant should be useful for construction of adaptive-stepsize strategies (see, for example, [3]).

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