. Austral. Math. Soc. 25 (Series A) (1978), 35–40

A PROPERTY OF HEREDITARILY LOCALLY CONNECTED CONTINUA RELATED TO ARCWISE ACCESSIBILITY

JOSEPH N. SIMONE

(Received 21 December 1976; revised 1 July 1977)
Communicated by N. Smythe

Abstract

A continuum (that is, a compact connected Hausdorff space) is hereditarily locally connected if each of its subcontinua is locally connected. It is shown that a continuum X is hereditarily locally connected if and only if for each connected open set U in X and each point p in the boundary of U, $U \cup \{p\}$ is locally connected. This result is used to prove that if X is an hereditarily locally connected continuum, U is a connected open subset of X, p is an element of the boundary of U and X is first countable at p, then p is arcwise accessible from U.

Subject classification (Amer. Math. Soc. (MOS) 1970): 54 F 15, 54 D 05

By a *continuum* we mean a compact connected Hausdorff space. A continuum is *hereditarily locally connected* if each of its subcontinua is locally connected. A family of subsets of a space X is called a *null family* if for each two open sets U and V in X with $\bar{U} \cap \bar{V} = \varnothing$, not more than a finite number of elements of the family meet both U and V. If $\{F_\alpha \mid \alpha \in A\}$ is a family of disjoint subsets of X, then $\{F_\alpha \mid \alpha \in A\}$ is said to have *property D* if $F_\alpha \cap \operatorname{Cl} \bigcup \{F_\beta \mid \beta \neq \alpha\} = \varnothing$ for all α in A. Simone (to appear) has shown that a continuum X is hereditarily locally connected if and only if every family of disjoint continua in X with property D is a null family. This characterization of hereditarily locally connected continua will be used in the proofs that follow.

Our first theorem is a generalization of a well-known *metric* theorem due to Whyburn (1942, p. 90).

THEOREM 1. *If X is an hereditarily locally connected continuum' then every family of disjoint connected open subsets of X is a null family.*

PROOF. Since each two points in a connected open set U in a locally connected continuum are contained in a continuum contained in U (Hocking and Young, 1961, p. 110), it follows immediately from the above characterization of hereditarily locally connected continua that every family of disjoint connected open subsets of X is a null family.

If X is a space and $A \subseteq B \subseteq X$, then we use the notation ∂B to denote the boundary of B in X, and the notation $\partial_B A$ to denote the boundary of A in the subspace B. Before stating our main result, we prove the following lemma.

Lemma 2. *Let X be a connected regular space, and let U be a connected proper open subset of X. If for each point p in ∂U, $U \cup \{p\}$ is locally connected, then for each set A such that $U \subseteq A \subseteq \bar{U}$, A is locally connected.*

PROOF. Let $U \subseteq A \subseteq \bar{U}$, $p \in A$ and W an A-open set containing p. Suppose that $p \in U$. Since U is a proper subset of X and X is connected, there exists a point x in ∂U. Then $U \cup \{x\}$ is locally connected. Now $W \cap U$ is an X-open set containing p and contained in $U \cup \{x\}$. Therefore, there exists a connected $U \cup \{x\}$-open set B such that $p \in B \subseteq W \cap U$. However, clearly $B \subseteq U \subseteq A$, and hence B is A-open. Since $p \in B \subseteq W$, A is locally connected at p.

Next, suppose that $p \in \partial U$. By hypothesis $U \cup \{p\}$ is locally connected. Let $W^* = W \cap (U \cup \{p\})$. W^* is $U \cup \{p\}$-open and $p \in W^*$ and so there exists a connected $U \cup \{p\}$-open set V^* such that $p \in V^* \subseteq W^*$. Let W_1 be an open set such that $W = W_1 \cap A$ and let V_1 be a $\bar{U}$-open set such that $V^* = V_1 \cap (U \cup \{p\})$. Finally, let $V = V_1 \cap W_1$. Then V is $\bar{U}$-open and $V^* = V \cap (U \cup \{p\})$. Furthermore, it is clear that $p \in V \cap A \subseteq W$ and $V \cap A$ is A-open. We claim that $V \cap A$ is connected. Clearly $V^* \subseteq V \cap A$. We will show that $V \cap A \subseteq \overline{V^*}$. Suppose that $y \in V - \overline{V^*}$. Then $y \notin U$, and hence $y \in \partial U$. Now V is $\bar{U}$-open and $y \in V \cap (X - \overline{V^*})$. Since X is regular, there exists a $\bar{U}$-open set B such that $y \in B \subseteq V$ and $B \cap \overline{V^*} = \emptyset$. However, since $y \in \partial U$ it is clear that $U \cap B \neq \emptyset$. Let $z \in U \cap B$. Then $z \in B \subseteq V$ and $z \in U$, and therefore

$$z \in V \cap U \subseteq \overline{V^*}.$$

Hence, $z \in B \cap \overline{V^*}$ which is a contradiction. Therefore, $V \subseteq \overline{V^*}$, and hence $V \cap A \subseteq \overline{V^*}$. Since V^* is connected and $V^* \subseteq V \cap A \subseteq \overline{V^*}$, it follows that $V \cap A$ is connected. Therefore, A is locally connected at p, and hence A is locally connected.

THEOREM 3. *The continuum X is hereditarily locally connected if and only if for each connected open set U in X and each point p in ∂U, $U \cup \{p\}$ is locally connected.*

PROOF. Let X be hereditarily locally connected, and suppose that there exist a connected open set U in X and a point p in ∂U such that $U \cup \{p\}$ is not locally connected. Since $U \cup \{p\}$ is obviously connected im kleinen at each point of U, it follows that $U \cup \{p\}$ is not connected im kleinen at p. Let $U_1 = U \cup \{p\}$. There exists a U_1-open set W such that $p \in W$ and such that if $p \in B \subseteq W$ and B is U_1-open, then the component of some point of B in W does not contain p. Let V be a U_1-open set such that $p \in V$ and $\mathrm{Cl}_{U_1} V \subseteq W$, and let $\mathscr{U}$ denote the set of all U_1-open sets containing p and contained in V. For each B in $\mathscr{U}$, there exists a point x_B in

B such that the component K_B of x_B in W does not contain p. Now, for each B in $\mathscr{U}$, $p \notin K_B$ and hence K_B is a component of $W-\{p\}$. However, $W-\{p\}$ is X-open and X is locally connected so each K_B is open in X. Furthermore, since each K_B is a component of W, each K_B is W-closed.

Consider the net $\{K_B, B \in \mathscr{U}\}$, where $\mathscr{U}$ has been directed by reverse inclusion. Clearly, $p \in \liminf\{K_B, B \in \mathscr{U}\}$. We claim that $p \notin \overline{K_B}$ for each B in $\mathscr{U}$. For suppose that $p \in \overline{K_B}$ for some B in $\mathscr{U}$. Then $K_B \cup \{p\}$ is connected, contained in W and properly contains K_B which contradicts the fact that K_B is a component of W. Since $p \notin \overline{K_B}$ for each B in $\mathscr{U}$, it follows that $\{K_B \mid B \in \mathscr{U}\}$ is infinite. Suppose that $K_B \cap \partial_{U_1} V \neq \varnothing$ for each B in $\mathscr{U}$. V is U_1-open so there exists an open set V^* such that $V = V^* \cap U_1$. Now, it follows immediately that $\partial_{U_1} V \subseteq \partial V^*$ and therefore $K_B \cap \partial V^* \neq \varnothing$ for each B in $\mathscr{U}$. Hence, there exists a point y in

$$\limsup\{K_B, B \in \mathscr{U}\} \cap \partial V^*.$$

Let $\{K_{B_\alpha}, \alpha \in E\}$ be a convergent subnet of $\{K_B, B \in \mathscr{U}\}$ such that $y \in \lim\{K_{B_\alpha}, \alpha \in E\}$ (Frolík, 1960, p. 173). Clearly, $p \in \lim\{K_{B_\alpha}, \alpha \in E\}$. Recall now that V^* is an open set containing p. Let G_1 and G_2 be open sets containing p such that

$$p \in G_1 \subseteq \overline{G_1} \subseteq G_2 \subseteq \overline{G_2} \subseteq V^*.$$

Then G_1 and $X-\overline{G_2}$ are open sets and $\overline{G_1} \cap \overline{(X-\overline{G_2})} = \varnothing$. Furthermore, $p \in G_1$ and $y \in X-\overline{G_2}$ and hence there exists an α_0 in E such that $K_{B_\alpha} \cap G_1 \neq \varnothing$ and $K_{B_\alpha} \cap (X-\overline{G_2}) \neq \varnothing$ for all $\alpha \geqslant \alpha_0$. Since $p \notin \bar{K}_{B_\alpha}$ for all α in E, it follows, as before, that $\{K_{B_\alpha} \mid \alpha \geqslant \alpha_0\}$ is infinite. However, this contradicts Theorem 1.

Hence, there exists a B_0 in $\mathscr{U}$ such that $K_{B_0} \cap \partial_{U_1} V = \varnothing$. Since K_{B_0} is connected and $K_{B_0} \cap V \neq \varnothing$, it follows immediately that $K_{B_0} \subseteq \mathrm{Cl}_{U_1} V$. Furthermore, since $\mathrm{Cl}_{U_1} V \subseteq W$ we have that K_{B_0} is a component of $\mathrm{Cl}_{U_1} V$, and therefore, that K_{B_0} is closed in $\mathrm{Cl}_{U_1} V$. However, $\mathrm{Cl}_{U_1} V$ is closed in U_1, and hence K_{B_0} is closed in U_1. But K_{B_0} is X-open, and therefore K_{B_0} is open and closed in U_1 which contradicts the fact that U_1 is connected. We conclude that $U \cup \{p\}$ is locally connected.

Suppose, now, that for each connected open set U in X and each point p in ∂U, $U \cup \{p\}$ is locally connected. We claim that X is locally connected. For since X is a continuum, X contains a noncut point p. Then $X-\{p\}$ is a connected open set and $p \in \partial(X-\{p\})$, and hence $X = (X-\{p\}) \cup \{p\}$ is locally connected.

Assume that X is not hereditarily locally connected. Then there exist an infinite family $\{G_n\}_{n=1}^{\infty}$ of disjoint subcontinua of X with property D, and open sets W and V such that $\bar{W} \cap \bar{V} = \varnothing$ and $G_n \cap W \neq \varnothing$ and $G_n \cap V \neq \varnothing$ for all n. Since $\{G_n\}_{n=1}^{\infty}$ has property D, it follows immediately that there exists a sequence $\{U_n\}_{n=1}^{\infty}$ of disjoint open sets such that $G_n \subseteq U_n$ for each n. Now $\bar{W}$ is compact, and therefore there exists a point p in $\bar{W} \cap \limsup G_n$. Let W^* be a connected open set such that $p \in W^*$ and $\overline{W^*} \subseteq X - \bar{V}$. Since $U_n \cap U_m = \varnothing$ if $n \neq m$, it is clear that $p \notin G_n$ for each n, and therefore that infinitely many G_n meet W^*. Let $\{G_{n_i}\}_{i=1}^{\infty}$ be the subsequence of elements of $\{G_n\}_{n=1}^{\infty}$ which meet W^*. Now X is locally connected, so for each n there exists a connected open set V_n such that $G_n \subseteq V_n$ and $\overline{V_n} \subseteq U_n$. Let

$$U = W^* \cup \bigcup_{i=1}^{\infty} V_{n_i}.$$

U is a connected open set. Since $\bar{V}$ is compact and each G_{n_i} meets V, there exists a point q in $\bar{V} \cap \limsup G_{n_i}$. Clearly $q \notin \overline{W^*}$. Furthermore, for each $i \neq j$, $V_{n_i} \cap G_{n_j} = \varnothing$, and hence $q \notin \bigcup_{i=1}^{\infty} V_{n_i}$. Therefore, $q \notin U$. By Lemma 2 and hypothesis, it follows that $\bar{U}$ is a locally connected continuum.

Now, it is easily shown that

$$\mathrm{Cl}\left(\bigcup_{i=1}^{\infty} V_{n_i}\right) = \bigcup_{i=1}^{\infty} \bar{V}_n \cup \limsup \bar{V}_{n_i},$$

and hence

$$\bar{U} = \bigcup_{i=1}^{\infty} \bar{V}_{n_i} \cup \limsup \bar{V}_{n_i} \cup \overline{W^*}.$$

However, each $\overline{V_n}$ is contained in U_n and $\{U_n\}_{n=1}^{\infty}$ is a sequence of disjoint open sets, and hence

$$\bar{V}_n \cap \limsup \bar{V}_{n_i} = \varnothing$$

for each j.

Now $q \in \bar{U} - \overline{W^*}$ and $\bar{U}$ is locally connected, so there exists a connected $\bar{U}$-open set B such that $q \in B$ and $\bar{B} \subseteq \bar{U} - \overline{W^*}$. But then,

$$\bar{B} = (\bar{B} \cap \limsup \bar{V}_{n_i}) \cup \bigcup_{i=1}^{\infty} \bar{B} \cap \bar{V}_{n_i}.$$

However, $q \in \bar{B} \cap \limsup \bar{V}_{n_i}$ and since B is a $\bar{U}$-open set containing q, $\bar{B} \cap \bar{V}_{n_i} \neq \varnothing$ for infinitely many i. Therefore, $\bar{B}$ can be written as a nondegenerate countable union of disjoint compact sets which contradicts the fact that $\bar{B}$ is a continuum. Hence, X is hereditarily locally connected.

Tymchatyn (to appear) has recently obtained some characterizations of hereditarily locally connected continua related to the above theorem. By combining Lemma 2 and Theorem 3, we immediately obtain the following corollary which generalizes the equivalence of parts (1) and (3) of Theorem 2.2 of Nishiura and Tymchatyn (1976, p. 586) to nonmetric continua.

COROLLARY 4. *The continuum X is hereditarily locally connected if and only if for each connected open set U in X and each set A such that $U \subseteq A \subseteq \bar{U}$, A is locally connected.*

An *arc* is a continuum with exactly two noncut points. If K is an arc with noncut points p and q, then K is called an *arc from p to q*. Let X be a space, $A \subseteq X$ and

https://doi.org/10.1017/S144678870003891X Published online by Cambridge University Press

$p \in X - A$. The point p is said to be *arcwise accessible from* A if for each point x in A there exists an arc K from x to p such that $K \subseteq A \cup \{p\}$. Closely related to Theorem 3 is the following question.

QUESTION. *If X is an hereditary locally connected continuum and U is a connected open subset of X, then is each point p in ∂U arcwise accessible from U?*

Although this general problem remains open, by assuming that X is first countable at the point in question, we can answer the above problem in the affirmative for a large class of hereditarily locally connected continua.

THEOREM 5. *If X is an hereditarily locally connected continuum, U is a connected open subset of X, $p \in \partial U$ and X is first countable at p, then p is arcwise accessible from U.*

PROOF. Let $x \in U$ and let $U_1 = U \cup \{p\}$. By Theorem 3, U_1 is locally connected, and hence there exists a decreasing countable base $\{B_n\}_{n=1}^{\infty}$ at p of connected U_1-open sets. Since U_1 is locally connected and B_n is U_1-open, it follows that each B_n is locally connected.

Let $x_1 \in B_1 - \{p\}$. Now $x_1 \in U$ and hence there exists a continuum K_1 in U containing x and x_1. Let V_1 be the component of x_1 in $B_1 - \{p\}$. Since B_1 is connected and locally connected, it follows that $p \in \mathrm{Cl}_{B_1} V_1$ (Kuratowski, 1968, Theorem 19, p. 235). Thus $p \in \mathrm{Cl}_{U_1} V_1$. However, B_2 is a U_1-open set containing p and therefore $V_1 \cap B_2 \neq \emptyset$. Let $x_2 \in V_1 \cap B_2$. Now V_1 is open in U_1. However, $p \notin V_1$ and therefore $V_1 \subseteq U$. Hence, V_1 is a connected open set. Therefore, as before, there exists a continuum K_2 in V_1 containing x_1 and x_2. Let V_2 be the component of x_2 in $B_2 - \{p\}$. As above, $p \in \mathrm{Cl}_{U_1} V_2$ and hence there exists a point $x_3 \in V_2 \cap B_3$. Again, V_2 is a connected X-open set and therefore, there exists a continuum K_3 in V_2 containing x_2 and x_3.

Continue in this way by induction. Let $K^* = \mathrm{Cl} \bigcup_{n=1}^{\infty} K_n$. K^* is a continuum, and clearly x and p are in K^*. Furthermore, since for each n, $K_{n+1} \subseteq V_n \subseteq B_n$, it is obvious that $K^* \subseteq U \cup \{p\}$. Let K be an irreducible continuum in K^* from x to p. It is well known that every such irreducible continuum in an hereditarily locally connected continuum is an arc, and hence K is an arc from x to p. Since $K \subseteq U \cup \{p\}$, the proof is complete.

References

Z. Frolík (1960), "Concerning topological convergence of sets", *Czechoslovak Math. J.* **10** (85), 168–180.

J. G. Hocking and G. S. Young (1961), *Topology* (Addison–Wesley, Reading, Mass.).

K. Kuratowski (1968), *Topology II* (Academic Press, New York).

T. Nishiura and E. D. Tymchatyn (1976), "Hereditarily locally connected sets", *Houston J. Math.* **2**, 581–596.

J. N. Simone (to appear), "Concerning hereditarily locally connected continua", *Coll. Math.*
E. D. Tymchatyn (to appear), "Compactification of hereditarily locally connected spaces".
G. T. Whyburn (1942), *Analytic Topology* (Amer. Math. Soc., Providence, R.I.).

University of Missouri
Kansas City, Missouri 64110, USA

https://doi.org/10.1017/S144678870003891X Published online by Cambridge University Press