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NUCLEAR SPACES OF GENERALIZED TEST FUNCTIONS

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1. **Introduction.** It is well known that a large proportion of the locally convex spaces encountered in distribution theory are nuclear (Grothendieck [4], Treves [10], Schaeffer [8].) In [1] Beurling introduced spaces of test functions more general than those previously used. In this paper we shall show that many of these spaces, and resulting spaces of distributions, are also nuclear spaces.

We shall use the following notation and definitions. R denotes the real field, $\mathbb{C}$ the complex field. For any abstract space X, topology $\bar{G}$ on X, and "Carathéodory measure u on X [9]".

$$M_u = \{A \subset X : u(T) = u(T \cap A) + u(T - A) \text{ for all } T \subset X\}$$

u is a $\bar{G}$-Radón measure on X iff $\bar{G} \subset M_u$, for any $A \subset X$,

$$u(A) = \inf\{u(G) : A \subset G \in \bar{G}\},$$

for every closed compact $K \subset X$,

$$u(K) < \infty$$

and for every $G \in \bar{G}$,

$$u(G) = \sup\{u(K) : K \subset G \text{ is closed and compact}\}.$$

For any positive integer n, λ^n denotes Lebesgue measure on R^n. We put $\lambda = \lambda^1$. For any f in $L_1(R^n)$, the Fourier transform f^* of f is defined by

$$f^*(x) = \int (\exp -i\langle x, y\rangle) f(y)\, d\lambda^n(y), \qquad x \in R^n,$$

2. D_h**-spaces.** We shall define a class of spaces introduced by Beurling in [1].

2.1 Definitions. Let n be a positive integer.

(1) $\bar{M}$ is the set of all continuous real-valued functions h on R^n such that

(i) $0 = h(0) \le h(x+y) \le h(x) + h(y)$, $x \in R^n$, $y \in R^n$,

(ii) $\int h(x)/(1+|h|)^{n+1}\, d\lambda^n(x) < \infty$,

(iii) for some $a \in R$ and $b > 0$,

$$h(x) \ge a + b \ln(1+|x|) \quad \text{for all } x \in R^n.$$

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(2) For any $h \in \bar{M}$, $t \in R$, and $f \in L_1(R^n)$,

$$|f|_{h,t} = \int |f^*(x)| \exp th(x) \, d\lambda^n(x).$$

(3) For any h in $\bar{M}$ and compact $K \subset R^n$

$$D_h(K) = \{f \in L^1(R^n) \colon \text{support } f \subset K \text{ and } |f|_{h,t} < \infty \text{ for all } t > 0\}$$

endowed with the locally convex topology generated by the seminorms $|\cdot|_{h,t}$, $t > 0$.

(4) For any h in $\bar{M}$ and open $G \subset R^n$

$$D_h(G) = \{f \in L^1(R^n) \colon \text{support } f \subset G \text{ is compact and } |f|_{h,t} < \infty \text{ for all } t > 0\}$$

endowed with the strict inductive topology ([8, p. 57]) determined by the family of subspaces

$$\{D_h(K) \colon K \subset G \text{ is compact}\}.$$

(5) For any $h \in \bar{M}$, $t > 0$, and $f \in L_1(R^n)$,

$$\|f\|_{h,t} = \sup_{x \in R^n} |f^*(x)| \exp th(x).$$

2.2 Remarks. Let $K \subset R^n$ be compact and $G \subset R^n$ be open.

(1) For every f in $L_1(R^n)$, f^* is continuous ([7, p. 9]).

(2) $D_h(K)$ is a Fréchet space [2].

(3) For any sequence C of compact subsets of R^n with $C_k \subset$ interior C_{k+1} for each integer k, and $\bigcup C_k = G$, $D_h(G)$ is the strict (countable) inductive limit ([8, p. 57]) of the spaces $D_h(C_n)$, and is necessarily Hausdorff.

(4) For every compact $K \subset R^n$ with nonempty interior, condition (ii) of Definition 2.1.1 implies that $D_h(K) \neq \{0\}$ ([2, Theorem 1.3.7]).

(5) By Corollary I.3.21 of [2] and conditions (i)–(iii) of Definition 2.1,

$$f \in D_h(G) \Rightarrow f \text{ is infinitely differentiable.}$$

(6) From Corollary 1.4.3 of [2] we have the following. For any $t > 0$ there exist positive real numbers A, B, u, v, such that $t < u < v$ and for all f in $D_h(K)$

$$A\,|f|_{h,t} \leq \|f\|_{h,u} \leq B\,|f|_{h,v}$$

We shall use the following characterization of nuclear spaces which is due to Pietsch [5, 8].

2.3 Definition. Let X be a Hausdorff locally convex space. X is a nuclear space iff for each neighbourhood U of the origin in X there exists another neighbourhood V of the origin, and a w^*-Radón measure μ with support $\mu \subset V^0$ such that

$$\{x \in X \colon \int |x^*(x)| \, d\mu(x^*) \leq 1\} \subset U$$

where

X^* is the algebraic dual of X,

w^* is the topology on X^* of pointwise convergence on X,

V^0 is the polar of V i.e. $V^0=\{x^* \in X^*: |x^*(x)|\leq 1 \text{ for all } x \in V\}$.

We shall now prove the following.

2.4 THEOREM. *For any $h \in \bar{M}$ and compact $K\subset R^n$, $D_h(K)$ is a nuclear space.*

Proof. Let $t>0$. By Remark 2.2.6, choose positive A, B, u, v, such that $t<u<v$ and for all f in $D_h(K)$

$$A\,|f|_{h,t}\leq \|f\|_{h,u}\leq B\,|f|_{h,v}.$$

For each $x \in R^n$ let

$$l_x: f\in D_h(K)\to f^*(x)\exp uh(x) \in R.$$

Let

$$\mu_0: A\subset R^n \to \int^* 1_A \exp(-(u-t)h(x))\,d\lambda^n(x),$$

where $\int^*$ denotes the outer integral ([9, Ch. 7]), and

$$1_A: x\in R^n \to \begin{cases} 1 & \text{if } x\in A,\\ 0 & \text{if } x\notin A.\end{cases}$$

For any w^*-open $G\subset (D_h(K))^*$ let

$$g(G)=\mu_0(l^{-1}[G])$$

and for any $A\subset (D_h(K))^*$,

$$\mu(A)=\inf\{g(G): A\subset G,\ G\subset (D_h(K))^* \text{ is } w^*\text{-open}\}.$$

We have that

$$\sup\{|l_x(f)|: |f|_{h,v}\leq 1\}\leq B, \qquad \text{for all } x\in R^n$$

and therefore

(1) $$\text{range } l\subset B\,\{f\in D_h(K): |f|_{h,v}\leq 1\}^0.$$

(2) Since f^* is continuous for every $f\in D_h(K)$, l is continuous with respect to the w^*-topology on $(D_h(K))^*$.

(3) $\mu_0(R^n)<\infty$; for there exists $a\in R$, $b>0$, such that

$$h(x)\geq a+b\ln(1+|x|),\ x\in R^n,$$

and therefore for every $x\in R^n$

$$\exp[-(u-t)h(x)]\leq \exp[-(u-t)(a+b\ln(1+|x|))]$$
$$=(\exp -a(u-t))(1/(1+|x|))^{b(u-t)},$$

from which one concludes that

$$\mu_0(R^n) < (\exp - a(u-t)) \int (1/(1+|x|)^{b(u-t)}\, d\lambda^n(x) < \infty,$$

using spherical coordinates ([3, p. 321]), and choosing u large enough.

Using standard measure theoretic techniques and (3) one may now show that μ_0 is Radón.

Using (2) and (3) one can then show that μ is a w^*-Radón measure, and further, by (1),

$$\text{support } \mu \subset B\,\{f \in D_h(K) : |f|_{h,v} \leq 1\}^0.$$

Then, for every $f \in D_h(K)$,

$$\begin{aligned} |f|_{h,t} &= \int |f^*(x)| \exp th(x)\, d\lambda^n(x) \\ &= \int |f^*(x)|\, (\exp uh(x))(\exp -(u-t)h(x))\, d\lambda^n(x) \\ &= \int |x^*(x)|\, d\mu(x^*). \end{aligned}$$

Since t was arbitrary it follows that $D_h(K)$ is nuclear.

Taking Theorem 2.4 together with known properties of nuclear spaces (as in [8]) we can deduce the following results.

NOTATION. For any topological vector space X, X'_b denotes the continuous dual of X endowed with the topology of uniform convergence on the bounded subsets of X.

2.5 THEOREMS. *Let $h \in \bar{M}$.*

(1) *For every open $G \subset R^n$, $D_h(G)$ is nuclear.*

(2) *For every compact $K \subset R^n$, $(D_h(K))'_b$ is nuclear.*

(3) *For every open $G \subset R^n$, $(D_h(G))'_b$ is nuclear.*

Proof of 2.5. By Remark 2.2.3, Theorem 2.4, and the fact that a countable inductive limit of nuclear spaces is nuclear ([8, p. 103]).

Proof of 2.5.2. By Remark 2.2.2 and Theorem 2.4, since the strong dual of a nuclear Fréchet space is nuclear ([8, p. 172, Theorem 9.6]).

Proof of 2.5.3. Let C be a sequence of compact subsets of R^n with $C_k \subset$ interior C_{k+1} for each integer k, and $\bigcup C_k = G$. For each k let $\bar{B}_k$ be the family of bounded subsets of $D_h(C_k)$ and let $\bar{B}_\infty$ be the family of bounded subsets of $D_h(G)$. Since $D_h(G)$ is a strict inductive limit of spaces $D_h(C_k)$, and for each k, $D_h(C_k)$ is a closed subspace of $D_h(C_{k+1})$, it follows that

$$\bar{B}_\infty = \bigcup \bar{B}_k.$$

Hence, noting that $\bar{B}_\infty$ is closed under finite unions, we have that $(D_h(G))'_b$ is the projective limit of the spaces $(D_h(C_k))'_b$ by the maps u_k^t, where u_k^t is the transpose of the canonical injection $u_k: D_h(C_k) \to D_h(G)$ ([8, p. 85, Proposition 15]). Since the spaces $(D_h(C_k))'_b$ are nuclear (Theorem 2.5.2) and the projective limit of nuclear spaces is again nuclear ([8, p. 103]) it now follows that $(D_h(G))'_b$ is nuclear.

2.6 Remark. We point out that the spaces $D_h(K)$, $D_h(G)$, being complete and barrelled (2.2.2, 2.2.3, [8, pp. 59, 61]) nuclear spaces, are necessarily Montel spaces ([10, pp. 356, 520]) and therefore so also are their strong duals ([10, p. 376]). In particular $D_h(K)$ and $D_k(G)$ are reflexive.

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