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An embedding theorem for ordered groups

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We show that if the normal closure of an element a, of an orderable group, G, is abelian, then G can be embedded in an orderable group, $G^{\#}$, which contains an n-th root of a for every positive integer, n. Furthermore, every order of G extends to an order of $G^{\#}$.

1. Preliminaries

1.1. A *partially ordered group* is a group, G, which is a partially ordered set under some partial order relation, $\leq$, the group operation and order relation being compatible in the sense that $g \leq h$ implies $agb \leq ahb$ for a, b, g and h in G. If, in addition, $(G, \leq)$ is a fully ordered set, then $(G, \leq)$ is a *fully ordered group* (*o-group*). A group, G, is an *orderable group* (*O-group*) if G can be made a fully ordered group. For details of the theory of ordered groups, the reader is referred to Fuchs [5] or Kokorin and Kopytov [8].

Throughout this paper, an *ordered group* will always be a fully ordered group and an *order of a group* will always be a full order. All identities of groups will be written, 1, and generally no notational distinction will be made between orders of different groups. N will denote the set of all strictly positive integers.

If two ordered groups, G and H, are isomorphic and the

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isomorphism, ϕ , satisfies $g > 1$ implies $g\phi > 1$ for all g in G , then ϕ is an *o-isomorphism*. So *o-automorphism* has the obvious meaning. If a group, G , can be embedded in an O-group, H , in such a way that every order of G extends to an order of H , then the embedding is called an *o*-embedding*.

A subgroup, B , of a group, G , is *isolated* if, for g in G and n in N , g^n belongs to B implies g belongs to B . The *isolated closure* of a subgroup, A , of G is the intersection of all isolated subgroups of G containing A and will be denoted by $I_G(A)$ (or, simply, $I(A)$ if no confusion arises). G is *divisible* if, for all g in G and n in N , the equation $x^n = g$ has a solution in G . A minimal, divisible extension of a group, G , is called a *completion of* G .

1.2. Every abelian O-group has a completion which is an abelian O-group (see [5], p. 36). In fact, every locally nilpotent O-group has a unique (up to o-isomorphism) locally nilpotent, orderable completion (see Mal'cev [10] and [11], and Kokorin and Kopytov [8], p. 58). More recently, Bludov and Medvedev [1] have shown that every metabelian O-group has a metabelian orderable completion. However, this completion is not, in general, unique (see [4]). In view of [1], we can generalize slightly a theorem of Minassian [12] and say that if an O-group, G , has a normal series $G > G_1 > G_2 > \ldots$ such that $\bigcap_{i=1}^{\infty} G_i = \{1\}$ and G/G_i , $i = 1, 2, \ldots$, is a locally nilpotent O-group or a metabelian O-group, then G has an orderable completion.

No more appears to be known at present about completing O-groups in one fell swoop, so to speak. In this paper, we show (§3, Corollary 1) that roots can be adjoined to certain elements of an arbitrary O-group, thereby partially answering a question of Neumann (see [5], p. 211, Problem 16). Namely, those elements contained in a normal abelian subgroup of the group. §3, Theorem 3, generalizes results of Conrad ([3], Theorem 3) and Kopytov [9]. In fact, the method used in §2 is almost identical to that employed by Kopytov [9]. (His theorem appears also in Fuchs [6], p. 83.) In §4, we present some properties of the embedding of Theorem 3.

1.3. We mention a result concerning the abelian completion of an

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abelian O-group.

LEMMA 1. *Let* $(A, \leq)$ *be an abelian o-group and let* $A^{\#}$ *be its abelian completion. Then* $\leq$ *extends uniquely to an order,* $\leq^{\#}$ *, of* $A^{\#}$ *and any o-automorphism,* ϕ *, of* $(A, \leq)$ *extends uniquely to an o-automorphism,* $\phi^{\#}$ *, of* $(A^{\#}, \leq^{\#})$.

We omit the proof as this lemma is virtually a restatement of a lemma of Conrad ([3], p. 518).

2. Completing a normal, abelian subgroup of an O-group

We begin with a definition. By *completing a subgroup,* U *, of a group,* V , we mean that V can be embedded in a group, W , in such a way that W contains a completion of (the image under the embedding of) U .

Suppose G is an O-group with normal, abelian subgroup, A . We wish to complete A and, for the moment, suppose that A is isolated.

For all g in G , denote by ϕ_g the restriction to A of the inner automorphism of G induced by g . (That is, $a\phi_g = g^{-1}ag$.) Let $A^{\#}$ be the abelian completion of A and for all a in $A^{\#}$, let $m(a)$ be a positive integer such that $a^{m(a)}$ is in A . Define $\phi_g^{\#} : A^{\#} \to A^{\#}$ by

$$a\phi_g^{\#} = \left(a^m\phi_g\right)^{1/m} \quad \text{where } m = m(a) . \tag{2.1}$$

$\phi_g^{\#}$ is the unique extension of ϕ_g to $A^{\#}$. We emphasize that this definition is independent of the choice of m in N such that a^m is in A .

We have the following:-

LEMMA 2. *(i) For all* g *and* h *in* G *,* $\phi_{gh}^{\#} = \phi_g^{\#}\phi_h^{\#}$.

(ii) For all a *in* A *,* $\phi_a^{\#} = 1$.

Proof. *(i)* Conrad proves this in his proof of Theorem 3.1 ([3], p. 519, lines 11-12).

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(ii) For all a in A, $\phi_a = 1$; so $b\phi_a^{\#} = \left(b^m\phi_a\right)^{1/m} = b$ for all b in $A^{\#}$ (where $m = m(b)$). //

Now we are ready to complete A. Let $G^{\#}$ be the (set theoretic) cartesian product, $G \times A^{\#}$, modulo the equivalence

(2.2) $(g, a) = (h, b)$ iff $h = gc$ and $b = c^{-1}a$ for some c in A.

It is easy to show that (2.2) *does* define an equivalence relation on $G \times A^{\#}$.

Define multiplication in $G^{\#}$ by

$$(g, a)(h, b) = \left(gh, \left(a\phi_h^{\#}\right)b\right). \tag{2.3}$$

To show that this definition is independent of the choice of g and h in G and a and b in $A^{\#}$, take any c and d in A. Then

$$\begin{aligned}
\left(gc, c^{-1}a\right)\left(hd, d^{-1}b\right) &= \left(gchd, \left(c^{-1}a\right)\phi_{hd}^{\#}d^{-1}b\right) && \text{(by (2.3))} \\
&= \left(gh\left(h^{-1}ch\right)d, \left(c^{-1}\phi_h\right)\left(a\phi_h^{\#}\right)d^{-1}b\right) && \text{(by Lemma 2)} \\
&= \left(ghd\left(c\phi_h\right), \left(c\phi_h\right)^{-1}d^{-1}\left(a\phi_h^{\#}\right)b\right) \\
& && \left(A^{\#} \text{ is abelian and } A \text{ is normal in } G\right) \\
&= \left(gh, \left(a\phi_h^{\#}\right)d\right) && \text{(by (2.2))} \\
&= (g, a)(h, d) && \text{(by (2.3))}.
\end{aligned}$$

So the definition of multiplication is satisfactory.

Associativity can be verified directly, $(1, 1)$ is an identity for $G^{\#}$ and an inverse of (g, a) is $\left(g^{-1}, a^{-1}\phi^{\#}_{g^{-1}}\right)$. So $G^{\#}$ is a group.

The map $g \mapsto (g, 1)$ embeds G in $G^{\#}$ and, since $(a, 1) = (1, a)$ for all a in A and since, for b in $A^{\#}$, the map $b \mapsto (1, b)$ embeds $A^{\#}$ in $G^{\#}$, we have the following:-

THEOREM 1. *The embedding of* G *into* $G^{\#}$ *given above completes the*

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normal, abelian, isolated subgroup, A *, of* G *. Furthermore,* $G^{\#}/A^{\#}$ *is isomorphic to* G/A *.*

Proof. It remains to prove the latter statement. $A^{\#}$ is normal in $G^{\#}$ because, for all (g, a) and $(1, b)$ in $G^{\#}$,

$$(g, a)^{-1}(1, b)(g, a) = \left(1, b\phi_g^{\#}\right) .$$

We merely observe that the obvious mapping, $(g, a)A^{\#} \mapsto gA$, is the required isomorphism of $G^{\#}/A^{\#}$ onto G/A . //

In order to discard the supposition that A is isolated, we need the following:-

LEMMA 3. *Let* A *be an abelian subgroup of the O-group,* G *. Then the isolated closure,* $I(A)$ *, of* A *in* G *is an abelian subgroup of* G *. If, in addition,* A *is normal, then* $I(A)$ *is normal.*

Proof. Let $B = \{g \in G : g^m \in A$ for some m in $N\}$. We show that B is an abelian subgroup of G and that $B = I(A)$. Take g and h in B and let m and n belong to N such that g^m and h^n are in A . Then $[g^m, h^n] = 1$, and so $[g, h] = 1$ (see [5], p. 38). Hence, $(gh^{-1})^{mn} = g^{mn}h^{-nm}$ which belongs to A . So gh^{-1} is in B and we have shown that B is an abelian subgroup of G .

To show that $B = I(A)$, take g in G such that g^n is in B for some n in N . Then there is m in N such that $g^{mn} = (g^n)^m$ is in A ; so g is in B . That is B is an isolated subgroup of G and, since $A \le B$, it follows that $I(A) \le B$. For all g in $G\backslash I(A)$, g^n is in $G\backslash I(A)$ and, hence, in $G\backslash A$ for all n in N (because $I(A)$ is isolated and $A \le I(A)$); so g is in $G\backslash B$ and it follows that $B \le I(A)$.

Now suppose A is normal. We show that B is normal. Take b in B and g in G , and let b^m belong to A for m in N . Then $(g^{-1}bg)^m = g^{-1}b^mg$ is in A – so $g^{-1}bg$ is in B and, hence, B is normal. //

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We are now in a position to prove:-

THEOREM 2. *If G is an O-group with normal, abelian subgroup, A, then A can be completed.*

Proof. Take the abelian completion, $I(A)^{\#}$, of $I(A)$ and let $G^{\#}$ be $G \times I(A)^{\#}$ modulo the appropriate equivalence (*cf.* (2.2)) and with the appropriate multiplication (*cf.* (2.3)). Then by Theorem 1, the embedding G into $G^{\#}$ completes $I(A)$ and, hence, completes A. //

[Observe that if A is not isolated, then (in view of our future requirements) $I(A)$ must be used in the construction of $G^{\#}$. Otherwise, there would be g in $G \backslash A$, a in $A^{\#} \backslash A$ and m in N such that $g^m = a^m$ is in A. That is, $(g, 1)^m = (1, a)^m$ while $(g, 1) \neq (1, a)$, an impossible situation in an O-group (see [5], p. 37, Proposition 9). Since we want to be able to order $G^{\#}$, such obvious hindrances must be removed.]

3. An order for $G^{\#}$

Take an O-group, G, with normal, abelian subgroup, A. By Lemma 3, we suffer no loss of generality by supposing that A is isolated, so embed G in $G^{\#}$ to complete A as in §2. Now take any order, $\leq$, of G. For (g, a) in $G^{\#}$, define

(3.1) $(g, a) > 1$ if, and only if, $g^m a^m > 1$ in G, where $m = m(a)$.

We must show that this definition is satisfactory in two senses. First, we must show that if a^m and a^n are in A, then $g^m a^m > 1$ implies $g^n a^n > 1$, and, second, that if $(g, a) > 1$, then $(gb, b^{-1}a) > 1$ for all b in A. (There is, of course, another sense in which this definition has to be satisfactory. Namely, that (3.1) makes $G^{\#}$ an o-group. This we show in due course.) We need the following:-

LEMMA 4. *Let $(V, \leq)$ be a partially ordered group. Take v and w in V and s in N. Then*

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(i) $vw \le wv$ *implies* $v^s w^s \le (vw)^s \le w^s v^s$,

(ii) $vw > 1$ *implies* $v^s w^s > 1$, *and*

(iii) *if* $(V, \le)$ *is fully ordered, then* $v^s w^s > 1$ *implies* $vw > 1$.

Proof. Straightforward induction proofs give *(i)* and *(ii)* while *(iii)* follows from *(i)*. (Compare *(i)* with Chehata [2], Lemma 9.) In fact, *(iii)* provides justification for trying a definition like (3.1). //

Now take a in $A^\#$, g in G and m , n in N such that a^m, a^n are in A . A standard euclidean algorithm argument shows that m and n are multiples of k in N , where k is the smallest positive integer such that a^k is in A . Let $m = rk$ and $n = sk$ where r, s are in N . Then

$$\begin{aligned} g^m a^m > 1 &\Rightarrow g^{rk} a^{rk} > 1 \\ &\Rightarrow g^k a^k > 1 \quad \text{(by Lemma 4 } (iii)\text{)} \\ &\Rightarrow g^{sk} a^{sk} > 1 \quad \text{(by Lemma 4 } (ii)\text{)} \\ &\Rightarrow g^n a^n > 1 . \end{aligned}$$

Now take $(g, a) > 1$ in $G^\#$ and any b in A . By (3.1), $(gb, b^{-1}a) > 1$ if, and only if, $(gb)^m (b^{-1}a)^m > 1$ in G where $m = m(b^{-1}a)$. Since b is in A , we may choose $m(b^{-1}a) = m(a)$. So, we must show that $(gb)^m b^{-m} a^m > 1$ (equivalently $b^{-m} a^m (gb)^m > 1$), knowing that $g^m a^m > 1$ (equivalently $a^m g^m > 1$), where $m = m(a) = m(b^{-1}a)$. Suppose $gb \ge bg$ in G . Then $(gb)^m \ge b^m g^m$ (Lemma 4 *(i)*). So,

$$\begin{aligned} b^{-m} a^m (gb)^m &\ge b^{-m} a^m b^m g^m \\ &= a^m g^m \quad (A \text{ abelian}) \\ &> 1 . \end{aligned}$$

Similarly, if $gb < bg$ in G , then $(gb)^m b^{-m} a^m > 1$. Hence, definition (3.1) makes sense. To show that (3.1) makes $G^\#$ an o-group, we need the following:-

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LEMMA 5. *For* (g, a) *in* $G^{\#}$, $(g, a)^{-1} > 1$ *if, and only if,* $g^m a^m < 1$ *in* G, *where* $m = m(a)$.

Proof. Observe that $\left(a\phi^{\#}_{g}\right)^m = a^m\phi_g$ is in A. So, we can always choose $m\left(a\phi^{\#}_{g}\right) = m(a)$ (and this choice we shall make in the following argument). Since $(g, a)^{-1} = \left(g^{-1}, a^{-1}\phi^{\#}_{g^{-1}}\right)$, it follows (by (3.1)) that $(g, a)^{-1} > 1$ if, and only if, $g^{-m}\left(a^{-1}\phi^{\#}_{g^{-1}}\right)^m > 1$ in G. Now $g^{-m}\left(a^{-1}\phi^{\#}_{g^{-1}}\right)^m = g^{-m}ga^{-m}g^{-1} = g^{-(m-1)}\left(g^m a^m\right)^{-1}g^{m-1}$; whence, $(g, a)^{-1} > 1$ if, and only if, $\left(g^m a^m\right)^{-1} > 1$ in G and the result follows. //

To show that $\left(G^{\#}, \leq\right)$ is an o-group, we show (*cf.* [5], p. 13) that for all x and y in $G^{\#}$:-

(i) $x > 1$ implies $x^{-1} \ngtr 1$,

(ii) $x \neq 1$ implies $x > 1$ or $x^{-1} > 1$,

(iii) $x > 1$ and $y > 1$ implies $xy > 1$, and

(iv) $x > 1$ implies $y^{-1}xy > 1$.

(i) Take $(g, a) > 1$ with $m = m(a)$. So $g^m a^m > 1$ in G ; hence $g^m a^m \nless 1$, and so $(g, a)^{-1} \ngtr 1$ (by Lemma 5).

(ii) Take any $(g, a) \neq 1$ in $G^{\#}$ with $m = m(a)$. If $g^m a^m > 1$ in G, then $(g, a) > 1$. If $g^m a^m < 1$ in G, then $(g, a)^{-1} > 1$ (Lemma 5). Now $g^m a^m \neq 1$ by the following argument:-

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$$g^m a^m = 1 \Rightarrow g^m = a^{-m}$$
$$\Rightarrow g^m \in A$$
$$\Rightarrow g \in A \qquad (A \text{ isolated in } G)$$
$$\Rightarrow g = a^{-1} \qquad ([5], \text{ p. } 57)$$
$$\Rightarrow (g, a) = 1 \quad \left(\text{by } (2.2)\right).$$

(iii) Take $(g, a) > 1$ and $(h, b) > 1$ in $G^{\#}$. Since $a^{m(a)m(b)}$ and $b^{m(b)m(a)}$ are in A, we may choose (and shall choose in the following argument) $m = m(a) = m(b)$. By definitions (2.3) and (3.1), we have $(g, a)(h, b) > 1$ if, and only if, $(gh)^m\left(h^{-1}a^m h\right)b^m > 1$ in G.

Suppose $gh \geq hg$. Then, transforming each side by h, we have $\left(h^{-1}gh\right)h \geq h\left(h^{-1}gh\right)$. So,

$$(gh)^m = \left(h\left(h^{-1}gh\right)\right)^m \geq h^m\left(h^{-1}gh\right)^m = h^m\left(h^{-1}g^m h\right).$$

Hence

$$(gh)^m\left(h^{-1}a^m h\right)b^m \geq h^m\left(h^{-1}g^m h\right)\left(h^{-1}a^m h\right)b^m$$
$$= b^{-m}\left[\left(b^m h^m\right)h^{-1}\left(g^m a^m\right)h\right]b^m$$
$$> 1 \quad \left(\text{because } b^m h^m > 1 \text{ and } g^m a^m > 1 \text{ in } G\right).$$

Similarly, if $gh < hg$, we have

$$(gh)^m\left(h^{-1}a^m h\right)b^m > \left(h^{-1}g^m h\right)h^m b^m\left(h^{-1}a^m h\right)$$
$$= h^{-1}a^{-m}h\left[h^{-1}\left(a^m g^m\right)h\left(h^m b^m\right)\right]h^{-1}a^m h$$
$$> 1 .$$

(iv) Take $(g, a) > 1$ and (h, b) in $G^{\#}$. Since $(h, b) = (h, 1)(1, b)$ and since $(uv)^{-1}w(uv) = v^{-1}\left(u^{-1}wu\right)v$ is an identity for groups, we may show, separately, that $(h, 1)^{-1}(g, a)(h, 1) > 1$ and $(1, b)^{-1}(g, a)(1, b) > 1$. Now $(h, 1)^{-1}(g, a)(h, 1) = \left(h^{-1}gh, a\phi_h^{\#}\right)$. Setting $m = m(a) = m\left(a\phi_h^{\#}\right)$, we have $\left(h^{-1}gh, a\phi_h^{\#}\right) > 1$ if, and only if, $h^{-1}g^m h\left(a^m\phi_h\right) > 1$ in G, where $h^{-1}g^m h\left(a^m\phi_h\right) = h^{-1}\left(g^m a^m\right)h > 1$ in G. So

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$(h, 1)^{-1}(g, a)(h, 1) > 1$. Now $(1, b)^{-1}(g, a)(1, b) = \left(g, \left(b\phi_g^{\#}\right)^{-1}ab\right)$.

Setting $m = m(a) = m(b)$, we have $\left(g, \left(b\phi_g^{\#}\right)^{-1}ab\right) > 1$ if, and only if,

$g^m\left(g^{-1}b^{-m}g\right)a^mb^m > 1$ in G , the latter being equivalent to

$a^mg^m[g, b^m] > 1$ in G . If $[g, b^m] \geq 1$ in G , then

$a^mg^m[g, b^m] \geq a^mg^m > 1$. Suppose $[g, b^m] < 1$ in G . Now

$$[g, b^m] < 1 \Rightarrow b^{-m}gb^m < g$$
$$\Rightarrow b^{-m}g^{m-1}b^m = \left(b^{-m}gb^m\right)^{m-1} \leq g^{m-1}$$

(only if $m = 1$ does equality hold).

So,

$$g^ma^m > 1 \Rightarrow g^m > a^{-m}$$
$$\Rightarrow g^{m-1} > a^{-m}g^{-1}$$
$$\Rightarrow b^{-m}g^{m-1}b^m > b^{-m}\left(a^{-m}g^{-1}\right)b^m$$
$$\Rightarrow g^{m-1} > b^{-m}\left(a^{-m}b^{-1}\right)b^m \quad \left(\text{since } g^{m-1} \geq b^{-m}g^{m-1}b^m\right)$$
$$\Rightarrow g^m > a^{-m}[b^m, g]$$
$$\Rightarrow a^mg^m[g, b^m] > 1 .$$

So, $(1, b)^{-1}(g, a)(1, b) > 1$ and, hence, $(h, b)^{-1}(g, a)(h, b) > 1$.

So, $\left(G^{\#}, \leq\right)$ is an o-group. Since the order of $G^{\#}$ extends that of G (that is, $(g, 1) > 1$ in $G^{\#}$ if, and only if, $g > 1$ in G), we have:-

THEOREM 3. *Let G be an O-group with normal, abelian subgroup, A . Then A can be completed by o^*-embedding G in an O-group, $G^{\#}$. If $A^{\#}$ is the completion of (the image under the embedding of) A , then $G^{\#}/A^{\#}$ is isomorphic to G/A .*

As a corollary, we have the result mentioned in §1.2.

COROLLARY 1. *Let G be an O-group, let a be an element of G , and let n be in N . If $\{a\}^G$ (the normal closure of $\{a\}$ in G) is*

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abelian, then G *can be* o^**-embedded in an* O*-group,* H *, in which there is a solution to the equation* $x^n = a$.

Observe that $\{a\}^G$ is abelian if, and only if, $[g, a, a] = 1$ for all g in G . (Here $[g, a, a]$ is the commutator $[[g, a], a]$ where $[g, a] = [g, 1a] = g^{-1}a^{-1}ga$. More generally, $[g, ka] = [[g, (k-1)a], a]$ for all $k \geq 2$ in N .) So Corollary 1 can be rephrased as:-

COROLLARY 1'. *Let* G, a *and* n *be as in Corollary* 1. *If* $[g, a, a] = 1$ *for all* g *in* G *, then* G *can be* o^**-embedded in an* O*-group,* H *, in which there is a solution to the equation* $x^n = a$.

Corollaries 1 and 1' suggest the questions:-

(1) What happens if the normal closure of a is

(i) (locally) nilpotent? or

(ii) metabelian?

(2) What happens if, for some $k > 2$ in N , $[g, ka] = 1$ for all g in G ?

I suspect that the answer to (1) (i) (effectively a question of Kokorin - see [7], Question 1.61) will be a theorem similar to Corollary 1, while the situations described in (1) (ii) and (2) seem less straightforward.

4. Some properties of the embedding $G \to G^\#$

4.1. We begin by showing that our method of completing a normal, abelian subgroup of an O-group is, essentially, the only way.

THEOREM 4. *Let* G *be an* O*-group with normal, abelian, isolated subgroup* A . *Then there is an* O*-group,* H *, which*

(i) completes A *and*

(ii) is generated by G *and* $I_H(A)$.

Any O*-group,* K *, satisfying (i) and (ii) is isomorphic to* H *, the restriction of the isomorphism to* G *being the identity. Furthermore, given any order of* K *, the isomorphism can be made an* o*-isomorphism in a natural manner.*

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Before proving this theorem, we mention that, in view of a result of Smirnov [13], our Theorem 4 is stronger than the similar theorem of Conrad ([3], Theorem 3). Smirnov shows that a maximal, normal, abelian subgroup of an O-group, V, need not be convex under any order of V.

Proof of Theorem 4. Clearly, $G^\#$ (as constructed in §2) satisfies *(i)* and *(ii)*. Let $H = G^\#$ and write elements of H as formal products, ga, with g in G and a in $I_H(A)$ (subject, of course, to an equivalence similar to (2.2)). Let K be any O-group satisfying *(i)* and *(ii)*, and, similarly, write elements of K in the form, gb, with g in G and b in $I_K(A)$. Since $I_H(A)$ and $I_K(A)$ are abelian completions of A, there is an isomorphism, χ, from $I_H(A)$ onto $I_K(A)$ satisfying $a\chi = a$ for all a in A. Define $\psi : H \to K$ by $(ga)\psi = g(a\chi)$. It is not difficult to show that ψ is an isomorphism from H onto K, and that the restriction of ψ to G is the identity.

Now take any order of K. This naturally induces an order of G which in turn induces an order of H (*cf.* (3.1)). Denote all these orders by $\le$ and take any $ga > 1$ in H. That is, $g^m a^m > 1$ in G, where $m = m(a)$. So

$$1 < g^m a^m = g^m(a^m\chi) = g^m(a\chi)^m .$$

By Lemma 4 *(iii)*, $(ga)\psi = g(a\chi) > 1$ in K, and so ψ is an o-isomorphism. //

4.2. For the remainder of this section, let G, A, $G^\#$ and $A^\#$ be as in §3.

Let $\Omega(G)$ and $\Omega(G^\#)$ denote the set of all full orders of G and $G^\#$ respectively. A group is an *O^*-group* if every partial order of the group extends to a full order of the group. A subgroup of an O-group, V, is *relatively* (respectively *absolutely*) *convex in* V if it is convex under at least one full order (respectively all full orders) of V. A normal subgroup, W, of a group, V, is *strongly isolated in* V if, for $v, v_1, v_2, \ldots, v_k$ in V, $v_1^{-1}vv_1v_2^{-1}vv_2 \ldots v_k^{-1}vv_k$ belongs to W implies v belongs to W.

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Proofs for the following rather motley theorem can be found in [4], Chapter 2.

THEOREM 5. *(i) There is a one-to-one mapping from* $\Omega(G)$ *onto* $\Omega(G^{\#})$.

(ii) If A *is relatively (respectively absolutely) convex in* G *, then* $A^{\#}$ *is relatively (respectively absolutely) convex in* $G^{\#}$.

(iii) If A *is strongly isolated in* G *, then* $A^{\#}$ *is strongly isolated in* $G^{\#}$.

(iv) If G *is an* O^**-group, then* $G^{\#}$ *is an* O^**-group.*

4.3. Finally, we turn to the case where G is solvable. Let $G = G^{(0)} > G^{(1)} > \ldots > G^{(l)} = \{1\}$ be the derived series of G . For arbitrary $g_0, g_1, \ldots, g_k$ in G and a in $A^{\#}$, define $[\![g_k, g_{k-1}, \ldots, g_0, a]\!]$ in $A^{\#}$ as follows:-

$[\![g_0, a]\!] = \left(a^{-1}\phi^{\#}_{g_0}\right)a$, and given that $b = [\![g_{k-1}, \ldots, g_0, a]\!]$ has been defined,

$$[\![g_k, g_{k-1}, \ldots, g_0, a]\!] = \left(b^{-1}\phi^{\#}_{g_k}\right)b \ .$$

Straightforward induction arguments prove the following:-

LEMMA 6. *(i) The* k*-th derived group of* $G^{\#}$ *can be generated by the set* $\left\{(x_k, 1), (1, [\![x_{k-1}, \ldots, x_0, a]\!]) : x_i \in G^{(i)}, a \in A^{\#}\right\}$.

(ii) $[\![x_k, \ldots, x_0, a]\!]^n = \left[\!\!\left[x_k, \ldots, x_0, a^n\right]\!\!\right]$ *for all integers,* n .

(iii) For all a *in* A *and* x_i *in* $G^{(i)}$ *,* $[\![x_k, \ldots, x_0, a]\!]$ *is in* $G^{(k+1)}$.

Note that this lemma is true for *any* O-group, G .

Now we can prove our final theorem.

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THEOREM 6. *If* G *is solvable of length* l *, then* $G^{\#}$ *is solvable of length* l *.*

Proof. It is sufficient to show that any two generators of the $(l-1)$-th derived group of $G^{\#}$ commute. By Lemma 6 *(i)*, and remembering that $G^{(l-1)}$ is abelian, we must show that, for all x_{l-1} in $G^{(l-1)}$ and $b = [\![x_{l-2}, \ldots, x_0, a]\!]$ $\left(x_i\right.$ is in $G^{(i)}$ and a is in $A^{\#}$ with a^m in $\left.A\right)$, $\left(x_{l-1}, 1\right)$ and $(1, b)$ commute. Now $[(x_{l-1}, 1), (1, b)] = (1, c)$ where $c = [\![x_{l-1}, \ldots, x_0, a]\!]$. Since $c^m = [\![x_{l-1}, \ldots, x_0, a^m]\!]$ is in $G^{(l)} = \{1\}$ (Lemma 6 *(ii)* and *(iii)*), and since O-groups are torsion-free, it follows that $c = 1$ and the proof is complete. //

References

[1] В.В. Блудов, Н.Я. Медведев [B.B. Bludov, N.Ja. Medvedev], "О пополнении упорядочиваемых метабелевых групп" [On the completion of an orderable metabelian group], *Algebra i Logika* 13 (1974), 369-373.

[2] C.G. Chehata, "On a theorem on ordered groups", *Proc. Glasgow Math. Assoc.* 4 (1958), 16-21.

[3] Paul Conrad, "Extensions of ordered groups", *Proc. Amer. Math. Soc.* 6 (1955), 516-528.

[4] Colin D. Fox, "The problem of adjoining roots to ordered groups", (PhD thesis, Australian National University, Canberra, 1974). See also Abstract: *Bull. Austral. Math. Soc.* 11 (1974), 157-158.

[5] L. Fuchs, *Partially ordered algebraic systems* (Pergamon, Oxford, London, New York, Paris, 1963).

[6] László Fuchs, *Teilweise geordnete algebraische Strukturen* (Akadémiai Kiadó, Budapest, 1966).

https://doi.org/10.1017/S0004972700023972 Published online by Cambridge University Press

[7] М.И. Каргаполов, Ю.И. Мерзляков, В.Н. Ремесленников [M.I. Kargapolov, Yu.I. Merzljakov, V.N. Remeslennikov], Коуровская тетрадь (Нерешенные задачи теории групп) [*Kourov notebook. Unsolved problems in the theory of groups*, 3rd edition, supplemented] (Izdat. Sibirsk. Otdel. Akad. Nauk SSSR, Novosibirsk, 1969).

[8] А.И. Кокорин, В.М. Копытов, Линейно упорядоченные группы (Nauka, Moscow, 1972).

A.I. Kokorin and V.M. Kopytov, *Fully ordered groups* (translated by D. Louvish. John Wiley & Sons, New York, Toronto; Israel Program for Scientific translations, Jerusalem, London, 1974).

[9] В.М. Копытов [V.M. Kopytov], "О пополнении центра упорядоченной группы" [On the completion of the centre of an ordered group], *Ural. Gos. Univ. Mat. Zap.* 4 (1963), 20-24.

[10] А.И. Мальцев [A.I. Mal'cev], "Нильпотентные группы без кручения" [Nilpotent torsion-free groups], *Izv. Akad. Nauk SSSR Ser. Mat.* 13 (1949), 201-212.

[11] А.И. Мальцев [A.I. Mal'cev], "О доупорядочении групп" [On the completion of group order], *Trudy Mat. Inst. Steklov.* 38 (1951), 173-175.

[12] Donald P. Minassian, "An embedding theorem for ordered groups", *Canad. J. Math.* 24 (1972), 944-946.

[13] Д.М. Смирнов [D.M. Smirnov], "Правоупорядоченные группы" [Right-ordered groups], *Algebra i Logika* 5 (6) (1966), 41-59.

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