

CONJUGATIONS ON BANACH ALGEBRAS

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An example is given of a complex unital Banach algebra carrying a continuum of conjugations, for each of which the selfconjugate subalgebra is strictly real.

1. INTRODUCTION

A *conjugation* on a complex Banach algebra \mathfrak{B} is a map $c: \mathfrak{B} \rightarrow \mathfrak{B}$, $x \mapsto x^c$, with the properties: for all $x, y \in \mathfrak{B}$ and $\lambda \in \mathbb{C}$, (i) $(x + y)^c = x^c + y^c$, (ii) $(\lambda x)^c = \bar{\lambda}x^c$, (iii) $(xy)^c = x^c y^c$, (iv) $x^{cc} = x$, (v) c is continuous. (Here $\bar{\lambda}$ denotes the complex conjugate of λ .) When \mathfrak{B} is unital with unit e , necessarily $e^c = e$. When \mathfrak{B} is commutative, c is a star operator. Given such a conjugation, the set

$$\mathfrak{B}^{(c)} := \{x \in \mathfrak{B} : x^c = x\}$$

is a real subalgebra of \mathfrak{B} , the *selfconjugate subalgebra for c* . If d is another conjugation on \mathfrak{B} distinct from c , then $\mathfrak{B}^{(c)}$ and $\mathfrak{B}^{(d)}$ are distinct.

An arbitrary complex Banach algebra \mathfrak{B} carries a conjugation if and only if it is equivalent (that is, isomorphic and renormable) to the complexification of a real Banach algebra. If c is a conjugation on \mathfrak{B} then \mathfrak{B} is equivalent to $(\mathfrak{B}^{(c)})_{\mathbb{C}}$, the complexification of $\mathfrak{B}^{(c)}$ (see [2, 4]).

A real Banach algebra \mathfrak{A} is called *strictly real* if $\text{Sp}_{\mathfrak{A}}(a) \subset \mathbb{R}$ for every $a \in \mathfrak{A}$. Recall that $\text{Sp}_{\mathfrak{A}}(a)$, the spectrum of a in \mathfrak{A} , is by definition the spectrum of the image of a in $\mathfrak{A}_{\mathbb{C}}$ under the canonical embedding of \mathfrak{A} in its complexification $\mathfrak{A}_{\mathbb{C}}$. Many aspects of Banach algebra theory take a simpler form when the algebra is strictly real; in particular, the algebra carries a partial order and its closure preorder, and much real-type analysis is possible on it: there are noticeable simplifications to the behaviour of the elementary functions; the resolvent is a Pick function; there are integral representation theorems in the dual space. See [1], [3]. As a consequence, there is special interest also in complex Banach algebras which are complexifications of strictly real algebras. The question arises: Do there exist complex Banach algebras \mathfrak{B} which are expressible as a complexification of a strictly real Banach algebra in more than one way. In [2] an example was given of an algebra \mathfrak{B} carrying two distinct but commuting conjugations

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for each of which the selfconjugate subalgebra is strictly real. The purpose of the present note is, by a modification and extension of that example, to give an example of a commutative unital Banach algebra \mathfrak{B} carrying a *continuum* of conjugations, for each of which the selfconjugate subalgebra is strictly real.

2. THE EXAMPLE

\mathfrak{B} is defined to be the unital Banach algebra with a single generator s for which $s^2 = 0$; each element has a unique representation in the form $x = \mu e + \nu s$ ($\mu, \nu \in \mathbb{C}$), and we can take $\|x\| := |\mu| + |\nu|$. Define

$$\mathfrak{J} = \{\mu e + \nu is : \mu, \nu \in \mathbb{R}\}.$$

\mathfrak{J} is a real subalgebra of \mathfrak{B} ; its complexification $\mathfrak{J}_{\mathbb{C}} = \mathfrak{J} \times \mathfrak{J}$, with elements written (z_1, z_2) , is complex isomorphic to \mathfrak{B} by the continuous map $\theta: (z_1, z_2) \mapsto z_1 + iz_2$, and if $\mathfrak{J}_{\mathbb{C}}$ is renormed by writing $\|((z_1, z_2))\| := \|z_1 + iz_2\|$ then θ becomes an isometry. Henceforth we write \mathfrak{B} in place of $\mathfrak{J}_{\mathbb{C}}$, for convenience. The set of all conjugations on \mathfrak{B} is given by the following lemma.

LEMMA 1. *A map $d: \mathfrak{B} \rightarrow \mathfrak{B}$ is a conjugation on \mathfrak{B} if and only if there exists a function $\delta: \mathfrak{B} \rightarrow \mathfrak{J}$, such that:*

- (a) $(z_1, z_2)^d = (\delta(z_1, z_2), \delta(-z_2, z_1))$ for all $z_1, z_2 \in \mathfrak{J}$,
- (b) δ is real linear,
- (c) $\delta(e, 0) = e, \delta(0, e) = 0$; and for some real numbers τ_1, τ_2 with $\tau_1^2 + \tau_2^2 = 1, \delta(is, 0) = \tau_1 is, \delta(0, is) = \tau_2 is$.

PROOF: It can be verified that any such δ determines a conjugation d . Conversely, let d be any given conjugation on \mathfrak{B} . It will determine two functions $\delta, \varepsilon: \mathfrak{B} \rightarrow \mathfrak{J}$, by the formula $(z_1, z_2)^d = (\delta(z_1, z_2), \varepsilon(z_1, z_2))$. Property (i) of Section 1 shows that δ and ε are additive, and (ii) shows further that δ is real linear and $\varepsilon(z_1, z_2) = \delta(-z_2, z_1)$: this gives (a) and (b). Since d preserves the unit $(e, 0)$ of \mathfrak{B} , $(\delta(e, 0), \delta(0, e)) = (e, 0)$; the first two equations of (c) follow. Supposing that

$$\delta(is, 0) = \rho_1 e + \tau_1 is, \quad \delta(0, is) = \rho_2 e + \tau_2 is$$

and substituting in $((is, 0)^2)^d = ((is, 0)^d)^2$, we find that $\rho_1 = \rho_2 = 0$. Property (iv) with (a) gives $\delta(\delta(z_1, z_2), \delta(-z_2, z_1)) = z_1$ for all $z_1, z_2 \in \mathfrak{J}$, and the choice $z_1 = is, z_2 = 0$ here shows that $\tau_1^2 + \tau_2^2 = 1$. □

With d, δ and the τ 's as in the lemma we find that, for all $\mu_1, \mu_2, \nu_1, \nu_2 \in \mathbb{R}$,

$$(*) \quad (\mu_1 e + \nu_1 is, \mu_2 e + \nu_2 is)^d = (\mu_1 e + (\tau_1 \nu_1 + \tau_2 \nu_2)is, -\mu_2 e + (\tau_2 \nu_1 - \tau_1 \nu_2)is).$$

LEMMA 2. *The selfconjugate subalgebra of \mathfrak{B} for the conjugation d of Lemma 1 is, if $-1 < \tau < 1$,*

$$\mathfrak{B}^{(d)} = \{(\mu e + \nu is, \tau_2^{-1}(1 - \tau_1)\nu is) : \mu, \nu \in \mathbb{R}\},$$

or $\mathfrak{B}^{(d)} = \{(\mu e, \nu is) : \mu, \nu \in \mathbb{R}\}$ if $\tau_1 = -1$, $\mathfrak{B}^{(d)} = \{(\mu e + \nu is, 0) : \mu, \nu \in \mathbb{R}\}$ if $\tau_1 = 1$.

PROOF: Verification, from (*). □

It remains to show

THEOREM. $\mathfrak{B}^{(d)}$ in Lemma 2 is strictly real.

PROOF: This involves working in the complexification $(\mathfrak{B}^{(d)})_{\mathbb{C}}$. A typical element of this algebra has the form

$$p = ((\mu_1 e + \nu_1 is, \tau_2^{-1}(1 - \tau_1)\nu_1 is), (\mu_2 e + \nu_2 is, \tau_2^{-1}(1 - \tau_1)\nu_2 is)),$$

(considering first the case $\tau_2 \neq 0$). Suppose that $\lambda = \alpha + i\beta$ belongs to the resolvent set of p ; this is so if and only if there exists an element q in the algebra such that

$$(\lambda((e, 0), (0, 0)) - p)q = ((e, 0), (0, 0)).$$

This equation when reduced to its components gives a system of linear equations in the 4 real coefficients appearing in q , the equations being independent of τ_1 and τ_2 , and solvable if and only if $(\alpha - \mu_1)^2 + (\beta - \mu_2)^2 \neq 0$. But we are concerned only with the case when p is the embedded image in $(\mathfrak{B}^{(d)})_{\mathbb{C}}$ of an element of $\mathfrak{B}^{(d)}$, that is, when $\mu_2 = \nu_2 = 0$; in this case the spectrum of p is $\{\mu_1\}$, which is real. This proves that $\mathfrak{B}^{(d)}$ is strictly real when $\tau_2 \neq 0$, and the case $\tau_2 = 0$ is verified in the same way. □

Thus each pair of reals τ_1, τ_2 with $\tau_1^2 + \tau_2^2 = 1$ determines an algebra of the sought kind, and these algebras are pairwise distinct.

The union of all these algebras is the set of all elements of \mathfrak{B} with real spectrum; any two distinct algebras intersect in $\mathbb{R}e$. If σ_1, σ_2 with $\sigma_1^2 + \sigma_2^2 = 1$ is another parameter pair distinct from τ_1, τ_2 , the two corresponding conjugations commute on \mathfrak{B} if and only if $\sigma_1 = -\tau_1$ and $\sigma_2 = -\tau_2$.

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