3
Interactions

In the previous two chapters, we have described kink solutions in several models but these solutions have mostly been discussed in isolation. In any real system, there is a variety of kinks and antikinks, in addition to small excitations (particles) of the fields. The interactions of kinks with other kinks and with particles play an important role in the evolution of the system. The motion of kinks is also accompanied by the radiation of particles. Ambient particles in the system scatter off kinks, and kinks collide with each other, and perhaps annihilate into particles. As discussed in Section 1.9, in some models a kink-antikink pair can bind together to form a non-dissipative solution which is called a "breather." In other models, approximate breather solutions have been found, which play an important role in the scattering of a kink and an antikink. These topics are discussed in the following sections.

3.1 Breathers and oscillons

So far we have been considering kinks, which are static solutions to the equations of motion. In the sine-Gordon model of Eq. (1.51), a one-parameter family of non-static, dissipationless solutions is also known. These are bound states of a kink and an antikink and are called breathers. The breather solution was described briefly in Section 1.9 and can be re written as

$$\phi_{\rm b}(t,x;v) = \frac{4}{\beta}\tan^{-1}\left[\frac{\sin(v\sqrt{\alpha}t/\sqrt{1+v^2})}{v\cosh(\sqrt{\alpha}x/\sqrt{1+v^2})}\right] \tag{3.1}$$

where v is a free parameter (see Fig. 3.1). We will have more to say about breathers when we quantize kinks in Chapter 4 as they play a very fundamental role in the novel duality between the sine-Gordon model and the massive Thirring model (see Section 4.7).

Breather solutions are not known to exist in the $\lambda\phi^4$ model [135]. However, numerical studies of the scattering of a Z_2 kink and antikink revealed the existence of

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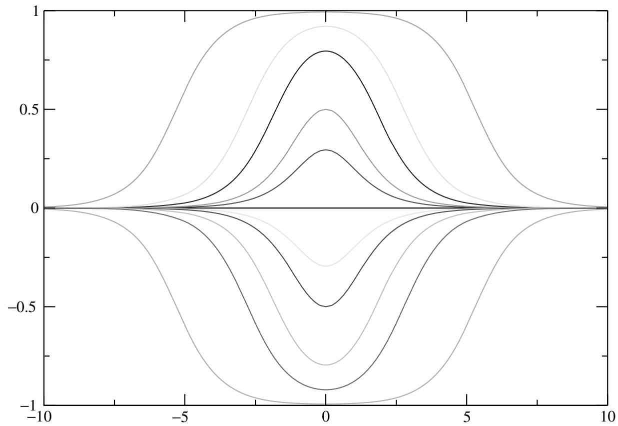

Figure 3.1 The sine-Gordon breather shown at various times during one oscillation period. At certain times, the field profile is that of a separated kink and an antikink. At other times, the kink and the antikink overlap and cannot be distinguished.

extremely long-lived, oscillating bound states of kinks and antikinks [4, 19, 26, 64]. The existence of kink-antikink bound states has been interpreted as a resonance phenomenon between the natural excitation frequency of the kink profile (shape mode) and the frequency of oscillation of the bound kink-antikink system. Radiation from a time-dependent scalar field configuration will be suppressed if the oscillation frequency of the configuration is small compared to the mass of the radiation quanta and this can be used to understand the longevity of oscillons (Farhi, 2005, private communication).

The simplest hypothesis is that oscillons are approximate breather solutions since a region of the sine-Gordon potential and the $\lambda\phi^4$ potential have very similar shapes. We can compare the two potentials when the sine-Gordon potential has been shifted so that it has a maximum at $\phi = 0$. The parameter β in the sine-Gordon model is chosen so that the first positive minimum is at $\phi = +\eta$. α is fixed by requiring that the masses of small excitations in the true vacua, given by the second derivative of the potential, are equal in the two models. Then the two potentials are given by

$$V_{Z_2}(\phi) = \frac{\lambda}{4}(\phi^2 - \eta^2)^2 \tag{3.2}$$

$$V_{\rm sG}(\Phi) = \frac{\alpha}{\beta^2}(1 - \cos(\beta(\phi - \eta))) \tag{3.3}$$

with

$$\alpha = 2\lambda\eta^2, \qquad \beta = \frac{\pi}{\eta} \tag{3.4}$$

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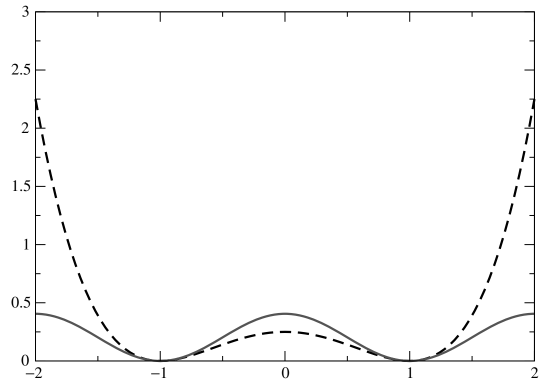

Figure 3.2 The $\lambda\phi^4$ potential (broken curve) and the shifted sine-Gordon potential (solid curve) when the parameters are chosen so that the vacua occur at the same values of ϕ and the curvatures of the potentials at the vacua are also equal.

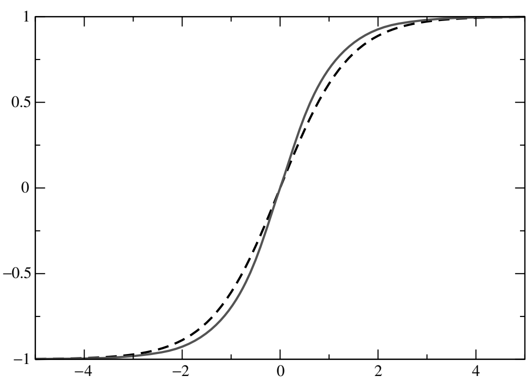

Figure 3.3 The profiles of the kinks in the $\lambda\phi^4$ model (broken curve) and the shifted sine-Gordon model (solid curve) with potentials as shown in Fig. 3.2.

The two potentials can be compared in the vicinity of their true vacuum at $\phi = \eta$. Then

$$V_{Z_2}(\phi) = \lambda\eta^2(\phi - \eta)^2 + \lambda\eta(\phi - \eta)^3 + \frac{\lambda}{4}(\phi - \eta)^4 \tag{3.5}$$

and

$$V_{\mathrm{sG}}(\phi) = \lambda\eta^2(\phi - \eta)^2 - \frac{2\pi^2\lambda}{4!}(\phi - \eta)^4 + O((\phi - \eta)^6) \tag{3.6}$$

In Fig. 3.2 we show these two potentials and in Fig. 3.3 we compare the kink profiles.

We will return to the breather and its role in the quantum sine-Gordon model at the end of Section 4.7.

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3.2 Kinks and radiation

By "radiation" we mean propagating excitations of small amplitude of a field, which in this chapter will be taken to be the same field that makes up the kink. Asymptotically, these excitations have the usual plane wave form: $\exp(i(\omega t \pm kx))$. In the kink background, these "scattering states" are found as solutions to the equation of motion for fluctuations about the kink. If we denote the kink solution by $\phi_k(x)$, the fluctuation field $\psi(t, x)$ is

$$\psi(t, x) = \phi(t, x) - \phi_k(x) \tag{3.7}$$

We will assume $|\psi| \ll \langle\phi\rangle$, where $\langle\phi\rangle$ is the vacuum expectation value of ϕ. To find the scattering states, we take $\psi = f(x)e^{-i\omega t}$ where it is understood that the real or imaginary part should be taken – in other words, the physical modes are $[f(x)e^{-i\omega t} \pm f^*(x)e^{+i\omega t}]$. Perturbing the Lagrangian for ϕ (first line of Eq. (1.2)), we find that $f(x)$ satisfies the (linearized) equation of motion

$$Hf \equiv -f'' + U(x)f = \omega^2 f \tag{3.8}$$

where

$$U(x) \equiv V''(\phi_k(x)) \equiv \left.\frac{\partial^2 V}{\partial \phi^2}\right|_{\phi=\phi_k} \tag{3.9}$$

The scattering states around a static kink are obtained by solving the Schrödinger-type equation, Eq. (3.8), which for some potentials, falls in the general class of equations discussed in Appendix C.

We now consider the Z_2 kink for which the potential U is obtained from Eqs. (3.9) and (1.2) to be

$$U(x) = \lambda\left(3\phi_k^2 - \eta^2\right) \tag{3.10}$$

We now list the eigenmodes of Eq. (3.8). (We will encounter them again in Chapter 4.) First, there are two bound states, also known as "discrete" modes:

$$\omega_0 = 0, \qquad f_0 = \mathrm{sech}^2 z \tag{3.11}$$

$$\omega_1 = \frac{\sqrt{3}}{2} m_\psi, \qquad f_1 = \sinh z\, \mathrm{sech}^2 z \tag{3.12}$$

where $z = x/w = m_\psi x/2$. The $\omega = 0$ mode is called the "translation mode" and the second is the "shape mode." Then there is a continuum of states for $m_\psi < \omega < \infty$ which are the scattering states:

$$f_k = e^{ikx}[3\tanh^2 z - 1 - w^2k^2 - i\, 3wk \tanh z] \tag{3.13}$$

The dispersion relation is

$$\omega_k^2 = k^2 + m_\psi^2 \tag{3.14}$$

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We are now interested in processes that involve both a kink and the scattering states (radiation). For example, if a kink accelerates, it will emit radiation. What is the radiated power? The answer will depend on the forces that make the kink accelerate and whether or not these forces deform the structure of the kink.[1] We shall examine the radiation from kink shape deformations and other interactions of kinks and radiation after a brief diversion in the next section.

3.3 Structure of the fluctuation Hamiltonian

In this section we will show two interesting properties of the fluctuation Hamiltonian, H, defined in Eq. (3.8). The first is that the potential $U(x)$ has a very special form that implies that the Hamiltonian can be factored. The second is that there exists a "partner Hamiltonian" with (almost) the same spectrum as the original Hamiltonian.

The special form of U follows from the fact that the kink has a translation zero mode (see Section 1.1). Hence there exists an eigenstate with $\omega = 0$. Denote this "translation mode" by $\psi_{\rm t}$. Hence

$$H\psi_{\rm t} = (-\partial^2 + U(x))\psi_{\rm t} = 0 \tag{3.15}$$

Therefore

$$U(x) = \frac{\psi_{\rm t}''}{\psi_{\rm t}} \tag{3.16}$$

which can also be rewritten as

$$U(x) = f' + f^2, \quad f = (\ln(\psi_{\rm t}))' \tag{3.17}$$

For the particular cases of the Z_2 and sine-Gordon kinks, not only is $U(x)$ of the form in Eq. (3.17) but it is also reflectionless. Then, an incident wave is fully transmitted and the reflection coefficient vanishes. In this case, the only non-trivial characteristic of scattering states is that the waves get a phase shift owing to the presence of $U(x)$. This property will be useful when we quantize the kink in Section 4.1.

The Hamiltonian H with a potential of the form in Eq. (3.17) has the important property that it can be factored

$$H = A^+A \equiv (+\partial + f)(-\partial + f) \tag{3.18}$$

Therefore the equation for the eigenstates is simply

$$Hf = A^+Af = \omega^2 f \tag{3.19}$$

[1] In the case of domain walls in three spatial dimensions, the curvature of the wall is itself responsible for acceleration. This motion leads to the emission of scalar and gravitational radiation and will be discussed in Chapter 8.

The factorization has the consequence that one can readily construct a "partner" Hamiltonian, H_-, that has almost an identical eigenspectrum as H. This partner Hamiltonian is $H_- = AA^+$. If f_i is an eigenstate of H with eigenvalue ω_i^2, then Af_i is an eigenstate of H_- with the same eigenvalue. This argument works for all eigenstates except the one for which $Af_i = 0$. Hence H has a single extra eigenstate with $\omega = 0$.[2]

The potential $U(x)$ determines the spectrum of excitations around a soliton. The factorizability of the Hamiltonian is useful in the problem of reconstructing $V(\phi)$ from the spectrum of fluctuations (i.e. the set of ω^2) using inverse scattering methods [165].

3.4 Interaction of kinks and radiation

As remarked below Eq. (3.17) the potentials $U(x)$ for both the Z_2 and the sine-Gordon kinks are rather special since they are reflectionless. All that happens is that the transmitted wave gets phase shifted. This is equivalent to a time delay in the propagation of the wave through the kink.

From the solution for the scattering states given in Eq. (3.13) for the Z_2 kink we find a momentum dependent phase shift

$$\delta_{\mathrm{k}}|_{Z_2} = 2\tan^{-1}\left(\frac{3wk}{w^2k^2 - 2}\right) \tag{3.22}$$

This corresponds to a time delay

$$\tau_{\mathrm{k}}\bigg|_{Z_2} = \frac{\delta_{\mathrm{k}}}{\omega}\bigg|_{Z_2} = \frac{2}{\sqrt{k^2 + m_\psi^2}}\tan^{-1}\left(\frac{3wk}{w^2k^2 - 2}\right) \tag{3.23}$$

Similarly the phase shift and time delay in the case of the sine-Gordon kink are

$$\delta_{\mathrm{k}}|_{\mathrm{sG}} = \pi - 2\tan^{-1}\left(\frac{k}{m_\psi}\right) \tag{3.24}$$

$$\tau_{\mathrm{k}}\bigg|_{\mathrm{sG}} = \frac{\delta_{\mathrm{k}}}{\omega}\bigg|_{\mathrm{sG}} = \frac{1}{\sqrt{k^2 + m_\psi^2}}\left[\pi - \tan^{-1}\left(\frac{k}{m_\psi}\right)\right] \tag{3.25}$$

[2] The two partner Hamiltonians can also be combined to form a supersymmetric Hamiltonian, H_{ss}

$$H_{\mathrm{ss}} = \begin{pmatrix} A^+A & 0 \\ 0 & AA^+ \end{pmatrix} = \{Q, Q^+\} \equiv QQ^+ + Q^+Q \tag{3.20}$$

where

$$Q = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}, \qquad Q^+ = \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix} \tag{3.21}$$

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While there is no reflection of radiation of the same field that makes up the kink in the Z_2 and sine-Gordon cases, there can be reflection of fluctuations of other fields [53]. As an example [171], consider a second scalar field χ included in the Z_2 model so that the full Lagrangian becomes

$$L = L_\phi + \frac{1}{2}(\partial_\mu \chi)^2 - \frac{m_\chi^2}{2}\chi^2 - \frac{\sigma}{2}\phi^2\chi^2 \tag{3.26}$$

where L_ϕ is the Lagrangian for the Z_2 model (Eq. (1.2)). Then the scattering modes of χ in the presence of a Z_2 kink are found by solving

$$\partial_t^2\chi - \partial_x^2\chi + m_\chi^2\chi + \sigma\phi_{\mathrm{k}}^2\chi = 0 \tag{3.27}$$

Substituting $\phi_{\mathrm{k}} = \eta\tanh(x/w)$ and $\chi = \exp(-\mathrm{i}\omega t)f(x)$, we get

$$\partial_X^2 f + (\nu^2 - \bar{\sigma}\,\mathrm{sech}^2(X))f = 0 \tag{3.28}$$

where $X \equiv x/w$, $\nu^2 = w^2(\omega^2 - m_\chi^2 - \sigma\eta^2)$, $\bar{\sigma} = \sigma\eta^2 w^2$. (Recall that $w = \sqrt{2/\lambda\eta^2}$.)

Equation (3.28) is a special case of the differential equation described in Appendix C. The scattering state is found for real values of ν and has the asymptotics: $f \to \mathrm{e}^{\mathrm{i}kx}$ for $x \to \infty$, and for $x \to -\infty$:

$$f \to \frac{\Gamma(1-\mathrm{i}kw)\Gamma(-\mathrm{i}k)\mathrm{e}^{\mathrm{i}kx}}{\Gamma(1/2+\gamma-\mathrm{i}k)\Gamma(1/2-\gamma-\mathrm{i}k)} + \frac{\Gamma(1-\mathrm{i}kw)\Gamma(\mathrm{i}k)\mathrm{e}^{-\mathrm{i}kx}}{\Gamma(1/2+\gamma)\Gamma(1/2-\gamma)} \tag{3.29}$$

where $k = \nu/w$ and $\gamma = \sqrt{\bar{\sigma} + 1/4}$.

The reflection coefficient can be read off from the asymptotic behavior of $f(x)$ as $x \to -\infty$ and has been evaluated in Section 12.3 of [113]

$$R = \frac{1+\cos(2\pi\gamma)}{\cosh(2\pi k) + \cos(2\pi\gamma)} \tag{3.30}$$

The transmission coefficient is

$$T = \frac{2\sinh^2(\pi k)}{\cosh(2\pi k) + \cos(2\pi\gamma)} = 1 - R \tag{3.31}$$

From the asymptotic expression in Eq. (3.29), it is also possible to calculate the time delay of the reflected and transmitted waves owing to the kink. For example, if we write

$$\frac{\Gamma(1-\mathrm{i}kw)\Gamma(-\mathrm{i}k)}{\Gamma(1/2+\gamma-\mathrm{i}k)\Gamma(1/2-\gamma-\mathrm{i}k)} = |T|^{1/2}\mathrm{e}^{\mathrm{i}\delta_{\mathrm{k}}} \tag{3.32}$$

where T is the transmission coefficient above, then the time delay of the transmitted wave is given by $\delta_{\mathrm{k}}/\omega$.

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3.5 Radiation from kink deformations

A static kink does not emit any radiation. Nor does it emit radiation if it is moving at constant velocity (see Eq. (1.10)). However, if the kink is accelerating (owing to some external force), or its shape is deformed, it can emit radiation in the form of scalar particles [106, 107]. In $3+1$ dimensions, acceleration and deformations arise since the kinks (domain walls) are moving under their own tension, except in the very special cases of static solutions. The radiation emitted from curved domain walls has not been calculated analytically, though the problem has been studied numerically [182]. In the case of $1+1$ dimensional Z_2 kinks that are undergoing periodic deformations, the radiation has been found analytically in [110, 140], and we shall describe this calculation below.

Following [110], we simplify notation by setting $\lambda = 2$ and $\eta = 1$ in the Z_2 model so that $w = 1$ in these units (see Section 1.1). Then the field $\phi(x,t)$ is written in terms of the complete set of small excitations. This gives

$$\phi(x,t) = \phi_{\rm k}(x) + R(t) f_0(x) + A(t) f_1(x) + f(x,t) \tag{3.33}$$

where $\phi_{\rm k} = \tanh(x)$, f_0 and f_1 are the translation and shape modes respectively as given in Eqs. (3.11) and (3.12), $R(t)$ and $A(t)$ are their time-dependent amplitudes, and the function $f(x,t)$ contains all the continuum states around the kink. The frequency of oscillation of $R(t)$ is $\omega_0 = 0$ and of $A(t)$ is $\omega_1 = \sqrt{3}$. These values were derived for linearized fluctuations about the kink. Non-linearities will modify $\omega_1 = \sqrt{3}$ but we assume that such modifications are small.

We will work in the rest frame of the kink and so

$$R(t) = 0 \tag{3.34}$$

The idea now is to insert Eq. (3.33) in the equation of motion for ϕ with some choice of the amplitude $A(t)$ which is assumed to be small, and then to find the solution for the scattering states, $f(x,t)$, which form the radiation.

Insertion of Eq. (3.33) in Eq. (1.4) gives

$$\begin{aligned}(\ddot{A} + 3A) f_1 + \ddot{f} - f'' + 2\big(3\phi_{\rm k}^2 - 1\big) f = &-6(f + \phi_{\rm k}) f_1^2 A^2 \\ &- 6(f + 2\phi_{\rm k}) f f_1 A - 2 f_1^3 A^3 \\ &- 6\phi_{\rm k} f^2 - 2 f^3\end{aligned} \tag{3.35}$$

where the equations satisfied by $\phi_{\rm k}$ and f_1 have been used. Assuming that A is small, and that f is $O(A^2)$ or smaller, the leading order equation is $\ddot{A} + 3A = 0 + O(A^2)$. Then to order A^2, the equation for f is

$$(\ddot{A} + 3A) f_1 + \ddot{f} - f'' + 2\big(3\phi_{\rm k}^2 - 1\big) f = -6\phi_{\rm k} f_1^2 A^2 \tag{3.36}$$

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The f-independent terms on the right-hand side of Eq. (3.35) are source terms which cause radiation. Hence f will not be zero at order A^2. The terms will also cause the amplitude, A, of the shape mode to depart from the purely oscillatory behavior. To determine how much of the source affects radiation and how much affects the shape mode, note that f_1 and f are orthogonal

$$\int \mathrm{d}x f_1(x) f(x,t) = 0 \tag{3.37}$$

So we can decompose the equation into a direction parallel to f_1 in mode space and orthogonal to it. One assumption we have to make is that the back-reaction of the radiative modes on the shape mode is higher order in A. For example, Eq. (3.37) does not by itself imply that f'' and f_1 are orthogonal. Then, multiplying Eq. (3.36) by f_1 and integrating over all space gives

$$\ddot{A} + 3A = -6A^2 \int \mathrm{d}x \phi_{\mathrm{k}} f_1^2 \equiv -6\alpha A^2 \tag{3.38}$$

provided we have normalized f_1 so that

$$\int \mathrm{d}x [f_1(x)]^2 = 1 \tag{3.39}$$

Explicit evaluation gives

$$\alpha = \frac{3\pi}{32} \tag{3.40}$$

The equation orthogonal to f_1 is

$$\ddot{f} - f'' + 2\big(3\phi_{\mathrm{k}}^2 - 1\big) f = -6\phi_{\mathrm{k}} f_1^2 A^2 + 6\alpha f_1 A^2 \tag{3.41}$$

and this will determine the radiation from the deformed kink once we have specified A.

The leading order solution for A is

$$A = A_0 \cos(\sqrt{3}t) \tag{3.42}$$

Hence

$$A^2 = \frac{A_0^2}{2}[\cos(2\sqrt{3}t) + 1] \tag{3.43}$$

This form implies that the source for f in Eq. (3.41) has a time-dependent piece and another time-independent piece. Since the equation is linear in f, only the time-dependent piece proportional to $\cos(2\sqrt{3}t)$ is important. Setting

$$f(x,t) = \mathrm{Re}(\mathrm{e}^{\mathrm{i}\omega t} F(x)) \tag{3.44}$$

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the equation that needs to be solved is

$$-F'' + \left(6\phi_{\mathrm{k}}^2 - 2 - \omega^2\right)F = \frac{3}{2}\left(\alpha f_1 - \phi_{\mathrm{k}} f_1^2\right)A_0^2 \mathrm{e}^{\mathrm{i}(\omega_0-\omega)t} \tag{3.45}$$

where $\omega_0 = 2\sqrt{3}$. Since the left-hand side is time-independent, this only has solutions for

$$\omega = \omega_0 = 2\sqrt{3} \tag{3.46}$$

and then all the solutions of the homogeneous equation are known (see Appendix C; [113, 126]). The solutions of the homogeneous equation with plane wave asymptotics are

$$F_q(x) = \left(3\phi_{\mathrm{k}}^2 - 1 - q^2 - 3iq\phi_{\mathrm{k}}\right)\mathrm{e}^{\mathrm{i}qx} \tag{3.47}$$

where $q = \sqrt{\omega^2 - 4}$. Knowing all the solutions of the homogeneous equation, it is possible to explicitly construct the (retarded) Green's function suitable for outgoing radiation.

$$G(x,y) = \begin{cases} -F_{-q}(y)F_q(x)/W, & (x < y) \\ -F_q(y)F_{-q}(x)/W, & (x > y) \end{cases} \tag{3.48}$$

where W is the Wronskian

$$W = F_q(x){F_{-q}}'(x) - {F_q}'(x)F_{-q}(x) \tag{3.49}$$

The Wronskian is a constant and its value can be found by using the explicit solutions

$$W = -2\mathrm{i}q(q^2+1)(q^2+4) \tag{3.50}$$

The solution of the inhomogeneous equation (3.45) is found by convoluting the source with the Green's function

$$F(x) = \int_{-\infty}^{+\infty} \mathrm{d}y\; G(x,y)\frac{3}{2}\left[\alpha f_1(y) - \phi_{\mathrm{k}}(y) f_1^2(y)\right]A_0^2 \tag{3.51}$$

With a little more manipulation, we obtain the radiation field in the $x \to +\infty$ limit

$$f(x,t) = \mathrm{Re}\left[\frac{-3A_0^2\mathrm{e}^{\mathrm{i}(\omega t - qx)}}{2\mathrm{i}q(2-q^2-3iq)}\int_{-\infty}^{+\infty} \mathrm{d}y\; \phi_{\mathrm{k}}(y) f_1^2(y) F_q(y)\right] \tag{3.52}$$

with $\omega = \omega_0 = 2\sqrt{3}$ and $q = \sqrt{\omega^2 - 4} = 2\sqrt{2}$. The integral can be done explicitly leading to

$$f(x,t) = \frac{\pi q(q^2-2)}{32\sinh(\pi q/2)}\sqrt{\frac{q^2+4}{q^2+1}}A_0^2\cos(\omega t - qx - \delta) \tag{3.53}$$

The phase δ can be read off from Eq. (3.52) because the integral is purely imaginary and does not contribute

$$\delta = \tan^{-1}\left(\frac{3q}{q^2 - 2}\right) \tag{3.54}$$

Now that we have the solution for the radiation field, we can find the energy flux by using the T_{0i} components of the energy-momentum tensor in Eq. (1.39). Including a factor of 2 to account for the radiation toward $x \to -\infty$, we obtain the radiated power [110]

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -0.020A_0^4 \tag{3.55}$$

The back-reaction of the radiation on the deformation amplitude can be estimated on the grounds of energy conservation. In [110] the results above are compared to the results of a numerical evolution of the deformations using the full non-linear equations with good agreement.

3.6 Kinks from radiation

By time reversing kink and antikink annihilation, it should be possible to obtain kink-antikink creation from incoming radiation. However, the stream of incoming radiation would have to be sent with just the correct phase relationship and energy. Such initial conditions occupy zero volume in the space of all initial conditions. A more physical problem is to identify the set, or a large subset, of initial conditions for the incoming radiation that will lead to kink-antikink creation. This problem is unsolved. Yet certain interesting results have been obtained in [110] in the "gradient flow" approximation in which the second time derivative terms in the equation of motion are neglected.

Consider the collision of two kinks in the presence of a pre-existing kink [110], as depicted in Fig. 3.4. As the kink-antikink-kink ($k\bar{k}k$) system evolves, a kink-antikink annihilate, and we are left with a kink whose shape is excited. Reversing this process, if we start with a kink whose shape is excited, it can produce a kink-antikink pair. In [110], this relation between the shape mode and the creation of a kink-antikink pair was explored.

3.7 Scattering of kinks

The sine-Gordon model is a famous example of a completely integrable system [48]. Sine-Gordon kinks are examples of "solitons" in the strict mathematical sense in which when two or more solitons (or anti-solitons) scatter, they simply pass through

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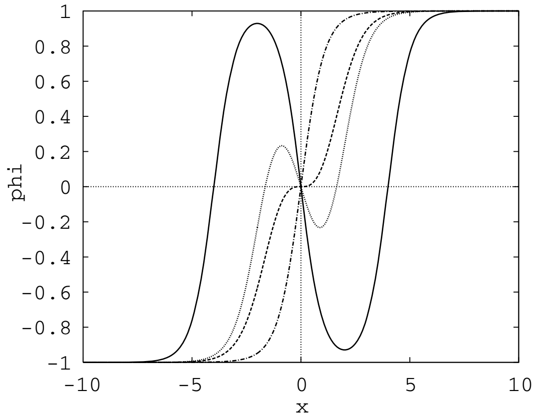

Figure 3.4 A kink collides from the left with another kink coming in from the right in the presence of an antikink in the middle. The time evolution of the field is shown in succession by the solid, dotted, dashed, and dashed-dotted curves. The evolution shows that a kink and antikink annihilate leaving behind a kink whose shape mode is excited (dotted and dashed curves). With further evolution, the shape mode will dissipate and an unexcited kink will remain as seen in the dashed-dotted curve. [Figure reprinted from [110].]

each other. The only consequence of the scattering is that there is a phase shift, or equivalently, a time delay. The time delay may be understood by realizing that the force between two kinks in the sine-Gordon model is attractive. Hence the kinks collide and form a bound state for some time. The time delay may be viewed as the time spent by the kinks in the form of a bound state. A crucial aspect of the scattering is that there is no dissipation. More details can be found, for example, in [48].

Kink scattering in the Z_2 model has a more complex character. In this case, we cannot have kink-kink scattering because two Z_2 kinks cannot be adjacent to each other. Instead, we need only consider kink-antikink scattering. This has been the subject of significant investigation [26, 4]. When a kink-antikink collide, the only possibilities are that they reflect back or they annihilate (see Fig. 3.5).

As we might expect, at very low incoming velocities, a bound state is formed and annihilation inevitably occurs, while at very high velocities, reflection takes place. The remarkable discovery of numerical studies of kink-antikink scattering is that the change from annihilation to reflection does not happen at just one critical value of the incoming velocity. Instead there are bands of incoming velocity at which annihilation occurs, while at other values of the incoming velocity the kink and antikink are reflected. The plot in Fig. 3.6 shows these results.

The unexpected dependence of kink-antikink scattering on the incoming velocity has been examined closely in [26, 4]. The behavior is understood as a resonance effect between oscillations of the mode that describes the shape distortions of the

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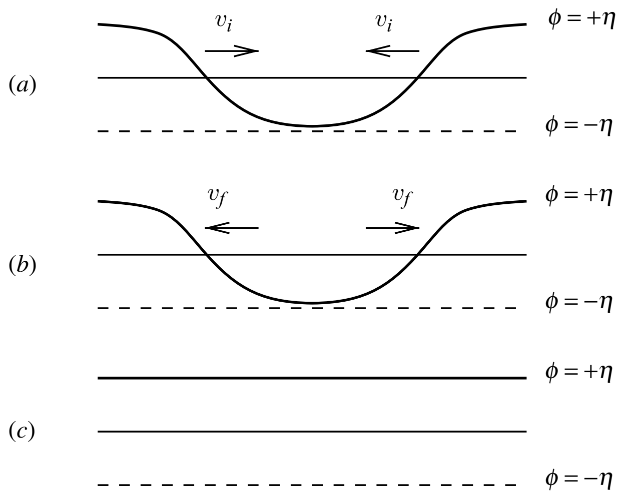

Figure 3.5 A kink and an antikink with incoming velocity v_i are shown in panel (a). The two possible outcomes of the scattering are shown in panels (b) and (c). In panel (b), the kinks scatter and reflect. Their outgoing velocity v_f need not be equal to v_i. In panel (c), the kink and antikink have annihilated and radiated away their energy, leaving behind the trivial vacuum. In both outcomes, the scattering is likely to be accompanied with radiation that has not been depicted.

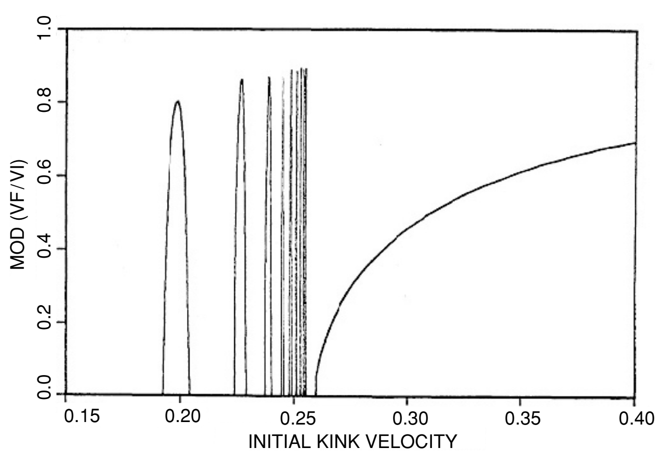

Figure 3.6 The ratio of outgoing to incoming kink velocities after scattering versus the incoming velocity [26, 4]. When the outgoing velocity is plotted to be zero, a kink-antikink bound state is formed that decays to the vacuum by radiation. Notice that the kink-antikinks annihilate in certain bands in the initial velocity. [Figure reprinted from [26].]

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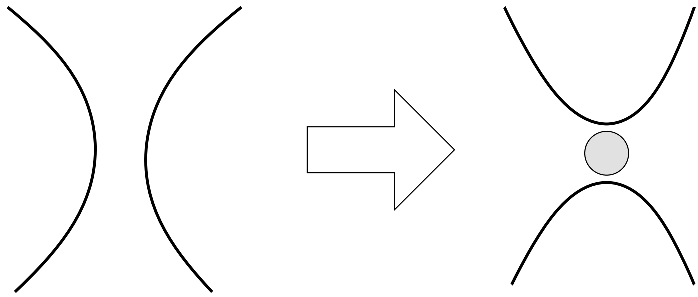

Figure 3.7 Two curved domain walls collide and intercommute. At the collision point, there is lots of radiation owing to annihilation or owing to the formation of a closed domain wall that then collapses and decays into radiation. To imagine the walls in three dimensions, rotate the figures along the horizontal axis. In the initial state the two curved walls are disconnected from each other while in the final state, the wall is in the shape of a "wormhole," with a sphere in the middle.

kinks and the oscillations of the kinks as a whole owing to kink-antikink interactions. We shall not describe the details of the analysis here.

The scattering of $SU(5) \times Z_2$ kinks has been studied numerically in [121]. In this case, there is an additional degree of freedom, namely the non-Abelian charge of the colliding kinks (or "color") and there are a variety of initial conditions that can be considered. For the stable variety of kinks – the $q = 2$ kinks (see Section 3.2) – the scattering of kinks and antikinks of the same color is qualitatively similar to that of Z_2 kinks. If the colors are different, however, there is a repulsive force between the kinks and they are observed to bounce back elastically.

3.8 Intercommuting of domain walls

We finally consider the collision of two domain walls. The outcome is found by numerical evolution of the equations of motion. As the walls come together, they reconnect along the curve of intersection [136] as shown in Fig. 3.7. This process is called "intercommuting" or, simply, "reconnection."

3.9 Open questions

1. Suppose we want to create a well-separated Z_2 kink-antikink pair by colliding small amplitude plane waves (particles) in the $\phi = +\eta$ vacuum. What conditions must be imposed on the incoming waves? What is the space of initial conditions that leads to domain wall formation? Can the initial conditions be implemented in a practical setting (e.g. accelerator experiments)?
2. Can a domain wall lattice be generalized to other defects, e.g. a lattice of strings and monopoles?

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3. Study the interaction of domain walls and strings/magnetic monopoles in a model that contains both types of defects e.g. the $SU(5) \times Z_2$ model has walls and magnetic monopoles.
4. Construct a sine-Gordon-like breather field configuration in the $\lambda\phi^4$ model. This will not be an exact solution of the field equation of motion. Hence it will radiate. Calculate the radiated power. Are there circumstances in which the radiated power is very small?
5. Can the analysis of radiation from kink deformations be extended to the case of oscillating domain walls? The simplest procedure would be to decompose the field as in Eq. (3.33) and to include a suitable external (harmonic) potential that drives the translation mode only. This will cause the kink to oscillate as a whole without deformations. However, the oscillations will source the shape mode and the radiation, and an analysis of the kind in Section 3.5 seems feasible.
6. Can the analysis of radiation from kink deformations be extended to the case of spherical domain walls?
7. How can the radiation analysis be extended to vortex solutions in two or more spatial dimensions?