Appendix C

Residue forms and classical Morse theory

In our first two appendices, we developed the mathematics needed to prove that the Cauchy integral representation

$$a_{\boldsymbol{r}} = \left(\frac{1}{2\pi i}\right)^s \int_C F(\boldsymbol{z})\, \boldsymbol{z}^{-\boldsymbol{r}-\boldsymbol{1}} d\boldsymbol{z}$$

for the coefficients of a convergent Laurent series $F(\boldsymbol{z}) = \frac{P(\boldsymbol{z})}{Q(\boldsymbol{z})} = \sum_{\boldsymbol{r}} a_{\boldsymbol{r}} \boldsymbol{z}^{\boldsymbol{r}}$ depends only on the singular homology class of the chain C and the de Rham cohomology class of the form $F(\boldsymbol{z})\, \boldsymbol{z}^{-\boldsymbol{r}-\boldsymbol{1}} d\boldsymbol{z}$ in the domain of holomorphicity

$$\mathcal{M} = \left\{ \boldsymbol{z} \in \mathbb{C}^d : Q(\boldsymbol{z}) \prod_{j=1}^{d} z_j \neq 0 \right\}$$

of the integrand.

In this appendix, we begin to discuss how such a representation allows us to manipulate Cauchy integrals into a form where we can derive asymptotic information. First, we discuss intersection classes and residue forms, which illustrate how to convert the Cauchy integral into an integral lying "on" the singular set $\mathcal{V} \subset \mathbb{C}^d$ of F. After introducing these concepts, we discuss how to use Morse theory to manipulate integrals over chains in $\mathcal{V}$ into representations that will ultimately allow us to use saddle point approximations. Morse theory is a large subject, and our treatment is restricted to the core topics we need: height functions, attachments, homology groups, and homotopy type. In this appendix we focus on the case where $\mathcal{V}_* := \mathcal{V} \cap \mathbb{C}_*^d$ is a complex manifold. Appendix D, our final appendix, describes extensions of this material to general algebraic sets (and their complements) using *stratified* Morse theory.

C.1 Intersection classes

Before describing how to generalize residues from the classical univariate setting to several variables, we first need to describe the domains of integration over which we can take multidimensional residues of differential forms with singularities on $\mathcal{V}$. These domains of integration will be *intersection classes*.

The intuition behind intersection classes is captured in Figure C.1. A torus T on one side of $\mathcal{V}$ expands to a torus T' on the other side of $\mathcal{V}$. Mathematically, this expansion could be obtained by expanding each coordinate at a constant rate, or by a more complicated deformation, or perhaps not by a deformation at all but through a *cobordism*, meaning some $(d + 1)$-chain whose boundary is $T' - T$. In getting from T to T' this expansion crosses $\mathcal{V}$; if the crossing is transverse, as it will be generically, it sweeps out a $(d - 1)$-chain $\gamma \subseteq \mathcal{V}$. For the intersection class to be well defined for our purposes, the homology class of γ in $H_{d-1}(\mathcal{V}_*)$ should depend only on the homology classes of T and T' in $\mathcal{M}$.

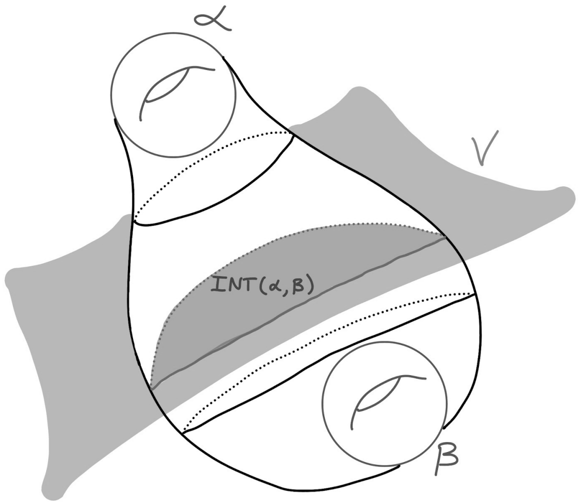

Figure C.1 The intersection class of a cobordism from α to β.

The concepts involved in defining an intersection class are analytic in nature, so we work with analytic functions instead of restricting ourselves to polynomials. Let $\mathcal{V} = \mathcal{V}_Q$ be the complex manifold in $\mathbb{C}^d$ defined by the vanishing of an analytic function $Q(z)$ on $\mathbb{C}^d$ *whose gradient does not vanish* on $\mathcal{V}$. Our first step in constructing the intersection class is to derive a diffeomorphism between a neighborhood of $\mathcal{V}$ in $\mathbb{C}^d$ and a product $A \times B$, where A is a connected

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open set in $\mathcal{V}$ and B is a neighborhood of the origin in $\mathbb{C}$. This is accomplished by considering the embedded complex manifold $\mathcal{V}$ in its ambient space $\mathbb{C}^d$. The tangent bundle of $\mathcal{V}$ may be identified with a sub-bundle of the tangent space to $\mathbb{C}^d$ by sending $\boldsymbol{v}$ to $e_*(\boldsymbol{v})$, where $e : \mathcal{V} \to \mathbb{C}^d$ is the embedding of $\mathcal{V}$ into $\mathbb{C}^d$.

Recall from Appendix A that for any $\boldsymbol{w} \in \mathbb{C}^d$ there is a natural identification ϕ of the tangent space $T_{\boldsymbol{w}}\mathbb{C}^d$ with $\mathbb{C}^d$ using the standard basis $\partial/\partial z_1, \ldots, \partial/\partial z_d$ for the holomorphic tangent space. Intuitively, we decompose $\mathbb{C}^d$ near $\boldsymbol{w}$ by taking the tangent plane to $\mathcal{V}$ at $\boldsymbol{w}$ and its orthogonal complement. Formally, the ***embedded tangent space*** and the ***embedded normal space*** of $\mathcal{V}$ at $\boldsymbol{w}$ are the subsets $S_{\boldsymbol{w}} := \{\boldsymbol{w}+\phi(e_*(X)) : X \in T_{\boldsymbol{w}}\mathcal{V}\}$ and $S'_{\boldsymbol{w}} := \{\boldsymbol{w}+\boldsymbol{v} : \boldsymbol{v} \in N_{\boldsymbol{w}}\mathcal{V}\}$ of $\mathbb{C}^d$, respectively, where $N_{\boldsymbol{w}}\mathcal{V} \subseteq \mathbb{C}^d$ is the orthogonal complement to $\phi(e_*(T_{\boldsymbol{w}}\mathcal{V}))$.

Under our assumptions, $N_{\boldsymbol{w}}\mathcal{V}$ is the one-dimensional complex vector space (or two-dimensional real vector space) described in local coordinates as the span of the vector $(\nabla Q)(\boldsymbol{w})$. The ***total space of the normal bundle*** to $\mathcal{V}$ is the set $\{(\boldsymbol{w}, \boldsymbol{v}) \in \mathcal{V} \times \mathbb{C}^d : \boldsymbol{v} \in N_{\boldsymbol{w}}\mathcal{V}\}$ pairing elements of $\mathcal{V}$ and normal vectors.

Lemma C.1 (Collar Lemma)**.** *Under our running assumption that the gradient of Q is nonvanishing on $\mathcal{V}$, there is an open neighborhood of $\mathcal{V}$ in $\mathbb{C}^d$ that is diffeomorphic to the total space of the normal bundle to $\mathcal{V}$ under a diffeomorphism that maps $\boldsymbol{w} \in \mathcal{V}$ to the vector $(\boldsymbol{w}, \mathbf{0})$.*

Proof Because ∇Q is nonvanishing on $\mathcal{V}$, the gradient ∇Q is non-zero in a neighborhood of $\mathcal{V}$ and thus defines a complex line bundle whose integral surfaces have real dimension two. If U is any sufficiently small neighborhood of $\mathcal{V}$, we let $a : U \to \mathcal{V}$ be the map sending $\boldsymbol{z} \in U$ to the unique point of $\mathcal{V}$ on whose integral curve it lies; see Figure C.2. The map ψ sending $\boldsymbol{z} \in U$ to $\psi(\boldsymbol{z}) = (a(\boldsymbol{z}), \rho(\boldsymbol{z}))$ is the desired diffeomorphism, where $\rho(\boldsymbol{z})$ is the projection of $\boldsymbol{z} - a(\boldsymbol{z})$ onto the affine set $S'_{a(\boldsymbol{z})}$, because $\rho(\boldsymbol{z}) \in N_{a(\boldsymbol{z})}\mathcal{V}$ by construction and the kernels of da and $d\rho$ are transverse on $\mathcal{V}$ (they are orthogonal subspaces), hence also transverse in a sufficiently small neighborhood of $\mathcal{V}$. □

Lemma C.1 implies that for any k-chain γ in $\mathcal{V}$ we can define a $(k+1)$-chain $\circ\gamma$, which we call a ***tube around*** γ, by taking the union of small circles in the fibers of the normal bundle with centers in γ. The radii of these disks should be positive and small enough to fit into the domain of the collar map, but can (continuously) vary with the point on the base. Different choices of the radii of these circles lead to homologous tubes. Similarly, we let $\bullet\gamma$ denote the union of solid disks in the fibers of the normal bundle with centers in γ. The elementary rules for boundaries of products imply

$$\partial(\circ\gamma) = \circ(\partial\gamma) \quad \text{and} \quad \partial(\bullet\gamma) = \circ\gamma \cup \bullet(\partial\gamma)\,.$$

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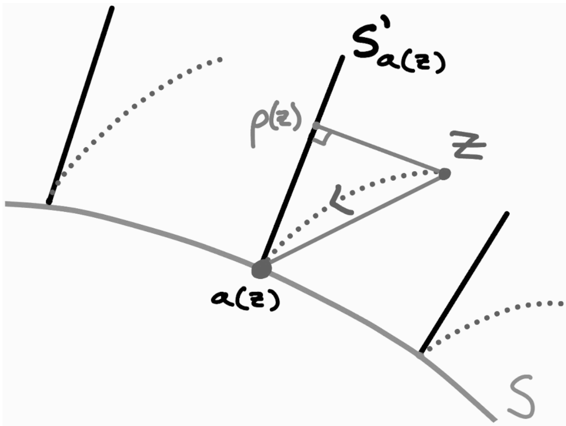

Figure C.2 Integral curves (dotted) making up the normal bundle, and a decomposition of z into $(a(z), \rho(z))$.

Because $\circ$ commutes with ∂, cycles map to cycles, boundaries map to boundaries, and the map $\circ$ on the singular chain complex of $\mathcal{V}$ induces a map from $H_{k-1}(\mathcal{V}_*)$ to $H_k(\mathcal{M})$, where $\mathcal{M} = \mathbb{C}_*^d \setminus \mathcal{V}$. To simplify notation, we also denote this map on homology by $\circ$.

We are now ready to define intersection classes after recalling a few constructions from differential geometry. Two submanifolds $A, B \subset \mathbb{C}^d$ are said to ***intersect transversely*** if for all $w \in A \cap B$ the tangent spaces of A and B at w jointly span $\mathbb{C}^d$. Two classic results of differential geometry state that if A and B intersect transversely then $A \cap B$ is a manifold, and that if B is fixed and A is any manifold then A can be slightly perturbed into a manifold A' that intersects B transversely (i.e., transversality is a *generic* property) – see [Hir76, Chapter 3], for instance.

Theorem C.2 (intersection classes)**.** *Define* $\circ : H_{d-1}(\mathcal{V}) \to H_d(\mathcal{M})$ *as above. Under our running assumption that* $\mathcal{V}$ *is a manifold,*

(i) $\circ$ *is injective and its image is the kernel of the map* ι_* *induced by the inclusion* $\mathcal{M} \xrightarrow{\iota} \mathbb{C}_*^d$.

(ii) Given $\alpha \in \ker(\iota_*)$ *one may compute the inverse* $\mathcal{I}(\alpha) = \circ^{-1}(\alpha)$ *by intersecting* $\mathcal{V}_*$ *with any* $(d+1)$*-chain in* $\mathbb{C}_*^{d+1}$ *whose boundary is* α *and for which the intersection with* $\mathcal{V}_*$ *is transverse.*

When $\alpha = C - C'$ in Theorem C.2, where C and C' are two d-cycles in $\mathcal{M}$ homologous in $\mathbb{C}_*^d$, we call $\mathbf{INT}[C, C'; \mathcal{V}] = \mathcal{I}(C - C')$ the ***intersection class*** of C and C'. We usually use the intersection class when $C = T$ and $C' = T'$ are

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tori and T' can be deformed to points where the Cauchy integral representing a multivariate sequence is asymptotically negligible.

Proof of Theorem C.2 The *Thom–Gysin long exact sequence* implies exactness of a sequence

$$0 \to H_{d-1}(\mathcal{V}_*) \xrightarrow{\circ} H_d(\mathcal{M}) \to H_d(\mathbb{C}_*^d). \tag{C.1.1}$$

This may be found in [Gor75, page 127] (where, in the notation of that source, $W = \mathbb{C}_*^d$) though in the particular situation at hand it goes back to Leray [Ler59]. Injectivity of $\circ$ follows from exactness at $H_{d-1}(\mathcal{V}_*)$ while the rest of part (*i*) follows from exactness at $H_d(\mathcal{M})$.

For part (*ii*), we begin by showing that the map I which takes a subset S of $\mathbb{C}_*^d$ transverse to $\mathcal{V}_*$ and returns $I(S) = S \cap \mathcal{V}_*$ induces a well-defined map from $\ker(\iota_*)$ to $H_{d-1}(\mathcal{V}_*)$. Because transversality is generic, given any $\alpha \in \ker(\iota_*)$ there exist $(d+1)$-chains intersecting $\mathcal{V}_*$ transversely whose boundary is α. If $\mathcal{D}$ is such a chain and $C = I(\mathcal{D})$ then C is a cycle, since

$$\partial C = \partial(\mathcal{D} \cap \mathcal{V}_*) = (\partial \mathcal{D}) \cap \mathcal{V}_* = \alpha \cap \mathcal{V}_* = \emptyset\,.$$

Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two such chains, and define $C_j = \mathcal{D}_j \cap \mathcal{V}_*$. Observe that $\mathcal{D}_1 - \mathcal{D}_2$ is null homologous because there is no $(d+1)$-homology in $\mathbb{C}_*^d$. Thus, $\mathcal{D}_1 - \mathcal{D}_2 = \partial\,\mathbf{H}$ for some $(d+1)$-chain $\mathbf{H}$ in $\mathbb{C}_*^d$. Choosing $\mathbf{H}$ transverse to $\mathcal{V}_*$,

$$C_1 - C_2 = I(\mathcal{D}_1 - \mathcal{D}_2) = \partial(\mathbf{H} \cap \mathcal{V}_*)$$

is a boundary in $\mathcal{V}_*$. Thus, the class $[I(\mathcal{D})]$ in $H_{d-1}(\mathcal{V}_*)$ is the same for any $\mathcal{D}$ with $\partial\mathcal{D} = \alpha$. If $\alpha = \circ(\gamma)$ then taking $\mathcal{D} = \bullet(\gamma)$ gives $I(\mathcal{D}) = \gamma$, showing that I does in fact invert $\circ$ and thus computes $\mathcal{I}$. □

Remark C.3. Because $\mathbb{C}_*$ is topologically a circle, the homology group $H_d(\mathbb{C}_*^d)$ is cyclic and generated by a product of small circles about the coordinate axes, and $H_k(\mathbb{C}_*^d)$ vanishes for $k > d$. The kernel of ι_* consists of the classes that don't *link* the origin in $\mathbb{C}_*^d$, i.e., the classes σ for which the integer invariant $\ell(\sigma) = (2\pi i)^{-d} \int_\sigma dz_1/z_1 \wedge \cdots \wedge dz_d/z_d$ vanishes. This holds, for example, if $\sigma = T - T'$, where T and T' are standard oriented tori around the origin, since $\ell(T) = \ell(T') = 1$.

There is a version of the intersection class in relative homology as well. This will be useful to us when we integrate over a difference of tori, one being the starting domain of integration in the Cauchy integral and the other being a “large” torus, because it helps us ignore whether we have chosen a large enough torus to avoid $\mathcal{V}_*$ at points that are asymptotically negligible. We omit the proof of this construction, which is similar to the proof of Theorem C.2.

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Corollary C.4. *Let Y be a closed subspace of $\mathbb{C}^d_*$ and let α and β be relative cycles in the pair $(\mathcal{M}, \mathcal{M} \cap Y)$ that are homologous in $(\mathbb{C}^d_*, Y)$. There is a well-defined intersection class $\mathbf{INT}[\alpha,\beta;\mathcal{V}]_Y \in H_{d-1}(\mathcal{V}_*, Y)$ such that if $\mathbf{H}$ is any $(d+1)$-chain in $\mathbb{C}^d_*$ with $\partial\mathbf{H} = \alpha' - \beta' + \gamma$, where $[\alpha'] = [\alpha]$ and $[\beta'] = [\beta]$ in $H_d(\mathcal{M}, \mathcal{M} \cap Y)$ and $\gamma \in Y$, and if $\mathbf{H}$ intersects $\mathcal{V}$ transversely, then $\mathbf{H} \cap \mathcal{V}$ is a relative cycle in the class $\mathbf{INT}[\alpha,\beta;\mathcal{V}]_Y$.* □

By the excision property of homology, the pair $(\mathcal{M}, \mathcal{M} \cap Y)$ is homotopy equivalent to the pair $(\mathcal{M} \setminus Y^\circ, \partial Y)$. This allows us to extend Corollary C.4 to the case where α and β can intersect $\mathcal{V}$, but only in the interior of Y.

Corollary C.5. *Let Y be a closed subspace of $\mathbb{C}^d_*$ and let α and β be relative cycles homologous in $(\mathbb{C}^d, Y)$ intersecting $\mathcal{V}$ only in the interior of Y. There is a well-defined intersection class $\mathbf{INT}[\alpha,\beta;\mathcal{V}]_Y \in H_{d-1}(\mathcal{V}_*, \mathcal{V}_* \cap Y)$, depending only on the class of $\alpha - \beta$ in $H_d(\mathcal{M}, \mathcal{M} \cap Y)$, such that if $\mathbf{H}$ is a $(d+1)$-chain in $\mathbb{C}^d_*$ with $\partial\mathbf{H} = \alpha - \beta + \gamma$, where γ is supported on the interior of Y, and if the intersection of $\mathbf{H}$ with $\mathcal{V}$ is transverse away from the interior of Y, then $\mathbf{H} \cap \mathcal{V}$ is a relative cycle representing the class $\mathbf{INT}[\alpha,\beta;\mathcal{V}]_Y$.* □

In the special case of Corollary C.5, where $\beta = 0$ and Y is the set of points at height c or less, we denote the ***relative intersection class*** by $\mathbf{INT}[\alpha;\mathcal{V}]_{\leq c}$.

C.2 Residue forms and the residue integral theorem

Integrating a differential form over a difference of chains can often be reduced to integrating a *residue form* over an intersection cycle. Because residues depend on local behavior, we work with subsets of $\mathbb{C}^d$ that are *locally* defined by analytic functions. An ***analytic hypersurface*** (or simply hypersurface) is a set $\mathcal{V} \subset \mathbb{C}^d$ such that for any $\mathbf{w} \in \mathcal{V}$ and any sufficiently small neighborhood $\mathcal{D}$ of $\mathbf{w}$ in $\mathbb{C}^d$ there is an analytic function $Q_{\mathcal{D}}$ on $\mathcal{V} \cap \mathcal{D}$ such that $\mathcal{V} \cap \mathcal{D} = \{\mathbf{z} \in \mathcal{D} : Q_{\mathcal{D}}(\mathbf{z}) = 0\}$. If the function $Q_{\mathcal{D}}$ can be chosen to have nonvanishing gradient on $\mathcal{V} \cap \mathcal{D}$ then we say $\mathcal{V}$ is a ***smooth analytic hypersurface at $\mathbf{w}$***, and we call $\mathcal{V}$ a ***smooth analytic hypersurface*** if it is a smooth analytic hypersurface at every point.

Although the theory of multivariate residues is much more involved than its univariate counterpart, we require it only for differential forms whose singularities lie on unions of smooth analytic hypersurfaces. We build our results in four degrees of generality, starting with forms having smooth simple poles, then forms with smooth higher order poles, followed by forms with transversely intersecting smooth sheets of simple poles, and concluding with forms having transversely intersecting smooth sheets with higher order poles.

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C.2.1 Residue forms for smooth simple poles

Fix a smooth analytic hypersurface $\mathcal{V}$ defined locally by analytic functions $Q_{\mathcal{D}}$ as above, and recall our notation $\mathcal{M} = \mathbb{C}_*^d \setminus \mathcal{V}$ and $d\boldsymbol{z} = dz_1 \wedge \cdots \wedge dz_d$. We say that a d-form $\omega \in E^d(\mathcal{M})$ has ***smooth poles of order*** k on $\mathcal{V}$ if for any $\boldsymbol{a} \in \mathcal{V}$ there is a sufficiently small neighborhood $\mathcal{D}$ of $\boldsymbol{a}$ in $\mathbb{C}_*^d$ such that $Q_{\mathcal{D}}(\boldsymbol{z})^k \omega$ extends to a holomorphic form on $\mathcal{D}$ but $Q_{\mathcal{D}}(\boldsymbol{z})^j \omega$ does not extend to such a holomorphic form for any $0 \leq j < k$. A form with smooth poles of order one is said to have *simple poles*.

Proposition C.6. *Let ω be a holomorphic d-form with smooth simple poles on $\mathcal{V}$, represented as a quotient $\omega = P(\boldsymbol{z})/Q(\boldsymbol{z})\, d\boldsymbol{z}$ of analytic functions on $\mathcal{M} \cap \mathcal{D}$ for some domain $\mathcal{D} \subset \mathbb{C}^d$ on which the gradient of Q does not vanish. If $\mathcal{W} = \mathcal{V} \cap \mathcal{D}$ and $\iota : \mathcal{W} \hookrightarrow \mathcal{D}$ is the inclusion of $\mathcal{W}$ into $\mathbb{C}^d$ then there is a $(d-1)$-form θ on $\mathcal{D}$ solving $dQ \wedge \theta = P\, d\boldsymbol{z}$ and any such solution restricts to a unique $(d-1)$-form $\operatorname{Res}(\omega) = \iota^*\theta$ on $\mathcal{W}$ called the **residue of** ω on $\mathcal{W}$.*

Remark C.7. Our definition of the residue is both ***natural***, meaning it does not depend on the particular polynomials P and Q used to represent $\omega = P(\boldsymbol{z})/Q(\boldsymbol{z})\, d\boldsymbol{z}$ on $\mathcal{D}$, and ***functorial***, meaning $\operatorname{Res}(f^*\omega) = f^* \operatorname{Res}(\omega)$ for smooth functions f. Because the residue is natural, we can define $\operatorname{Res}(\omega)$ on all of $\mathcal{V}$ by defining it locally over the elements of a cover of $\mathcal{V}$ by sufficiently small domains using Proposition C.6.

Proof We must show both that $dQ \wedge \theta = P\, d\boldsymbol{z}$ always has a holomorphic solution, and that the restriction of any such solution to $\mathcal{V}$ is unique. Uniqueness follows from Exercise A.18 at the end of Appendix A: if θ_1 and θ_2 are two solutions then $dQ \wedge (\theta_1 - \theta_2) = 0$, hence Exercise A.18 implies $\iota^*\theta_1 = \iota^*\theta_2$. The existence of a solution follows from the following proposition, which expresses the residue explicitly in local coordinates in sufficiently small neighborhoods, by combining residues in local neighborhoods as discussed in Remark C.7. □

Proposition C.8. *Under the hypotheses of Proposition C.6, if the partial derivative $\partial Q/\partial z_k$ is nonvanishing on $\mathcal{D}$ for some fixed $k \leq d$ and $\boldsymbol{r} \in \mathbb{Z}^d$ then*

$$\operatorname{Res}\left(\boldsymbol{z}^{-\boldsymbol{r}}\omega\right) = (-1)^{k-1} \frac{\boldsymbol{z}^{-\boldsymbol{r}} P(\boldsymbol{z})}{Q_{z_k}(\boldsymbol{z})} d\boldsymbol{z}_{\hat{k}}\,, \tag{C.2.1}$$

where $d\boldsymbol{z}_{\hat{k}} = dz_1 \wedge \cdots \wedge dz_{k-1} \wedge dz_{k+1} \wedge \cdots \wedge dz_d$.

Proof If $k = 1$ and θ is the right-hand side of (C.2.1), then

$$dQ \wedge \theta = \left(\sum_{j=1}^{d} Q_{z_j}(\boldsymbol{z}) dz_j\right) \wedge \left(\frac{\boldsymbol{z}^{-\boldsymbol{r}} P}{Q_{z_1}(\boldsymbol{z})}\, dz_2 \wedge \cdots \wedge dz_d\right) = \boldsymbol{z}^{-\boldsymbol{r}} P\, d\boldsymbol{z}\,,$$

as desired. In the general case, the sign $(-1)^{k-1}$ comes from the position of dz_k in the wedge product. □

Exercise C.1. Let $\omega = 1/Q(x,y)\,dxdy$, where $Q(x,y) = 1 - x - xy + y^2$. Find a formula for $\mathrm{Res}(\omega)$ on $\mathcal{V} = \mathcal{V}_Q$ in terms of dx only, and another in terms of dy only. Prove that the restrictions of these forms to $\mathcal{V}$ are equal.

Theorem C.9 (residue integral theorem)**.** *Suppose $\mathcal{V} = \mathcal{V}_Q = \{\boldsymbol{z} \in \mathbb{C}^d : Q(\boldsymbol{z}) = 0\}$ is defined globally by a function Q that is analytic on a neighborhood of $\mathcal{V}$ and has nonvanishing gradient on $\mathcal{V}$, and let ω be a holomorphic d-form on $\mathcal{M}$ with smooth simple poles on $\mathcal{V}$. If α and β are d-cycles in $\mathcal{M}$ whose projections to $H_d(\mathbb{C}_*^d)$ are equal then*

$$\int_\alpha \omega - \int_\beta \omega = 2\pi i \int_{\mathbf{INT}[\alpha,\beta;\mathcal{V}]} \mathrm{Res}(\omega)\,.$$

Proof Vanishing of $[\alpha - \beta]$ in $H_d(\mathbb{C}_*^d)$ by definition implies the existence of a $(d+1)$-chain $\mathbf{H}$ on $\mathbb{C}_*^d$ with boundary $\alpha - \beta$. Perturbing slightly if necessary, we can assume without loss of generality that $\mathbf{H}$ intersects $\mathcal{V}$ transversely. Letting N denote the intersection of $\mathbf{H}$ with a small neighborhood of $\mathcal{V}$ and $\Theta = \mathbf{H} - N$, the vanishing of holomorphic integrals of d-forms over boundaries (Theorem A.27) implies that the integral of the holomorphic d-form ω over $\partial\Theta$ vanishes. In other words,

$$\int_\alpha \omega - \int_\beta \omega = \int_{\partial N} \omega\,.$$

The Collar Lemma (Lemma C.1) implies that N is homotopic to a product $\sigma \times B_\varepsilon$, where $\sigma = \mathbf{H} \cap \mathcal{V}$. Thus ∂N is homotopic to $\partial(\sigma \times B_\varepsilon)$, which is equal to $\sigma \times \partial B_\varepsilon$ because σ is a cycle, giving

$$\int_{\sigma \times \partial B_\varepsilon} \omega = \int_\sigma \left(\int_{\partial B_\varepsilon} \omega \right).$$

Using functoriality of the residue, we may change coordinates so that $\mathcal{V}$ is the complex hyperplane defined by $z_1 = 0$. Thus we need only prove our claim in the case where $Q(\boldsymbol{z}) = z_1$. Writing $\omega = (P/z_1)dz_1 \wedge (dz_2 \wedge \cdots \wedge dz_d)$, the iterated integral is

$$\int_\sigma \left[\int_{\partial B_\varepsilon} \frac{P(\boldsymbol{z})}{z_1}\,dz_1 \right] dz_2 \wedge \cdots \wedge dz_d\,.$$

By standard univariate complex analysis, the inner integral at a point $(z_2, \ldots, z_d)$ is the residue with respect to t of the meromorphic function $P(t, z_2, \ldots, z_d)/t$ at the pole $(0, z_2, \ldots, z_d)$. This is equal to

$$(2\pi i) \int_\sigma P(0, z_2, \ldots, z_d)\,,$$

which in this special case is precisely $\int_\sigma \text{Res}(\omega)$. □

There is also a relative version of this result.

Theorem C.10 (relative residue integral theorem)**.** *Let $\mathcal{V}$ and ω be as in Theorem C.9. If Y is any closed subspace of $\mathbb{C}_*^d$ such that $H_d(\mathbb{C}_*^d, Y)$ vanishes, and if α is a d-cycle in $\mathcal{M}$, then*

$$\int_\alpha \omega = 2\pi i \int_{\mathbf{INT}[\alpha,0;\mathcal{V}]} \text{Res}(\omega) + \int_{C'} \omega \qquad \text{(C.2.2)}$$

for some chain C' supported on the interior of Y. In particular, if $\omega = \boldsymbol{z}^{-\boldsymbol{r}}\eta$ for some holomorphic form η on $\mathcal{M}$ and if Y is the set where the real part of $h_{\hat{\boldsymbol{r}}} = -\hat{\boldsymbol{r}} \cdot \log \boldsymbol{z}$ is at most c then, as $\lambda \to \infty$,

$$\int_\alpha \omega = 2\pi i \int_{\mathbf{INT}[\alpha,0;\mathcal{V}]} \text{Res}(\omega) + O\left(e^{\lambda c'}\right) \qquad \text{(C.2.3)}$$

for any $c' > c$.

Proof By the vanishing of $H_d(\mathbb{C}_*^d, Y)$ there is a $(d+1)$-chain $\mathbf{H}$ with $\partial\mathbf{H} = \alpha + \gamma$ and γ supported on the interior of Y. Let N denote the intersection of $\mathbf{H}$ with a neighborhood of $\mathcal{V}$. As before,

$$\int_\alpha \omega = \int_\gamma \omega + \int_{\partial N} \omega\,.$$

Letting $\sigma = \mathbf{H} \cap \mathcal{V}$, we recall that ∂N is homotopic to $\sigma \times B_\varepsilon$ plus a piece γ' in the interior of Y. Taking $C' = \gamma + \gamma'$, the rest of the proof of (C.2.3) is the same as that of Theorem C.9. The asymptotic estimate follows because $\left|\int_{C'} \omega\right| \le e^{\lambda c} \int_{C'} |\eta|$, as in the proof of Proposition B.10. □

C.2.2 Residue forms on smooth higher order poles

Let ω and ω' be holomorphic d-forms on $\mathcal{M}$ with simple poles on the smooth variety $\mathcal{V} = \mathcal{V}_Q$. If ω and ω' are cohomologous in $H^d(\mathcal{M})$ then $[\text{Res}(\omega)] = [\text{Res}(\omega')]$ in $H^{d-1}(\mathcal{V}_*)$, and the study of residue classes of forms with smooth higher order poles can be reduced to those with smooth simple poles using this property.

Lemma C.11 (Gelfand–Shilov reduction)**.** *If ω is a holomorphic d-form on $\mathcal{M}$ with smooth poles of order $k \geq 2$ on $\mathcal{V}$ and the representation $\omega = P(\mathbf{z})/Q(\mathbf{z})^k \, d\mathbf{z}$ holds on a domain $\mathcal{D}$ then*

$$\begin{aligned}\omega &= \frac{dQ}{Q^k} \wedge \psi + \frac{\theta}{Q^{k-1}} \\ &= d\left(\frac{-\psi}{(k-1)Q^{k-1}}\right) + \frac{\theta_1}{Q^{k-1}}\end{aligned}$$

for some holomorphic forms ψ and θ, where $\theta_1 = \theta + d\psi/(k-1)$. Thus, any d-form on $\mathcal{M}$ with smooth poles of order k is cohomologous to a d-form on $\mathcal{M}$ with smooth poles of order $k-1$.

Proof See [AY83, Lemma 17.1]. □

If ω is any d-form on $\mathcal{M}$ with smooth poles then Lemma C.11 implies ω is cohomologous to a d-form ω' on $\mathcal{M}$ with smooth simple poles, and we define the ***residue class*** $[\mathrm{Res}(\omega)]$ of ω to be the class $[\mathrm{Res}(\omega')] \in H^{d-1}(\mathcal{V}_*)$. To simplify notation we usually write $\mathrm{Res}(\omega)$ for the class $[\mathrm{Res}(\omega)]$. As our integrals of residues depend only on their cohomology classes, there is no harm in this abuse of notation. This inductive definition gives the following corollary of Theorem C.9.

Corollary C.12. *Suppose the assumptions of Theorem C.9 hold, except that ω can have smooth poles of any order on $\mathcal{V}$. If α and β are d-cycles in $\mathcal{M}$ whose projections to $H_d(\mathbb{C}^d_*)$ are equal then the identity*

$$\int_\alpha \omega - \int_\beta \omega = 2\pi i \int_{\mathbf{INT}[\alpha,\beta;\mathcal{V}]} \mathrm{Res}(\omega)$$

still holds.

Just as for smooth poles, there is an explicit formula for the residue of a form with higher order poles. We state the following theorem for the types of integrands that arise in our asymptotic analyses.

Lemma C.13. *Let $d\mathbf{z}_{\hat{k}}$ denote the $(d-1)$-form $dz_1 \wedge \cdots \wedge dz_{k-1} \wedge dz_{k+1} \wedge \cdots \wedge dz_d$. Wherever the functions $P(\mathbf{z})\mathbf{z}^{-\mathbf{r}}$ and $Q(\mathbf{z})$ are analytic and the partial derivative $Q_{z_k}(\mathbf{z})$ does not vanish,*

$$\mathrm{Res}\left(\mathbf{z}^{-\mathbf{r}} \frac{P(\mathbf{z})}{Q(\mathbf{z})^\ell}\, d\mathbf{z}\right) = \mathbf{z}^{-\mathbf{r}} \Phi_{r_k}(\mathbf{z}) \tag{C.2.4}$$

for a polynomial

$$\Phi_{r_k}(\mathbf{z}) = \left[(-1)^{k-1}\binom{-r_k}{\ell-1} z_k^{-(\ell-1)} \frac{P(\mathbf{z})}{Q_{z_k}(\mathbf{z})^\ell} + O\left(r_k^{\ell-2}\right)\right] d\mathbf{z}_{\hat{k}}$$

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in r_k of degree $\ell - 1$ whose coefficients are analytic functions of $\boldsymbol{z}$ explicitly given in terms of derivatives of P and Q.

Proof We induct on ℓ, with the case $\ell = 1$ handled by Proposition C.8. Assume for an induction that the lemma holds for $\ell - 1$. Because the residue of an exact form is zero, we let

$$\eta = (-1)^{k-1} \boldsymbol{z}^{-\boldsymbol{r}} \frac{P(\boldsymbol{z})}{(\ell - 1) Q(\boldsymbol{z})^{\ell-1} Q_{z_k}(\boldsymbol{z})} d\boldsymbol{z}_{\hat{k}}$$

and examine

$$\begin{aligned} 0 &= \operatorname{Res}(d\eta) \\ &= \operatorname{Res}\left[\boldsymbol{z}^{-\boldsymbol{r}} \frac{P_{z_k}(\boldsymbol{z})}{(\ell - 1) Q(\boldsymbol{z})^{\ell-1} Q_{z_k}(\boldsymbol{z})} d\boldsymbol{z} + \boldsymbol{z}^{-\boldsymbol{r}} \frac{-r_k P(\boldsymbol{z}) z_k^{-1}}{(\ell - 1) Q(\boldsymbol{z})^{\ell-1} Q_{z_k}(\boldsymbol{z})} d\boldsymbol{z} \right. \\ &\qquad \left. - \boldsymbol{z}^{-\boldsymbol{r}} \frac{P(\boldsymbol{z})}{Q(\boldsymbol{z})^{\ell}} d\boldsymbol{z} - \boldsymbol{z}^{-\boldsymbol{r}} \frac{P(\boldsymbol{z}) Q_{z_k, z_k}(\boldsymbol{z})}{(\ell - 1) Q(\boldsymbol{z})^{\ell-1} Q_{z_k}(\boldsymbol{z})} d\boldsymbol{z} \right]. \end{aligned}$$

Isolating the third term on the right yields

$$\begin{aligned} \operatorname{Res}\left(\boldsymbol{z}^{-\boldsymbol{r}} \frac{P(\boldsymbol{z})}{Q(\boldsymbol{z})^{\ell}} \, d\boldsymbol{z} \right) &= \operatorname{Res}\left(\boldsymbol{z}^{-\boldsymbol{r}} \frac{-r_k P(\boldsymbol{z}) z_k^{-1}}{(\ell - 1) Q(\boldsymbol{z})^{\ell-1} Q_{z_k}(\boldsymbol{z})} \, d\boldsymbol{z} \right) \\ &\quad + \operatorname{Res}\left(\boldsymbol{z}^{-\boldsymbol{r}} \frac{A(\boldsymbol{z})}{Q(\boldsymbol{z})^{\ell-1}} \, d\boldsymbol{z} \right), \end{aligned} \tag{C.2.5}$$

for an analytic function A independent of r_k. Applying the induction hypothesis to the first residue on the right-hand side of (C.2.5) shows that it equals

$$(-1)^{k-1} \left[\frac{-r_k}{\ell - 1} \binom{-r_k - 1}{\ell - 2} \boldsymbol{z}^{-\boldsymbol{r}} z_k^{-(\ell-2)} \frac{P(\boldsymbol{z}) z_k^{-1} / Q_{z_k}(\boldsymbol{z})}{Q_{z_k}(\boldsymbol{z})^{\ell-1}} + O\left(r_k^{\ell-3} \right) \right] d\boldsymbol{z}_{\hat{k}} \, ,$$

while applying the induction hypothesis to the second residue on the right-hand side of (C.2.5) proves that it is $O\left(r_k^{\ell-2}\right)$. Combining powers of $Q_{z_k}(\boldsymbol{z})$ and powers of z_k, and simplifying $\frac{-r_k}{\ell - 1} \binom{-r_k - 1}{\ell - 2} = \binom{-r_k}{\ell - 1}$, then gives the stated result. □

C.2.3 Iterated residue forms for simple poles on transverse sheets

In this section we summarize a generalization of residue forms to the case where $\mathcal{V}$ is the union of a finite number of smooth analytic hypersurfaces that intersect transversely. A full treatment of residues for forms with transverse poles can be found in [AY83, Section 16.5].

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Definition C.14. If $\mathcal{V} \subset \mathbb{C}^d$ is an analytic hypersurface then we call $\boldsymbol{w} \in \mathcal{V}$ a ***transverse multiple point*** of $\mathcal{V}$ if there exists a neighborhood $\mathcal{D}$ of $\boldsymbol{w}$ in $\mathbb{C}^d$ such that $\mathcal{D} \cap \mathcal{V} = \mathcal{D} \cap (\mathcal{V}_{Q_1} \cup \cdots \cup \mathcal{V}_{Q_k})$ for smooth analytic hypersurfaces $\{\mathcal{V}_{Q_j} : 1 \le j \le k\}$ defined by analytic functions $Q_j(\boldsymbol{z})$ whose gradients at $\boldsymbol{z} = \boldsymbol{w}$ are linearly independent. When this collection of analytic functions is understood and $\boldsymbol{m} \in \mathbb{N}^k$ then we write $\boldsymbol{Q}(\boldsymbol{z})^{\boldsymbol{m}} = Q_1(\boldsymbol{z})^{m_1} \cdots Q_k(\boldsymbol{z})^{m_k}$. If every point of $\mathcal{V}$ is a transverse multiple point then we call $\mathcal{V}$ a ***transverse analytic hypersurface***.

Example C.15. Every smooth analytic hypersurface is a transverse analytic hypersurface. ◁

Fix a transverse analytic hypersurface $\mathcal{V}$ and let ω be a d-form on $\mathcal{M} = \mathbb{C}^d \setminus \mathcal{V}$. We say that ω has a ***transverse pole*** (or ***transverse multiple point***) of order $\boldsymbol{m} \in \mathbb{N}^k$ at $\boldsymbol{w} \in \mathcal{V}$ if

- there exists a neighborhood $\mathcal{D}$ of $\boldsymbol{w}$ in $\mathbb{C}^d$ and analytic functions $Q_1, \dots, Q_k$ on $\mathcal{D}$ such that $\mathcal{D} \cap \mathcal{V} = \mathcal{D} \cap (\mathcal{V}_{Q_1} \cup \cdots \cup \mathcal{V}_{Q_k})$ and the gradients of the Q_i are linearly independent at $\boldsymbol{w}$ (in particular, they are all non-zero),
- there exists an analytic function P on $\mathcal{D}$ such that $\omega = P(\boldsymbol{z})/\boldsymbol{Q}(\boldsymbol{z})^{\boldsymbol{m}} d\boldsymbol{z}$ when $\boldsymbol{z} \in \mathcal{D} \cap \mathcal{M}$, and
- there is no possible choice of $\boldsymbol{Q}$ and P such that these properties hold with any coordinate of $\boldsymbol{m}$ decreased.

A transverse multiple point of order $\mathbf{1}$ is called a ***transverse simple pole*** (or ***transverse simple point***). The final item in this definition implies that the numerator P and denominator factors Q_k in the local representation of ω are coprime in the ring of germs of analytic functions, ensuring that $\boldsymbol{m}$ is the correct notion of order (no unwanted cancellation can occur).

Let $\boldsymbol{p}$ be a transverse simple pole of the d-form ω, with the local representation

$$\omega = \frac{P(\boldsymbol{z})}{Q_1(\boldsymbol{z}) \cdots Q_k(\boldsymbol{z})} d\boldsymbol{z}$$

in some neighborhood of $\boldsymbol{p}$. To simplify notation we write $\mathcal{V}_i = \mathcal{V}_{Q_i}$ and let $\mathcal{S} = \bigcap_{i=1}^k \mathcal{V}_i$ be the ***stratum*** of $\mathcal{V}$ containing $\boldsymbol{p}$. Because the gradients of the Q_i are linearly independent at $\boldsymbol{p}$, there exist coordinates $\boldsymbol{\pi} = \{\pi_1, \dots, \pi_{d-k}\}$ that locally analytically parametrize $\mathcal{S}$ near $\boldsymbol{p}$. In particular, writing $\boldsymbol{z}_{\boldsymbol{\pi}} = (z_{\pi_1}, \dots, z_{\pi_{d-k}})$ there exists a neighborhood $\mathcal{D}$ of $\boldsymbol{p}$ in $\mathbb{C}^d$ and analytic functions $\zeta_i(\boldsymbol{z}_{\boldsymbol{\pi}})$ on $\mathcal{D}$ for $i \notin \boldsymbol{\pi}$ such that $\boldsymbol{z} \in \mathcal{D}$ lies in $\mathcal{S}$ if and only if $z_i = \zeta_i(\boldsymbol{z}_{\boldsymbol{\pi}})$ for all $i \notin \boldsymbol{\pi}$.

As detailed later in these appendices, if $\mathcal{D}$ is sufficiently small then $\mathcal{M} \cap \mathcal{D}$ has a local product structure $\tilde{N} \times \mathcal{S}$. Because $\boldsymbol{p}$ is a transverse multiple point

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with k sheets, the factor $\tilde{N}$ is homotopy equivalent to a k-torus and we can represent the homology of $\tilde{N}$ using a product of k circles around $\boldsymbol{p}$. To make this explicit, we note that the map $\Psi : \mathcal{D} \to \mathbb{C}^d$ defined by

$$\Psi(\boldsymbol{z}) = (Q_1(\boldsymbol{z}), \ldots, Q_k(\boldsymbol{z}), z_{\pi_1} - p_{\pi_1}, \ldots, z_{\pi_{d-k}} - p_{\pi_{d-k}}) \tag{C.2.6}$$

is a bi-analytic change of coordinates taking $\mathcal{D} \cap \mathcal{S}$ to a neighborhood of the origin in $\{\mathbf{0}\} \times \mathbb{C}^{d-k}$. Let $T_\varepsilon \subseteq \mathbb{C}^k \times \{\mathbf{0}\}$ denote the product of circles of radius ε in each of the first k coordinates. If ε is sufficiently small then $T_\varepsilon \subset \Psi(\mathcal{D})$ and the cycle $T = \Psi^{-1}(T_\varepsilon)$ will be a generator for $H_k(\tilde{N})$. We give $\mathcal{D}$ the local product structure that Ψ^{-1} induces from the product structure on $\mathbb{C}^d$.

Definition C.16. If f is a differentiable function then the ***logarithmic gradient*** of f at $\boldsymbol{z}$ is

$$(\nabla_{\log} f)(\boldsymbol{z}) := \left(z_1 f_{z_1}(\boldsymbol{z}), \ldots, z_d f_{z_d}(\boldsymbol{z})\right).$$

For each $\boldsymbol{z} \in \mathcal{S}$, the ***augmented lognormal matrix*** is the $d \times d$ matrix

$$\mathbf{\Gamma}_\Psi(\boldsymbol{z}) = \begin{pmatrix} (\nabla_{\log} Q_1)(\boldsymbol{z}) \\ \vdots \\ (\nabla_{\log} Q_k)(\boldsymbol{z}) \\ z_{\pi_1} \boldsymbol{e}_{\pi_1} \\ \vdots \\ z_{\pi_{d-k}} \boldsymbol{e}_{\pi_{d-k}} \end{pmatrix},$$

where $\boldsymbol{e}_j$ denotes the jth elementary basis vector. Equivalently, $\mathbf{\Gamma}_\Psi = J_\Psi D$ where D is the diagonal matrix with entries $z_1, \ldots, z_d$ and J_Ψ is the Jacobian matrix of the map Ψ.

Remark. The definition of $\mathbf{\Gamma}_\Psi$ depends on the choice of factorization, each factor being determined only up to a complex multiple. Suitable normalizations are assumed later in the definition of the torus $\mathcal{T}$ following (10.3) and the determination of the orientation in the proof of Theorem 10.25.

Theorem C.17 (iterated residues)**.** *Under the setup discussed above, let ω be the holomorphic d-form $\omega = \frac{P(\boldsymbol{z})}{\prod_{j=1}^k Q_j(\boldsymbol{z})} d\boldsymbol{z}$ on $\mathcal{M} \cap \mathcal{D}$ and write $\mathcal{S}_\mathcal{D} = \mathcal{S} \cap \mathcal{D} = \mathcal{V}_1 \cap \cdots \cap \mathcal{V}_k \cap \mathcal{D}$.*

(i) **Iterated residue is well defined.** *The restriction to $\mathcal{S}_\mathcal{D}$ of any d-form θ on $\mathcal{D}$ satisfying*

$$dQ_1 \wedge \cdots \wedge dQ_k \wedge \theta = P\, d\boldsymbol{z} \tag{C.2.7}$$

is independent of the particular solution θ.

(ii) **Formula for the iterated residue.** *Denoting the* ***iterated residue*** *defined by this restriction by* $\operatorname{Res}(\omega; \mathcal{S}_{\mathfrak{D}})$*, there is a formula*

$$\operatorname{Res}\left(\frac{P(\boldsymbol{z})}{\prod_{j=1}^{k} Q_j(\boldsymbol{z})} d\boldsymbol{z}\,;\,\mathcal{S}_{\mathfrak{D}}\right) = \left.\frac{P(\boldsymbol{z})}{\det J_{\Psi}(\boldsymbol{z})}\right|_{\substack{z_i=\zeta_i(\boldsymbol{z}_{\boldsymbol{\pi}})\\ \textit{for all } i\notin\boldsymbol{\pi}}} dz_{\pi_1} \wedge \cdots \wedge dz_{\pi_{d-k}}\,. \tag{C.2.8}$$

(iii) **Residue integral identity.** *Let* σ *be any* $(d-k)$*-chain in* $\mathcal{S}_{\mathfrak{D}}$ *and* $T = \Psi^{-1}(T_\varepsilon)$ *be as above. Then*

$$\frac{1}{(2\pi i)^k}\int_{T\times\sigma} \frac{P(\boldsymbol{z})\,d\boldsymbol{z}}{\prod_{j=1}^{k} Q_j(\boldsymbol{z})} = \int_{\sigma}\operatorname{Res}\left(\frac{P(\boldsymbol{z})}{\prod_{j=1}^{k} Q_j(\boldsymbol{z})}\,;\,\mathcal{S}_{\mathfrak{D}}\right). \tag{C.2.9}$$

(iv) **Formula for Cauchy integral.** *In particular,*

$$\frac{1}{(2\pi i)^k}\int_{T\times\sigma} \frac{\boldsymbol{z}^{-\boldsymbol{r}-\mathbf{1}}P(\boldsymbol{z})}{\prod_{j=1}^{k} Q_j(\boldsymbol{z})} d\boldsymbol{z} = \int_{\sigma} \left.\frac{\boldsymbol{z}^{-\boldsymbol{r}}P(\boldsymbol{z})}{\det \boldsymbol{\Gamma}_{\Psi}(\boldsymbol{z})}\right|_{\substack{z_i=\zeta_i(\boldsymbol{z}_{\boldsymbol{\pi}})\\ \textit{for all } i\notin\boldsymbol{\pi}}} dz_{\pi_1}\wedge\cdots\wedge dz_{\pi_{d-k}}\,. \tag{C.2.10}$$

Proof We first prove all four parts under the assumption that $Q_j(\boldsymbol{z}) = z_j$ for all $1 \le j \le k$. Setting $\pi_i = k+i$ for all $1 \le i \le d-k$, the form $\theta = P(\boldsymbol{z})\,dz_{k+1} \wedge \cdots \wedge dz_d$ satisfies (C.2.7). As in the proof of Proposition C.6 above, the result of Exercise A.17 implies that $\iota^*\theta$ is well defined, yielding (i). The formula (C.2.8) is also evident in this case: J_Ψ is the identity matrix, hence (C.2.8) agrees with our choice of θ after setting $z_i = 0$ for $1 \le i \le k$, proving (ii). For (iii), we write the left-hand side as an iterated integral

$$\frac{1}{(2\pi i)^k}\int_{\sigma}\int_{\gamma_1}\cdots\int_{\gamma_k} \frac{P(\boldsymbol{z})\,d\boldsymbol{z}}{\prod_{j=1}^{k} Q_j(\boldsymbol{z})}\,,$$

where γ_j is the circle of radius ε about the origin in the jth coordinate. Applying the univariate residue theorem to each of the inner k integrals leaves

$$\int_{\sigma} P(\boldsymbol{z})\,dz_{k+1} \wedge \cdots \wedge dz_d\,,$$

proving (iii). Finally, (iv) follows from (iii) by replacing $P(\boldsymbol{z})$ with $\boldsymbol{z}^{-\boldsymbol{r}-\mathbf{1}}P(\boldsymbol{z})$ in (C.2.8) and absorbing one factor of each z_j in the denominator when going from $\det J_\Psi$ to $\det \boldsymbol{\Gamma}_\Psi$.

For the general case, map by Ψ and use functoriality. The fact that Res is well defined and functorial follows from the same argument as in the proof of Proposition C.6. Applying the case already proved to the image space and pulling back by Ψ^{-1}, it remains only to observe that $dz_{k+1} \wedge \cdots \wedge dz_d$ pulls back to $\left.\frac{1}{\det J_\Psi(\boldsymbol{z})}\right|_{z_i=\zeta_i(\boldsymbol{z}_{\boldsymbol{\pi}})\,:\, i\notin\boldsymbol{\pi}} dz_{\pi_1}\wedge\cdots\wedge dz_{\pi_{d-k}}$, and $P(\mathbf{0})$ pulls back to $P(\boldsymbol{z})|_{z_i=\zeta_i(\boldsymbol{z}_{\boldsymbol{\pi}})\,:\, i\notin\boldsymbol{\pi}}$.

□

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Remark. The residue depends on Q_j only via its gradient. The sign of the residue form depends on the order of the factors in the denominator, and we account for this when using residue forms to determine asymptotics.

When the stratum $\mathcal{S}$ is a single point (meaning $k = d$) the residue at $\boldsymbol{p}$ is just a number, simplifying the conclusions of Theorem C.17 as follows.

Corollary C.18. *Suppose the hypotheses of Theorem C.17 hold in the special case where $k = d$, so that the residue of ω at $\boldsymbol{p}$ is a number θ_0. Then*

(i) $P(\boldsymbol{p})\, d\boldsymbol{z} = \theta_0\ (dQ_1 \wedge \cdots \wedge dQ_d)\ (\boldsymbol{p})$,

(ii) $$\operatorname{Res}\left(\frac{P(\boldsymbol{z})}{\prod_{j=1}^d Q_j(\boldsymbol{z})} d\boldsymbol{z}\, ;\, \boldsymbol{p}\right) = \frac{P(\boldsymbol{p})}{\det J_\Psi(\boldsymbol{p})},$$

(iii) $$\frac{1}{(2\pi i)^d} \int_{\Psi^{-1}(T_\varepsilon)} \frac{P(\boldsymbol{z})\, d\boldsymbol{z}}{\prod_{j=1}^d Q_j(\boldsymbol{z})} = \operatorname{Res}\left(\frac{P(\boldsymbol{z})}{\prod_{j=1}^d Q_j(\boldsymbol{z})}\, ;\, \boldsymbol{p}\right),$$

(iv) $$\frac{1}{(2\pi i)^d} \int_{\Psi^{-1}(T_\varepsilon)} \frac{\boldsymbol{z}^{-\boldsymbol{r}-\boldsymbol{1}} P(\boldsymbol{z})}{\prod_{j=1}^s Q_j(\boldsymbol{z})} d\boldsymbol{z} = \frac{\boldsymbol{p}^{-\boldsymbol{r}-\boldsymbol{1}} P(\boldsymbol{p})}{\det J_\Psi(\boldsymbol{p})} = \frac{\boldsymbol{p}^{-\boldsymbol{r}} P(\boldsymbol{p})}{\det \boldsymbol{\Gamma}_\Psi(\boldsymbol{p})}.$$

Example C.19 (two lines in $\mathbb{C}^2$)**.** Let

$$Q(x, y) = \left(1 - \frac{1}{3}x - \frac{2}{3}y\right)\left(1 - \frac{2}{3}x - \frac{1}{3}y\right)$$

so that $\mathcal{V}_Q$ has a transverse multiple point at $(x, y) = (1, 1)$. The gradients of the factors of Q are $(1/3, 2/3)$ and $(2/3, 1/3)$, which are also their logarithmic gradients when $x = y = 1$. The determinant of $\boldsymbol{\Gamma}_\Psi$ is therefore one of $\pm 1/3$, the sign choice depending on the order in which we choose the factors. Up to sign, the iterated residue of $Q(x, y)^{-1} dx \wedge dy$ at $(1, 1)$ is thus the number 3. ◁

Example C.20 (dimension three with two factors)**.** Consider the generating function

$$F(x, y, z) = \frac{16}{(4 - 2x - y - z)(4 - x - 2y - z)},$$

whose singular set consists of two planes meeting at the complex line $\mathcal{S} = \{(1, 1, 1) + \lambda(-1, -1, 3) : \lambda \in \mathbb{C}\}$. In this case we can parametrize $\mathcal{S}$ globally by any of its three coordinates (i.e., we can take $\mathcal{D} = \mathbb{C}^3$). Choosing the third

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coordinate, making $\pi_1 = 3$, we obtain

$$J_\Psi(x,y,z) = \begin{bmatrix} -2 & -1 & -1 \\ -1 & -2 & -1 \\ 0 & 0 & 1 \end{bmatrix},$$

whence $\det \mathbf{\Gamma}_\Psi = 3$ and

$$\operatorname{Res}(F(x,y,z)\,dx \wedge dy \wedge dz; \mathcal{S}) = \frac{16}{3}\,dz\,.$$

Choosing one of the first two coordinates leads to an equivalent answer: the first two rows of J_Ψ are unchanged while the third row becomes either $(1,0,0)$ or $(0,1,0)$, ultimately giving the representations $-16dx$ and $16dy$. These are all equal, up to sign, as 1-forms on $\mathcal{S}$. ◁

C.2.4 Iterated residue forms for higher order poles on transverse divisors

In Section C.2.2 above we used Gelfand–Shilov reduction (Lemma C.11) to define a residue for higher order smooth poles in terms of the residue for smooth simple poles. A version of Gelfand–Shilov reduction also works for iterated residues leading, through a computation analogous to the ones used to establish Theorem C.17, except messier, to the following result.

Proposition C.21. *Let $\mathcal{S}$ be a smooth codimension k variety in $\mathbb{C}^d_*$ defined by the vanishing of k analytic functions $Q_1,\dots,Q_k$, let U denote the module over holomorphic functions of all meromorphic forms that can be written as $\psi/\prod_{j=1}^k Q_j^{n_j}$, where ψ is holomorphic in a neighborhood of $\mathcal{S}$ in $\mathbb{C}^d$, and let $R = U/E$, where E is generated by the forms $\{\operatorname{Res}(d\eta) : \eta \in R\}$. Then every class in R has a representative in which each power n_j is equal to 1.* □

The rest of this section is devoted to the statement and proof of Theorem C.24, an explicit formula for the residue in the specific case we use in this text. We begin with a lemma indicating what form the answer will take.

Lemma C.22. *Let $f, f_1, \dots, f_d$ be smooth functions of $\boldsymbol{u} \in \mathbb{C}^k$. Then*

$$\left(\frac{\partial}{\partial \boldsymbol{u}}\right)^{\boldsymbol{n}} f(\boldsymbol{u}) f_1(\boldsymbol{u})^{r_1} \cdots f_d(\boldsymbol{u})^{r_d} = f(\boldsymbol{u}) f_1(\boldsymbol{u})^{r_1} \cdots f_d(\boldsymbol{u})^{r_d} \Phi(\boldsymbol{r}, \boldsymbol{u}),$$

where Φ is a polynomial in $\boldsymbol{r}$ of degree $|\boldsymbol{n}| = n_1 + \cdots + n_k$. The leading term of Φ is $\mathcal{K}(\boldsymbol{r},\boldsymbol{u})^{\boldsymbol{n}} = \prod_{j=1}^k \mathcal{K}_j(\boldsymbol{r},\boldsymbol{u})^{n_j}$, where

$$\mathcal{K}_j(\boldsymbol{r},\boldsymbol{u}) = \sum_{i=1}^d r_i \frac{\partial \log f_i}{\partial u_j}\,.$$

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Proof We show by induction that

$$\left(\frac{\partial}{\partial \boldsymbol{u}}\right)^{\boldsymbol{n}} f(\boldsymbol{u}) f_1(\boldsymbol{u})^{r_1} \cdots f_d(\boldsymbol{u})^{r_d} = f(\boldsymbol{u}) f_1(\boldsymbol{u})^{r_1} \cdots f_d(\boldsymbol{u})^{r_d} \left[\mathcal{K}(\boldsymbol{r}, \boldsymbol{u})^{\boldsymbol{n}} + Q(\boldsymbol{r}, \boldsymbol{u})\right] \tag{C.2.11}$$

for all $\boldsymbol{n}$, where Q is a polynomial in $\boldsymbol{r}$ of degree less than $|\boldsymbol{n}|$. When $\boldsymbol{n} = \boldsymbol{0}$ this holds with $Q = 0$. Assuming this holds for $\boldsymbol{n}$, taking the logarithm and differentiating with respect to u_j gives, after some algebraic simplification, that $\left(\frac{\partial}{\partial \boldsymbol{u}}\right)^{\boldsymbol{n}+\delta_j} f(\boldsymbol{u}) f_1(\boldsymbol{u})^{r_1} \cdots f_d(\boldsymbol{u})^{r_d}$ equals the right-hand side of (C.2.11) when $\mathcal{K}(\boldsymbol{r}, \boldsymbol{u})^{\boldsymbol{n}} + Q(\boldsymbol{r}, \boldsymbol{u})$ is replaced by

$$\frac{\partial \log f}{\partial u_j} + \mathcal{K}_j \cdot (\mathcal{K}^{\boldsymbol{n}} + Q) + \frac{\partial}{\partial u_j}(\mathcal{K}^{\boldsymbol{n}} + Q)\,.$$

The terms in this expression other than $\mathcal{K}_j \cdot \mathcal{K}^{\boldsymbol{n}} = \mathcal{K}^{\boldsymbol{n}+\delta_j}$ are polynomials in $\boldsymbol{r}$ of degree at most $|\boldsymbol{n}|$, completing the induction. □

We now specialize to our context.

Corollary C.23. *Let* Ψ *be the parametrization defined in* (C.2.6) *with Jacobian matrix* $J_\Psi(\boldsymbol{z})$*. If we parametrize* $\boldsymbol{z} = \boldsymbol{z}(\boldsymbol{u})$ *for variables* $\boldsymbol{u} \in \mathbb{C}^k$ *and* $\boldsymbol{m}$ *is a vector of positive integers then*

$$\left(\frac{\partial}{\partial \boldsymbol{u}}\right)^{\boldsymbol{m}-\boldsymbol{1}} \left(\frac{\boldsymbol{z}(\boldsymbol{u})^{-\boldsymbol{r}} P\left(\Psi^{-1}(\boldsymbol{u})\right)}{\prod_{j=1}^d z_j(\boldsymbol{u})\ \det J_\Psi\left(\Psi^{-1}(\boldsymbol{u})\right)}\right) = \frac{\boldsymbol{z}(\boldsymbol{u})^{-\boldsymbol{r}} P\left(\Psi^{-1}(\boldsymbol{u})\right) \mathcal{P}(\boldsymbol{r}, \boldsymbol{u})}{\prod_{j=1}^d z_j(\boldsymbol{u})\ \det J_\Psi\left(\Psi^{-1}(\boldsymbol{u})\right)}\ , \tag{C.2.12}$$

where

$$\mathcal{P}(\boldsymbol{r}, \boldsymbol{u}) = \left[\prod_{j=1}^k \left(\sum_{i=1}^d r_i \frac{\partial \log z_i(\boldsymbol{u})}{\partial u_j}\right)^{m_j - 1} + \ R(\boldsymbol{r}, \boldsymbol{u})\right] \tag{C.2.13}$$

for some polynomial R *in* $\boldsymbol{r}$ *of degree less than* $|\boldsymbol{m}| - k$*.*

Theorem C.24. *Under our running assumptions, the iterated residue has a computable expression*

$$\operatorname{Res}\left(\boldsymbol{z}^{-\boldsymbol{r}-\boldsymbol{1}} \frac{P(\boldsymbol{z})}{\prod_{j=1}^k Q_j(\boldsymbol{z})^{m_j}} d\boldsymbol{z}\,;\, S_{\mathcal{D}}\right) \ = \ \boldsymbol{z}^{-\boldsymbol{r}} \frac{\mathcal{P}(\boldsymbol{r}, \boldsymbol{z})}{\prod_{j \in \boldsymbol{\pi}} z_j}\Bigg|_{z_i = \zeta_i(\boldsymbol{z}_{\boldsymbol{\pi}})\,:\, i \notin \boldsymbol{\pi}} d\boldsymbol{z}_{\boldsymbol{\pi}}\,, \tag{C.2.14}$$

where $\mathcal{P}(\boldsymbol{r}, \boldsymbol{z})$ *is a polynomial in* $\boldsymbol{r}$ *of degree* $|\boldsymbol{m}| - k$*. The leading term of* $\mathcal{P}(\boldsymbol{r}, \boldsymbol{z})$ *is*

$$\mathcal{P}(\boldsymbol{r}, \boldsymbol{z}) \sim \frac{(-1)^{|\boldsymbol{m}-\boldsymbol{1}|}}{(\boldsymbol{m}-\boldsymbol{1})!} \ \frac{P(\boldsymbol{z})}{\det \boldsymbol{\Gamma}_\Psi(\boldsymbol{z})} \left(\boldsymbol{r}\boldsymbol{\Gamma}_\Psi^{-1}\right)^{\boldsymbol{m}-\boldsymbol{1}}\,, \tag{C.2.15}$$

where $\boldsymbol{\Gamma}_\Psi$ *is the matrix from Definition C.16, the notation* $(\boldsymbol{r}\boldsymbol{\Gamma}_\Psi^{-1})^{\boldsymbol{m}-\boldsymbol{1}}$ *stands* *for* $\prod_{i=1}^k (\boldsymbol{r}\boldsymbol{\Gamma}_\Psi^{-1})_i^{m_i-1}$, *and* $(\boldsymbol{m}-\boldsymbol{1})! = \prod_{i=1}^k (m_i-1)!$. *When* $k = d$, *the formula* (C.2.14) *simplifies slightly to*

$$\operatorname{Res}\left(\boldsymbol{z}^{-\boldsymbol{r}-\boldsymbol{1}} \frac{P(\boldsymbol{z})}{\prod_{j=1}^d Q_j(\boldsymbol{z})^{m_j}} d\boldsymbol{z}\,;\, \boldsymbol{p} \right) = \boldsymbol{p}^{-\boldsymbol{r}} \mathcal{P}(\boldsymbol{r}, \boldsymbol{p})\,. \tag{C.2.16}$$

Remark. Recall that the factors $\{Q_i : 1 \le i \le k\}$ are defined only up to transformations multiplying each Q_i by a complex number λ_i, satisfying $\prod_{i=1}^k \lambda_i^{m_1} = 1$. This multiplies $\det \boldsymbol{\Gamma}_\Psi$ by $\prod_{i=1}^k \lambda_i$ and divides $(\boldsymbol{r}\boldsymbol{\Gamma}_\Psi^{-1})^{\boldsymbol{m}-\boldsymbol{1}}$ by $\prod_{i-1}^k \lambda_i^{m_i-1}$, thus leaving the ratio $(\boldsymbol{r}\boldsymbol{\Gamma}_\Psi^{-1})^{\boldsymbol{m}-\boldsymbol{1}} / \det \boldsymbol{\Gamma}_\Psi$ which appears in (C.2.15) invariant. Later, when we need to compute orientations, it will be convenient to normalize each Q_i to have constant term 1, simultaneously normalizing P to have constant term $a_{\boldsymbol{0}}$.

Example C.25. Let a and b be positive integers and consider the function

$$F(x, y, z) = \frac{16}{(4 - 2x - y - z)^a \, (4 - x - 2y - z)^b}\,,$$

generalizing the function in Example C.20. Choosing to parametrize the line $\mathcal{S}$ defined by the common zero sets of the denominator factors of F by the coordinate z, we have the matrix

$$\boldsymbol{\Gamma}_\Psi(x, y, z) = \begin{bmatrix} -2x & -y & -z \\ -x & -2y & -z \\ 0 & 0 & 1 \end{bmatrix},$$

whence $\det \boldsymbol{\Gamma}_\Psi = 3xy$ and, writing $\boldsymbol{r} = (r, s, t)$,

$$\boldsymbol{r}\boldsymbol{\Gamma}_\Psi^{-1} = \left(\frac{sx - 2ry}{3xy}, \frac{ry - 2sx}{3xy}, \frac{3txy - ryz - sxz}{3xy} \right).$$

Since we can parametrize $x = y = g(z)$ on $\mathcal{S}$ where $g(z) = (4 - z)/3$, we have

$$\det \boldsymbol{\Gamma}_\Psi|_{x=y=g(z)} = \frac{(4-z)^2}{3}$$

and

$$(\boldsymbol{r}\boldsymbol{\Gamma}_\Psi^{-1})^{\boldsymbol{m}-\boldsymbol{1}}\Big|_{x=y=g(z)} = \left(\frac{2r - s}{z - 4}\right)^{a-1} \left(\frac{2s - r}{z - 4}\right)^{b-1},$$

where $\boldsymbol{m} = (a, b)$. Thus

$$\operatorname{Res}\left[\boldsymbol{z}^{-\boldsymbol{r}-\boldsymbol{1}} F(\boldsymbol{z})\, d\boldsymbol{z}; \mathcal{S}\right] = x^{-r} y^{-s} z^{-t-1} \left[\mathcal{P}_0(z) + O\left((r+s)^{a+b-3}\right)\right] dz\,, \tag{C.2.17}$$

where, taking into account $(-1)^{|\boldsymbol{m}-\boldsymbol{1}|} = (-1)^{(a-1)+(b-1)}$ to change factors of $z-4$

into $4 - z$, we see

$$\mathcal{P}_0(z) = \frac{48}{(4-z)^2\,(a-1)!\,(b-1)!}\left(\frac{2r-s}{4-z}\right)^{a-1}\left(\frac{2s-r}{4-z}\right)^{b-1}.$$

◁

Proof of Theorem C.24. Fix any index t with $1 \le t \le k$ and let η be the $(k-1)$-form defined by

$$\eta = \frac{\tilde{P}(\boldsymbol{u})}{\boldsymbol{u}^{\boldsymbol{m}-\delta_t}} d\boldsymbol{u}_{\hat{t}}\,,$$

where $d\boldsymbol{u}_{\hat{t}}$ denotes the form $du_1 \wedge \cdots \wedge du_{t-1} \wedge du_{t+1} \wedge \cdots \wedge du_k$ and $\tilde{P}$ is an analytic function to be chosen later. Direct computation shows

$$d\eta = \frac{(\partial/\partial u_t)\tilde{P}(\boldsymbol{u})}{\boldsymbol{u}^{\boldsymbol{m}-\delta_{jt}}} d\boldsymbol{u} - \frac{(m_t-1)\tilde{P}(\boldsymbol{u})}{\boldsymbol{u}^{\boldsymbol{m}}} d\boldsymbol{u}\,,$$

and the fact that $\operatorname{Res}[d\eta] = 0$ implies

$$\operatorname{Res}\left[\frac{\tilde{P}(\boldsymbol{u})}{\boldsymbol{u}^{\boldsymbol{m}}} d\boldsymbol{u}\right] = \frac{1}{m_t - 1}\ \operatorname{Res}\left[\frac{(\partial/\partial u_t)\tilde{P}(\boldsymbol{u})}{\boldsymbol{u}^{\boldsymbol{m}-\delta_{jt}}} d\boldsymbol{u}\right]$$

(all residues with respect to forms in $\boldsymbol{u}$ are taken around the origin, which we suppress for readability). Applying this maneuver $m_t - 1$ times for each $1 \le t \le k$ then yields

$$\operatorname{Res}\left[\frac{\tilde{P}(\boldsymbol{u})}{\boldsymbol{u}^{\boldsymbol{m}}} d\boldsymbol{u}\right] = \frac{1}{(\boldsymbol{m}-\boldsymbol{1})!}\ \operatorname{Res}\left[\frac{(\partial/\partial\boldsymbol{u})^{\boldsymbol{m}-\boldsymbol{1}}\tilde{P}(\boldsymbol{u})}{u_1\cdots u_k} d\boldsymbol{u}\right], \tag{C.2.18}$$

and using Theorem C.17 on the right-hand side of (C.2.18) implies

$$\operatorname{Res}\left[\frac{\tilde{P}(\boldsymbol{u})}{\boldsymbol{u}^{\boldsymbol{m}}} d\boldsymbol{u}\right] = \frac{1}{(\boldsymbol{m}-\boldsymbol{1})!}\left(\frac{\partial}{\partial\boldsymbol{u}}\right)^{\boldsymbol{m}-\boldsymbol{1}}\tilde{P}(\boldsymbol{u})\,. \tag{C.2.19}$$

By the definition of the map Ψ we can parametrize $\boldsymbol{z}$ on $\mathcal{S}_{\mathcal{D}}$ by $\boldsymbol{z}(\boldsymbol{u}) = \Psi^{-1}(\boldsymbol{0}, \boldsymbol{u})$ for $\boldsymbol{u}$ in a neighborhood of the origin in $\mathbb{C}^{d-k}$. To simplify notation we write $\Psi^{-1}(\boldsymbol{0}, \boldsymbol{u})$ as $\Psi^{-1}(\boldsymbol{u})$, understanding the first k coordinates are implicitly zero. We now select

$$\tilde{P}(\boldsymbol{u}) = \frac{\boldsymbol{z}(\boldsymbol{u})^{-\boldsymbol{r}-\boldsymbol{1}} P(\Psi^{-1}(\boldsymbol{u}))}{J_\Psi(\Psi^{-1}(\boldsymbol{u}))}\,,$$

which is chosen so that

$$\Psi^*\left(\frac{\tilde{P}(\boldsymbol{u})}{\boldsymbol{u}^{\boldsymbol{m}}} d\boldsymbol{u}\right) = \boldsymbol{z}^{-\boldsymbol{r}-\boldsymbol{1}}\frac{P(\boldsymbol{z})}{\boldsymbol{Q}(\boldsymbol{z})^{\boldsymbol{m}}} d\boldsymbol{z}\,.$$

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Functoriality of the residue, combined with (C.2.19), now implies

$$\begin{aligned}
&\operatorname{Res}\left[\boldsymbol{z}^{-\boldsymbol{r}-\mathbf{1}}\frac{P(\boldsymbol{z})}{\boldsymbol{Q}(\boldsymbol{z})^{\boldsymbol{m}}}d\boldsymbol{z};\mathcal{S}_{\mathcal{D}}\right] \\
&= \frac{1}{(\boldsymbol{m}-\mathbf{1})!}\left(\frac{\partial}{\partial \boldsymbol{u}}\right)^{\boldsymbol{m}-\mathbf{1}}\left(\frac{\boldsymbol{z}(\boldsymbol{u})^{-\boldsymbol{r}-\mathbf{1}}P(\Psi^{-1}(\boldsymbol{u}))}{J_\Psi(\Psi^{-1}(\boldsymbol{u}))}\right)\Bigg|_{(\mathbf{0},\boldsymbol{u})=\Psi(\boldsymbol{z})} d\boldsymbol{z}_{\boldsymbol{\pi}}\,. \qquad \text{(C.2.20)}
\end{aligned}$$

Applying Corollary C.23 to the right-hand side of (C.2.20) and noting that

$$\boldsymbol{\Gamma}_\Psi = \left(\frac{\partial u_i}{\partial \log z_j(\boldsymbol{u})}\right) \quad \text{implies} \quad \frac{\partial \log z_i(\boldsymbol{u})}{\partial u_j} = \left(\boldsymbol{\Gamma}_\Psi^{-1}\right)_{ij}$$

we obtain an expression

$$\left(\frac{\partial}{\partial \boldsymbol{u}}\right)^{\boldsymbol{m}-\mathbf{1}}\left(\frac{\boldsymbol{z}(\boldsymbol{u})^{-\boldsymbol{r}}P(\Psi^{-1}(\boldsymbol{u}))}{\prod_{j=1}^d z_j(\boldsymbol{u})J_\Psi(\Psi^{-1}(\boldsymbol{u}))}\right)\Bigg|_{(\mathbf{0},\boldsymbol{u})=\Psi(\boldsymbol{z})} = \boldsymbol{z}^{-\boldsymbol{r}}\frac{P(\boldsymbol{z})}{\det \boldsymbol{\Gamma}_\Psi(\boldsymbol{z})}\frac{\tilde{\mathcal{P}}(\boldsymbol{r},\boldsymbol{z})}{\prod_{j\in\boldsymbol{\pi}} z_j}$$

whose leading term is as stated. □

Remarks. The leading term (C.2.15) depends on the divisors Q_j only through their gradients. When the stratum $\mathcal{S}$ is a single point ($k = d$), the residue at $\boldsymbol{p}$ is a 0-form – i.e., a polynomial $\mathcal{P}(\boldsymbol{r})$ in $\boldsymbol{r}$.

C.3 Classical Morse theory

After using residues to replace our starting Cauchy integral with a residue integral over an intersection class σ in $\mathcal{V}$, we need to understand how to deform σ in $\mathcal{V}$. The possible deformations we can make, and which deformations will allow us to compute asymptotic behavior, depend on the topological properties of $\mathcal{V}$. Morse theory attempts to describe the topology of a space X by means of the geometry of X near critical points of a smooth, proper function $h : X \to \mathbb{R}$.

Our destination in this appendix is Theorems C.38 and C.39, which state that we may find a basis for each homology group $H_k(X)$ consisting of *quasi-local cycles* at the critical points of h: for each critical point $\boldsymbol{p}$ there will be a cycle with height bounded by $h(\boldsymbol{p})-\varepsilon$ except in an arbitrarily small neighborhood of $\boldsymbol{p}$. We establish this result by studying the sublevel sets $X_{\le a} := \{x \in X : h(x) \le a\}$ as a increases and showing that the homotopy type of X does not change (the Morse Lemma C.27) except at critical points, where a cell is attached (Theorem C.28). Along the way, a description of X as a cell complex is given in Theorem C.32. A description of the attachments in terms of relative homology is also given in the last section.

Our material here covers classical (smooth) Morse theory, which assumes that the space X under consideration is a manifold. More general spaces are handled in Appendix D.

Homotopy equivalence except at critical points

Let X be a smooth manifold and let $h : X \to \mathbb{R}$ be a smooth function; we think of h as giving the points on X a *height* (see Figure C.4 below). The ***critical points of the height function*** h are the points $\boldsymbol{p} \in X$ for which the differential $dh|_{\boldsymbol{p}}$ is zero on the tangent space $T_{\boldsymbol{p}}(X)$. The values $h(\boldsymbol{p})$ of h at its critical points $\boldsymbol{p}$ are called the ***critical values of the height function*** h. A critical point $\boldsymbol{p}$ is a ***nondegenerate critical point for*** h if the quadratic form given by the quadratic terms in the Taylor approximation for h at $\boldsymbol{p}$ has no zero eigenvalues. In coordinates, this means that the determinant of the Hessian matrix $\left[\frac{\partial^2 h}{\partial x_i \partial x_j}(\boldsymbol{p})\right]$ is non-zero when X is locally coordinatized by $x_1, \ldots, x_d$ near $\boldsymbol{p}$. While the Hessian matrix itself depends on the coordinates, its (non)singularity does not; see [Mil63, Section 2.1]. While it is traditional to require Morse functions to be proper and have distinct critical values, we will not require this.

Definition C.26 (Morse function)**.** A smooth function $h : X \to \mathbb{R}$ is called a ***Morse function*** if the critical points of h are nondegenerate. If h is a proper map (meaning the inverse image of any closed and bounded interval is compact) then we call h a ***proper Morse function***. If the critical values of h are distinct, then h is a ***Morse function with distinct critical values***.

Exercise C.2. In which of the following cases is h a proper Morse function on X?

(1) X is the surface of a doughnut lying on a table and h is height.
(2) X is the infinite cylinder $\{(x, y, z) : x^2 + y^2 = 1\}$ and h is the z coordinate.
(3) X is the unit sphere and h is the distance to the point $(-2, 0, 0)$.

Let X be a smooth manifold with proper Morse function h. If a is a real number, we let $X_{\leq a}$ denote the topological subspace $\{\boldsymbol{x} \in X : h(\boldsymbol{x}) \leq a\}$. The fundamental *Morse Lemma* states that the topology of $X_{\leq a}$ changes only when a is a critical value of h.

Lemma C.27 (Morse Lemma)**.** *Let $a < b$ be real numbers, suppose that the interval $[a, b]$ contains no critical values of h, and assume that $h^{-1}([a, b])$ is compact. Then the inclusion $X_{\leq a} \hookrightarrow X_{\leq b}$ is a homotopy equivalence.*

Proof The Morse Lemma is proven in [Mil63, Theorem 3.1] by constructing a homotopy on $X_{\leq b}$ that follows the orthogonal trajectories of the level-sets

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$h = c$ for constants $c \in [a, b]$. This is accomplished using a *downward gradient flow* constructed locally using the gradient of $h(\boldsymbol{x})$ (which never vanishes when $h(\boldsymbol{x}) \in [a, b]$ due to the absence of critical points). □

Exercise C.3. Let X be the torus embedded in $\mathbb{R}^3$ and let f be the distance from points on X to a fixed point not on X. Use the Morse Lemma to prove that f has a critical point on X that is either degenerate or is neither a maximum nor a minimum.

Attachment at critical points

Suppose now that there is precisely one critical point $\boldsymbol{p}$ with $h(\boldsymbol{p}) \in [a, b]$. The Hessian of h at $\boldsymbol{p}$ is a real symmetric matrix and therefore has real eigenvalues. We define the ***Morse index of*** h ***at*** $\boldsymbol{p}$ to be the number of negative eigenvalues of the Hessian. The Morse index can range from 0 at a local minimum to the dimension d of X at a local maximum. We now describe the topology of $X_{\leq b}$ as an attachment of a space Y to $X_{\leq a}$, where Y and the attaching map depend on the Morse index of h at $\boldsymbol{p}$. Following standard terminology in Morse theory, a *k-cell* (more properly a ***topological k-cell***) is a ball of dimension k (in Appendix A we used this term for k-simplices, but topologically a k-ball and k-simplex are equivalent).

Theorem C.28. *Suppose that $h^{-1}([a, b])$ is compact and contains precisely one critical point $\boldsymbol{p}$, with critical value $h(\boldsymbol{p})$ strictly between a and b. Then the space $X_{\leq b}$ has the homotopy type of $X_{\leq a}$ with a λ-cell attached along its boundary, where λ is the Morse index of the critical point $\boldsymbol{p}$ (a 0-cell is a point with empty boundary).*

Proof See [Mil63, Theorem 3.2]. □

Example C.29. Suppose X is the unit sphere in $\mathbb{R}^3$ and consider the height function $h(x, y, z) = z$ (when working in $\mathbb{R}^3$, we often set $h(x, y, z) = z$ so that the "height" function measures actual height). There are only two critical points of h, namely its minimum $(0, 0, -1)$ at height -1 and maximum $(0, 0, 1)$ at height 1.

Let us follow $X_{\leq a}$ as a increases from $-\infty$ to $+\infty$. For $a < -1$, the set $X_{\leq a}$ is empty. As a passes -1, Theorem C.28 states that a 0-cell is added with no identification, making $X_{\leq a}$, homotopically, a point. Geometrically, $X_{\leq a}$ with $a \in (-1, 1)$ is a small dish, which is contractible to a point. The only other attachment occurs at the top of the sphere. For $a < 1 \leq b$, the set $X_{\leq b} \setminus X_{\leq a}$ is a polar cap. Thus, geometrically as well as homotopically, a 2-cell is attached

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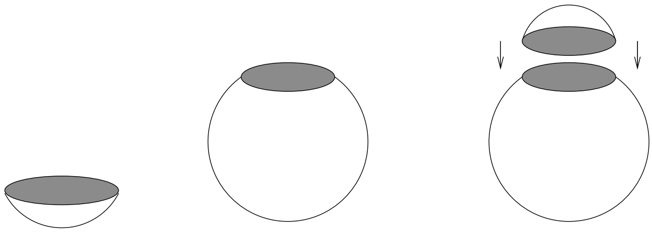

Figure C.3 Sublevel sets of a sphere, which form a contractible subset of the sphere until reaching its maximum, when a cap is attached to complete the sphere.

along its bounding circle. All spaces resulting from attaching a k-cell to a contractible space are homotopy equivalent to attaching a k-cell to a point. In the present case $k = 2$ and the resulting space is homotopy equivalent to a 2-sphere, recovering the homotopy class of X; see Figure C.3. We remark that analyzing the attachments recovers only the homotopy type, not the homeomorphism class. ◁

Example C.30. Let X be the torus in $\mathbb{R}^3$ obtained by rotating the circle $(x - 5)^2 + (y - 5)^2 = 1$ about the y-axis and let $h(x, y, z) = z$. The function h has four critical points, all on the z-axis: a maximum (Morse index 2) at $(0, 0, 6)$, a minimum (Morse index 0) at $(0, 0, -6)$, and saddle points (Morse index 1) at $(0, 0, 4)$ and $(0, 0, -4)$; see Figure C.4.

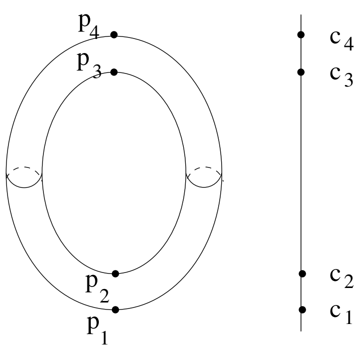

Figure C.4 The critical points on a torus for the standard height function.

For $-6 \le a < -4$ we see that, as in Example C.29, $X_{\le a}$ is homotopic to a point (geometrically, it is a dish). As a passes -4, the theorem tells us to add a 1-cell along its boundary. The only way of attaching a 1-cell to a point is to map both endpoints to the point, leaving a circle. Geometrically, if $-4 < a < 4$

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then the set $X_{\leq a}$ is a patch in the shape of a 2-cell with two disjoint segments of its boundary attached to two disjoint segments of the geometric boundary of $X_{\leq a}$; see Figure C.5.

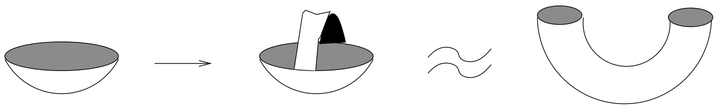

Figure C.5 Crossing the critical value c_2.

The critical point at height 4 adds another 1-cell modulo its boundary, making the homotopy type of $X_{\leq a}$ for $4 \leq a < 6$ the union of two circles touching at a point. Finally, crossing the top of the sphere a 2-cell B is added modulo its boundary. There is more than one choice for the homotopy type of the attachment, and keeping track only of the homotopy type throughout the process of attaching cannot resolve this choice – one must look at the geometry of the attachment. In this case, because the attachment is to a topological circle (the $6 - \varepsilon$ level set) and results in a nonintersecting surface in $\mathbb{R}^3$, there are only two possibilities for the attaching map in homology: ∂B is mapped to the circle in one of two orientations. Either choice results in a torus.

We remark that knowing the mapping of the last attachment in homology is not sufficient to compute the homotopy type of the space. For example, attaching a 2-cell to two circles joined at a point by mapping the boundary of the 2-cell to the common point produces a sphere with two circular handles, which is not homotopy equivalent to a torus. ◃

Although Theorem C.28 specifies the attachment pair when the topology of X changes, Example C.30 shows that the computation of the attaching map is not automatic. It will help to have some results that narrow down this computation to certain constructions local to the critical point: the homotopy in the Morse Lemma may be improved so that outside of a neighborhood of $\boldsymbol{p}$, every point is pushed down at least to a level $c - \varepsilon$.

Let $\boldsymbol{p}$ be a critical point with height $c = h(\boldsymbol{p})$ and suppose $a < c < b$ are such that $\boldsymbol{p}$ is the only critical point with height in $[a, b]$. Given $\varepsilon > 0$, let $B_\varepsilon(\boldsymbol{p})$ denote the ε-neighborhood of $\boldsymbol{p}$. We will see a formal version of the local Morse Lemma, in a more general context, in Appendix D. For now, we note that the local Morse Lemma implies that the homotopy type of $X_{\leq b}$ is the same as the homotopy type of $X_{\leq c-\varepsilon} \cup B_\varepsilon(\boldsymbol{p})$ for sufficiently small ε.

Definition C.31. Let $X_{c^+} := X_{\leq c-\varepsilon} \cup B_\varepsilon(\boldsymbol{p})$ for any sufficiently small $\varepsilon > 0$, and let $X^{\boldsymbol{p},\mathrm{loc}}$ denote the pair $(X_{c^+}, X_{\leq c-\varepsilon})$ depicted in Figure C.6.

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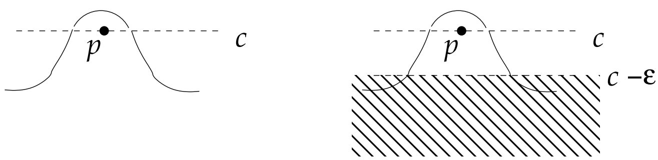

Figure C.6 The space X_{c^+} with points of height c represented by a dotted line.

The discussion above implies that the attachment pair $(X_{\le b}, X_{\le a})$ is homotopy equivalent to $X^{\boldsymbol{p},\mathrm{loc}}$. Suppose now that h is a Morse function whose critical values need not be distinct. If $[a, b]$ contains the unique critical value $c \in (a, b)$ then the homotopy pushes points down to $X_{\le c-\varepsilon}$ except in a neighborhood of the set of critical points whose value is $\boldsymbol{p}$. Since this set of critical points is discrete under our assumptions,

$$(X_{\le b}, X_{\le a}) \simeq \widetilde{\bigoplus_{\boldsymbol{p}:h(\boldsymbol{p})=c}} X^{\boldsymbol{p},\mathrm{loc}} \tag{C.3.1}$$

for sufficiently small $\varepsilon > 0$, where the tilde sum denotes the ***wedge*** of spaces, meaning a disjoint union with the second space in each pair identified. This equivalence states that $(X_{\le b}, X_{\le a})$ is homotopy equivalent to the wedge of the local pairs at all critical points $\boldsymbol{p}$ with value c. The reduced homology of a wedge is the direct sum of the reduced homologies of the individual spaces.

Exercise C.4. Intuitively, why is (C.3.1) a (reduced) *direct* sum? That is, explain why cycles in different summands cannot cancel each other.

The last step for this section is to put all this information together to produce a global topological picture of X. At the level of homotopy type, the result is that X has the topology of a cell complex, about which certain information is known.

Theorem C.32. *Let X be a manifold and $h : X \to \mathbb{R}$ be a differentiable function with no degenerate critical points. Suppose each sublevel set $X_{\le a}$ is compact. Then X has the homotopy type of a cell complex with one cell of dimension λ for each critical point of Morse index λ in $X_{\le a}$.*

Proof A full proof of Theorem C.32 can be found in [Mil63, Theorem 3.5], but we give a sketch here. The proof for the case of finitely many critical points with distinct critical values involves showing inductively that for any critical value c the homotopy equivalence between $X_{\le c-\varepsilon}$ and a cell complex may be extended, via the attachment of a cell, to a homotopy equivalence between $X_{\le c+\varepsilon}$ and a cell complex with one more cell. The restriction on distinct critical

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values is then removed by homotopically perturbing h so as to satisfy the conditions, and a limiting argument removes the assumption of a finite number of critical points. □

Example C.33. The 2-sphere from Example C.29 is a cell complex with one 2-cell and one 0-cell; as noted above, up to homotopy equivalence, there is only one choice for the attachment map. The 2-torus from Example C.30 is a cell complex with one 0-cell, two 1-cells, and one 2-cell. Up to homotopy equivalence, the one skeleton must be the wedge of two circles. There are a number of ways to attach a 2-cell to a wedge of two circles, and the right attaching map can be worked out by knowing what the boundary (i.e., the level set ε below the maximum height) looks like. ◁

Remark C.34. Let X be a complex d-manifold in $\mathbb{C}^n$ and, for $\boldsymbol{p} \in \mathbb{C}^n$, let $h_{\boldsymbol{p}}$ denote the function mapping $\boldsymbol{z} \in \mathbb{C}^n$ to the complex distance $\|\boldsymbol{z} - \boldsymbol{p}\| = (\sum_{j=1}^n |z_j - p_j|^2)^{1/2}$. Andreotti and Frankel's original proof of Proposition B.20 from Appendix B proved that $\boldsymbol{p}$ can be chosen to make $h_{\boldsymbol{p}}$ a Morse function by establishing that the set of $\boldsymbol{p}$ for which it is not a Morse function has positive codimension, then showing that the Morse index of any critical point on X for $h_{\boldsymbol{p}}$ is at most d.

C.4 Description at the level of homology

For our purposes it is useful to consider the successive attachments from the last section on the level of homology. Suppose that c is a critical value and (B, A) is any pair with the same homotopy type of the attachment $(X_{\leq c+\varepsilon}, X_{\leq c-\varepsilon})$. The long exact sequence has a portion

$$H_{n+1}(B, A) \xrightarrow{\partial_{n+1}} H_n(A) \xrightarrow{\iota_*} H_n(B) \xrightarrow{\pi_*} H_n(B, A) \xrightarrow{\partial_n} H_{n-1}(A),$$

which implies

$$\frac{H_n(A)}{\text{Image}(\partial_{n+1})} = \frac{H_n(A)}{\ker(\iota_*)} \cong \text{Image}(\iota_*) = \ker(\pi_*).$$

In particular, there is a short exact sequence

$$0 \to \frac{H_n(A)}{\text{Image}(\partial_{n+1})} \to H_n(B) \to \ker(\partial_n) \to 0$$

and, by Remark B.7, $H_n(B)$ decomposes as a direct sum of the kernel of ∂_n and the cokernel of ∂_{n+1}.

This decomposition allows us to construct a basis for the homology groups

$H_n(B)$ from knowledge of the homology of A and the boundary map ∂_*: starting with a basis for $H_n(A)$ we identify basis elements differing by elements in the image of ∂_{n+1} and then add new basis elements indexed by a basis for the kernel of ∂_n. These new basis elements have an explicit geometric description. The group $H_n(B,A)$ consists of equivalence classes of chains in B whose boundaries lie in A. If C is a chain in the kernel of ∂_n then the image $\partial_*([C])$ is the class of $\partial C \in H_{n-1}(A)$, which bounds some n-chain D in A. The inverse image of the class $[C]$ by π_* is the class of the chain $C - D$, which is a cycle because $\partial C = \partial D$. Heuristically, we write

$$\pi_*^{-1}([C]) = C - \partial_A^{-1}(\partial C) \tag{C.4.1}$$

and view $\pi_*^{-1}([C])$ as the relative cycle C in $Z_n(B,A)$, completed to an actual cycle in a way that stays within A.

Remark C.35. The choice of D in this construction is not natural (see Remark B.7). A particular composition of a space B as a subspace A attached to $C = \overline{B \setminus A}$ comes with an explicit inclusion map from ∂C to A, and this induces the ∂_* operator. There may, however, be more than one way to reassemble A, C, and ∂_* into B, giving homotopy equivalent spaces with different homology bases.

One further remark on notation: when attaching a space Y along Y_0, the pair (Y, Y_0) is commonly referred to as the ***attachment data*** or, in the case of Morse theory, the ***Morse data*** for the attachment. This data should really include the homotopy type of the attachment map, or else the homotopy type of X and the attachment data do not determine the homotopy type of the new space. On the level of homology what we need to know is the relative homology of the pair (Y, Y_0), which is the homology of the new space relative to the old space, *together with the ∂_* map*.

Filtered spaces

A ***filtered space*** X_n is the end of a nested sequence $X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n$ of topological spaces. We use the terminology of filtered spaces to describe how homology changes among sublevel sets $X_j = X_{\le a_j}$, and our first result concerns the homology of a chain that is successively pushed toward lower heights.

Lemma C.36 (Pushing Down Lemma)**.** *Let $X_0 \subseteq \cdots \subseteq X_n$ be a filtered space and let C be a non-zero homology class in $H_k(X_n, X_0)$ for some k. Then there is a unique positive $j \le n$ such that for some $C_* \in H_k(X_j, X_{j-1})$,*

$$\iota(C_*) = \pi(C) \neq 0 \text{ in } H_k(X_n, X_{j-1}), \tag{C.4.2}$$

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where ι is the map induced by the inclusion of pairs $(X_j, X_{j-1}) \to (X_n, X_{j-1})$ and π is the map induced by the projection of pairs $(X_n, X_0) \to (X_n, X_{j-1})$. If ι is an injection then C_* is unique.

Proof To prove uniqueness of j, suppose that (C.4.2) is satisfied for some minimal j with a chain C_*, and let $j < r \leq n$. The composition of the two maps

$$(X_j, X_{j-1}) \to (X_n, X_{j-1}) \to (X_n, X_{r-1})$$

induces the zero mapping on homology because any class in the image of the first map has a cycle representative in X_j. Letting π' denote projection of (X_n, X_0) to (X_n, X_{r-1}), we have $\pi'(C) = \pi'(\pi(C)) = \pi'(\iota(C_*)) = 0$ and therefore (C.4.2) cannot hold for $r > j$.

For existence we argue by induction on n. The case $n = 1$ is trivial because then $j = 1$ and $C_* = C$. Assume the result for $n-1$ and let C be a non-zero class in $H_k(X_n, X_0)$. If the image of C under the projection of (X_n, X_0) to (X_n, X_{n-1}) is non-zero then we may take C_* to be this image and j to be n. Assume therefore that C projects to zero. The short exact sequence of chain complexes for the pairs

$$0 \to (X_{n-1}, X_0) \to (X_n, X_0) \to (X_n, X_{n-1}) \to 0$$

induces the exact sequence

$$H_k(X_{n-1}, X_0) \to H_k(X_n, X_0) \to H_k(X_n, X_{n-1})\,.$$

By assumption C is in the kernel of the second map, hence is the image under the first map of some non-zero class C'. Applying the inductive hypothesis to C' yields some $j \leq n-1$ and a cycle $C_* \in H_k(X_j, X_{j-1})$ satisfying (C.4.2) with C' in place of C. The commuting diagram

$$\begin{tikzcd}
 & C' \in (X_{n-1}, X_0) \arrow[r, "\iota_3"] \arrow[d, "\pi_1"'] & C \in (X_n, X_0) \arrow[d, "\pi"'] \\
C_* \in (X_j, X_{j-1}) \arrow[r, "\iota_1"] & (X_{n-1}, X_{j-1}) \arrow[r, "\iota_2"] & (X_n, X_{j-1})
\end{tikzcd}$$

allows us to conclude $\pi(C) = \pi(\iota_3(C')) = \iota_2(\pi_1(C')) = \iota_2(\iota_1(C_*)) = \iota(C_*)$, verifying (C.4.2). □

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Building up by successive attachments

If we understand the topology of each pair (X_{k+1}, X_k) of consecutive elements in a filtration $X_0 \subseteq \cdots \subseteq X_n$ and we understand the homology groups of X_0 then, using the argument above and induction, we understand the homology groups of all X_k. Furthermore, if X is a smooth manifold with a proper height function h then the Morse Lemma implies that the topology of the continuum of spaces $\{X_{\leq t}\}$ is captured by a filtration of sets described by the critical values $c_0 < c_1 < \cdots < c_{n-1}$ of h. The ***Morse filtration*** of X with respect to h is the filtration defined by $X_j = X_{\leq c_j - \varepsilon}$ for $0 \leq j \leq n-1$ and $X_n = X_{\leq c_{n-1}+\varepsilon}$, where ε is any sufficiently small positive number.

When h has distinct critical values the pairs (X_{i+1}, X_i) are homotopy equivalent to $X^{\boldsymbol{p}_i,\mathrm{loc}}$, where $\boldsymbol{p}_i$ are the critical points listed in order of increasing height. In general, the successive pairs are homotopy equivalent to $\widetilde{\bigoplus_{h(\boldsymbol{p})=c_j}} X^{\boldsymbol{p},\mathrm{loc}}$ as c_j increases through all critical values. We could describe how to keep track of generators and relations for the homologies of X_j inductively on j in the general case, however what we will need is both more specialized (our spaces are complex algebraic or analytic varieties) and more general (our spaces may not be manifolds). Accordingly, we restrict the discussion here to one illustration, continuing our example of the torus to show what can happen.

Example C.37. In Example C.30 we examined a height function on the torus X with four critical points: one of Morse index 0, two of Morse index 1, and one of Morse index 2. All ∂_* maps vanish so the homology groups $H_0(X), H_1(X)$, and $H_2(X)$ are cyclic groups of rank 1, 2, and 1, respectively. The filtration consists of $X_0 = \emptyset$, X_1 which is contractible to a point, X_2 which is homeomorphic to a cylinder, X_3 which is homeomorphic to a punctured torus, and X_4 which is the whole torus.

As an illustration of the non-naturality of the homology basis in (C.4.1), consider the second 1-cell to be added. Let α be the homology class in $H_1(X_2)$ of the first 1-cell. Then the second 1-cell, which is a well-defined relative homology class β in $H_1(X_3, X_2)$, may be completed to an absolute class in $H_1(X_3)$ in many different ways, resulting in cycles differing by multiples of α. Geometrically, one may for example complete β to the circle defined by $x^2 + z^2 = 1$, or instead wrap around the torus any integer number of times. ◁

Exercise C.5. Let γ be the cycle pictured in Figure C.7, going around the torus between the critical points p_4 and p_3. Applying the Pushing Down Lemma with respect to the Morse filtration for the pictured height function, what is j and what cycle represents $\iota(C_*) = \pi(C)$?

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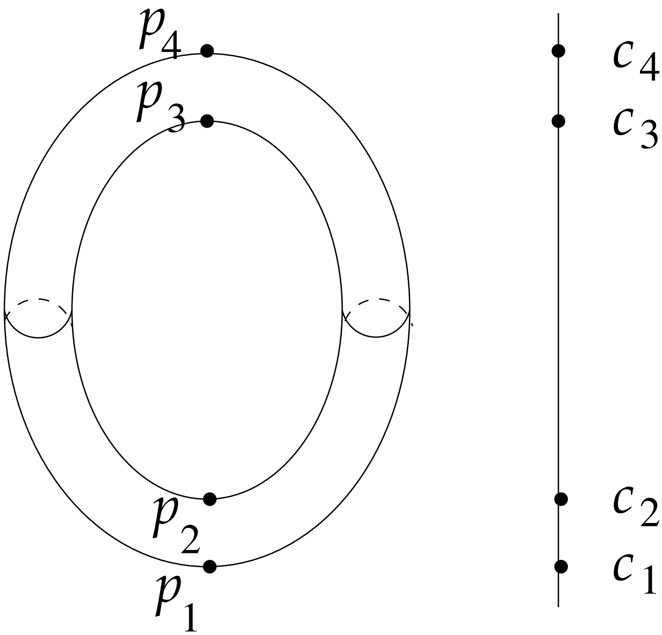

Figure C.7 A cycle on a torus.

Assume now that $X \subseteq \mathbb{C}_*^d$ is a smooth algebraic hypersurface (a real manifold of dimension $2d-2$) and that the specific height function $h(\boldsymbol{z}) = h_{\boldsymbol{r}}(\boldsymbol{z}) = -r_1 \log|z_1| - \cdots - r_d \log|z_d|$ is a proper Morse function on X. The purpose of introducing critical points at infinity in Chapter 7 is to remove the strong assumption that h is proper, however, to see how everything works, we now derive the results of Chapter 7 in this setting. The height function h is the real part of (a branch of) a holomorphic function $-\boldsymbol{r} \cdot \log \boldsymbol{z}$, and is thus harmonic, so all critical points for h on X have middle Morse index $d-1$. Let $\boldsymbol{p}^{(j)}$ for $1 \leq j \leq m$ enumerate the critical points, ordered so that the critical values $c_j = h(\boldsymbol{p}_j)$ are nondecreasing.

By the Morse Lemma, any cycle supported on $X_{<c_1}$ can be deformed via gradient flow to a cycle in $X_{\leq t}$ for t arbitrarily small, therefore integrals of $\boldsymbol{z}^{-\boldsymbol{r}} F(\boldsymbol{z}) d\boldsymbol{z}$ over such cycles decay faster than any exponential. For this reason, it suffices to describe the homology of $(X, -\infty)$, where $-\infty$ stands for $X_{\leq t}$ for any $t < c_1$. The final set of results in this appendix describes this homology.

Theorem C.38. *Suppose $h = h_{\boldsymbol{r}}$ is a proper Morse function on a smooth algebraic hypersurface X in $\mathbb{C}_*^d$. Then $H_k(X, -\infty)$ vanishes in dimensions $k \neq$*

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$d-1$. *The* $(d-1)$*-homology is given by* $H_{d-1}(X,-\infty) \cong \mathbb{C}^m$, *where* m *is the number of critical points. A basis* $\{\gamma_{\mathbf{p}}\}$ *may be chosen, indexed by the critical points* $\mathbf{p}$ *of* h *on* X, *with the property that* h *is maximized on* $\gamma_{\mathbf{p}}$ *at* $\mathbf{p}$. *By the isomorphism of smooth homology and singular homology, each* $\gamma_{\mathbf{p}}$ *may be chosen to be smooth.*

Proof First assume distinct critical values $c_1 < \cdots < c_m$ with corresponding critical points $\boldsymbol{p}^{(1)}, \ldots, \boldsymbol{p}^{(m)}$, and let X_j denote the space $X_{\leq c_j+\varepsilon}$. Inducting on j, we show that the conclusion of the theorem holds for X_j in place of X. First, we note that each pair (X_j, X_{j-1}) is homotopy equivalent to a $(d-1)$-ball B^{d-1} modulo its boundary, whose homology has rank 1 in dimension $d-1$ and zero in every other dimension. For the base step $j = 1$, where we take $X_0 = X_{\leq t}$ for any $t < c_1$, the conclusion is immediate.

Now assume the conclusion holds with $X = X_j$ for some $j < m$, and consider the short exact sequence of chain complexes of pairs

$$0 \to C_*(X_j, -\infty) \to C_*(X_{j+1}, -\infty) \to C_*(X_{j+1}, X_j) \to 0\,.$$

None of these pairs has any homology in dimensions higher than $d-1$, therefore the long exact sequence is as follows, with an arrow from the rightmost element of each row other than the bottom row to the leftmost element of the next row down.

$$\begin{array}{ccccccc}
0 & \to & H_{d-1}(X_j,-\infty) & \to & H_{d-1}(X_{j+1},-\infty) & \to & H_{d-1}(X_{j+1},X_j) \\
 & & H_{d-2}(X_j,-\infty) & \to & H_{d-2}(X_{j+1},-\infty) & \to & H_{d-2}(X_{j+1},X_j) \\
 & & H_{d-3}(X_j,-\infty) & \to & H_{d-3}(X_{j+1},-\infty) & \to & H_{d-3}(X_{j+1},X_j) \\
 & & \vdots & & \vdots & & \vdots \\
 & & H_0(X_j,-\infty) & \to & H_0(X_{j+1},-\infty) & \to & H_0(X_{j+1},X_j) \to 0.
\end{array}$$

Identifying each $H_k(X_{j+1}, X_j)$ as $\mathbb{C}$ if $k = d-1$ and 0 otherwise, and using the induction hypothesis, fills in most of this sequence:

$$\begin{array}{ccccccc}
0 & \to & \mathbb{C}^j & \to & H_{d-1}(X_{j+1},-\infty) & \to & \mathbb{C} \\
 & & 0 & \to & H_{d-2}(X_{j+1},-\infty) & \to & 0 \\
 & & 0 & \to & H_{d-3}(X_{j+1},-\infty) & \to & 0 \\
 & & \vdots & & \vdots & & \vdots \\
 & & 0 & \to & H_0(X_{j+1},-\infty) & \to & 0\,,
\end{array}$$

from which we deduce that $H_k(X_{j+1}, -\infty)$ has rank one more than that of $H_k(X_j, -\infty)$ when $k = d-1$ and rank zero otherwise. The extra generator comes from the attachment of B^{d-1} modulo its boundary, for which the generator γ_{j+1} may be chosen to maximize h at $\boldsymbol{p}^{(j)}$ (see Exercise C.10), thereby completing the induction. Finally, the assumption of distinct critical values may be removed via (C.3.1). □

Exercise C.6. Let X be a smooth complex algebraic variety of complex dimension d in $\mathbb{C}_*^m$ for $m > d$ and let h be a Morse function on X which is the real part of a complex analytic function. Suppose X has five critical points. Can you determine the homotopy type of the pair $(X_{\leq b}, X_{\leq a})$ when $a \to -\infty$ and $b \to +\infty$?

We now have the tools to state a result analogous to what Theorem C.38 tells us about X for $\mathcal{M} = \mathbb{C}_*^d \setminus X$. Recall the tube operator $\mathsf{o} : H_{d-1}(\mathcal{V}_*) \to H_d(\mathcal{M})$ from Theorem C.2 – this is not only an injection, but (as we will see in the next appendix) is an isomorphism on (X_{j+1}, X_j). An induction then gives the following result.

Theorem C.39. *Suppose the height function $h_{\mathbf{r}}$ is a proper Morse function on a smooth complex algebraic hypersurface X in $\mathbb{C}_*^d$ and let $\mathcal{M} = \mathbb{C}_*^d \setminus X$. Then there is a basis for $H_d(\mathcal{M}, -\infty)$ consisting of a single generator $\gamma_{\mathbf{p}}$ for each critical point $\mathbf{p}$ of $h_{\mathbf{r}}$, which is tube around a cycle reaching maximum height at $\mathbf{p}$ and is homotopy equivalent to $S^1 \times (B^{d-1}, \partial B^{d-1})$.* □

We cover Theorem C.39, and generalizations, in Appendix D.

Notes

Detailed treatments of multivariate residues are given in [Pha11; AY83], while the classic text on Morse theory, which we have based our presentation around, is [Mil63].

Additional exercises

Exercise C.7 (univariate residues via Stokes's Theorem)**.** Let f be a meromorphic function inside and on a closed contour γ such that f has no singularities on γ. The familiar residue theorem in one variable states that

$$\frac{1}{2\pi i}\int_\gamma f = \sum_a \mathrm{Res}(f; a),$$

where the sum is over the poles a of f inside γ. Derive this from Theorem C.9. What are α, β, ω, $\mathcal{V}$, and $\mathbf{INT}[\alpha, \beta; \mathcal{V}]$?

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Exercise C.8. Let P be a polynomial in two complex variables such that $P(0, y)$ has only simple, non-zero roots. Let α be a small torus of polyradius $(\varepsilon, \varepsilon)$ and let β be a torus of polyradius (ε, M), where M is much larger than any root of $P(0, y)$. Compute the intersection class $\mathbf{INT}[\alpha, \beta; \mathcal{V}]$. *Hint:* Use the obvious homotopy and parametrize $\mathcal{V}_*$ as $y = f_j(x)$ near each root y_j of $P(0, y)$.

Exercise C.9 (lumpy sphere)**.** Let X be a sphere with a lump, that is, a patch on the northern hemisphere where the surface is raised to produce a local, but not global, maximum of the height function. List the critical points of the lumpy sphere and determine the homotopy types of the attachments. This gives a description of the lumpy sphere as a cell complex different from the complex with just two cells. Use this to compute the homology and verify it is the same as for the non-lumpy sphere.

Exercise C.10. Let $\mathcal{M}$ be a manifold with Morse function h having distinct critical values and let $\boldsymbol{x}$ be a critical point of Morse index k. Let P be any submanifold of $\mathcal{M}$ diffeomorphic to an open k-ball about $\boldsymbol{x}$ such that h is strictly maximized on P at $\boldsymbol{x}$. Prove that P is a homology generator for the local homology group $H_k(\mathcal{M}^{h(\boldsymbol{x})+\varepsilon}, \mathcal{M}^{h(\boldsymbol{x})-\varepsilon})$. *Hint:* This is true of any embedded k-disk through $\boldsymbol{x}$ in $\mathcal{M}^{h(\boldsymbol{x})}$ that intersects the ascending $(n-k)$-disk transversely.

Exercise C.11. Let $\iota_d : \mathbb{CP}^{d-1} \hookrightarrow \mathbb{CP}^d$ denote the embedding $\iota_d(z_0 : \cdots : z_d) = (z_0 : \cdots : z_d : 0)$.

(i) Show that $\mathbb{CP}^d \setminus \text{Image}(\iota)$ is homeomorphic to a $(2d)$-ball.
(ii) Describe $\mathbb{CP}^d$ as $\mathbb{CP}^{d-1}$ with a $(2d)$-cell attached. What is the attachment map on homology?
(iii) Use induction on d to compute the homology of $\mathbb{CP}^d$.