Canad. Math. Bull. Vol. **41** (2), 1998 pp. 137–139

# GENERICITY OF CERTAIN CLASSES OF UNITARY AND SELF-ADJOINT OPERATORS

J. R. CHOKSI AND M. G. NADKARNI

ABSTRACT. In a paper [1], published in 1990, in a (somewhat inaccessible) conference proceedings, the authors had shown that for the unitary operators on a separable Hilbert space, endowed with the strong operator topology, those with singular, continuous, simple spectrum, with full support, form a dense $G_\delta$. A similar theorem for bounded self-adjoint operators with a given norm bound (omitting simplicity) was recently given by Barry Simon [2], [3], with a totally different proof. In this note we show that a slight modification of our argument, combined with the Cayley transform, gives a proof of Simon's result, with simplicity of the spectrum added.

Theorem 1 (p. 148) of Choksi-Nadkarni [1] (hereafter referred to as CN) states

THEOREM 1. *$P_c$, the set of non-atomic (i.e. continuous) probability measures on the circle $S^1$, is a dense $G_\delta$ in $P$ the set of all probability measures on the circle (in the weak* topology.).*

Since the set of measures in $P$ which vanish on a fixed open arc $I$ of the circle is closed, nowhere dense in $P$ we immediately have

COROLLARY (CN P. 148). *The set $\{\mu : \mu \in P_c, \mathrm{support}(\mu) = S^1\}$ is dense $G_\delta$ in $P$.*

If $J$ is any fixed closed arc in $S^1$, and $P(J)$, resp. $P_c(J)$, denote the probability measures, resp. non-atomic probability measures, with support in $J$, *exactly* the same proof shows.

THEOREM 1′. *$P_c(J)$ is a dense $G_\delta$ in $P(J)$. The set $\{\mu : \mu \in P_c(J), \mathrm{support}(\mu) = J\}$ is dense $G_\delta$ in $P(J)$.*

In any case these results are well-known. Note that the intersection of two dense $G_\delta$ sets is a dense $G_\delta$ since $P$ is a Baire space. We now use Theorem 2 of CN (p. 148), which states

THEOREM 2. *Fix $\nu \in P$. Then $\nu^\perp = \{\mu \in P_c : \mu \perp \nu\}$ is dense $G_\delta$ in $P_c$ and hence in $P$.*

We get from Theorem 1′ and Theorem 2,

THEOREM 2′. *The set $\{\mu : \mu \in P_c(J), \mathrm{support}(\mu) = J, \mu \perp \nu\}$ is dense $G_\delta$ in $P(J)$.*

We now use results on pp. 154–55 of CN, which themselves only use the results given above. Let $U$ be the set of unitary operators on a separable Hilbert space $H$ with the strong

Received by the editors September 3, 1996.
AMS subject classification: 47B15.

operator topology, $\mu_u$ the maximal spectral type of $U$. Note that the map $f\colon U \longrightarrow \mu_u$ is continuous from $\mathcal{U}$ (strong operator topology) to $\mathcal{P}$ (weak* topology). We use

PROPOSITION 1 (CN P. 154). *If* support$(\mu_u) = K \subseteq S^1$ *then the set of conjugates* $\{W^{-1}UW : W \in \mathcal{U}\}$ *is dense in* $\{V \in \mathcal{U} : \text{support}(\mu_v) \subseteq K\}$.

Now from this proposition and Theorem 2′ above we have immediately

THEOREM 4′. *If $J$ is a fixed, closed arc in $S^1$, $\nu$ a fixed probability measure in $S^1$, then* $\{U \in \mathcal{U} : \mu_u$ *continuous*, $\mu_u \perp \nu$, support$(\mu_u) = J\}$ *is dense $G_\delta$ in the set* $\{U \in \mathcal{U} : \text{support}(\mu_u) \subseteq J\}$.

Note that using Theorem 5 on p. 156 we could even add the condition '$U$ has simple spectrum' and still get a dense $G_\delta$ set of $U$.

We now deduce results on self-adjoint operators using the Cayley transform and its inverse

$$A = i(I + U)(I - U)^{-1}$$
$$U = (A - iI)(A + iI)^{-1}.$$

For each $a \in \mathbf{R}^+$, there exists $b$, $0 < b < \frac{1}{2}$ such that the above maps give bijections, continuous in the strong operator topology between the set of unitary operators $U$ with spectrum $\subseteq$ the arc $[e^{2\pi ib}, e^{2\pi i(1-b)}]$ and the set of self-adjoint operators $A$ with spectrum $\subseteq [-a, a]$. Also given $b$, $0 < b < \frac{1}{2}$, $\exists a \in \mathbf{R}^+$ such that the same thing holds. We now have at once from Theorem 4′, the following theorem of B. Simon ([2], Theorem 3, [3] Theorem 3.1).

THEOREM (B. SIMON). *Let $X_a$ be the family of all self-adjoint operators $A$ on a fixed separable Hilbert space $H$ with $\|A\| \leq a$. Equip $X_a$ with the strong operator topology. Then*

$$\{A : A \in X_a, A \textit{ has singular continuous spectrum, } \operatorname{spec}(A) = [-a, a]\}$$

*is dense $G_\delta$ in $X_a$.*

In fact a much stronger result holds: we can replace Lebesgue measure by any fixed measure $\nu$ on $[-a, a]$, and in addition (using our Theorem 5) include simplicity of the spectrum.

THEOREM. *Let $\nu$ be a fixed measure on* $[-a, a]$. *Then*

$$\{A : A \in X_a, \textit{ spectrum of } A \textit{ simple continuous}, \perp \nu, \operatorname{spec}(A) = [-a, a]\}$$

*is dense $G_\delta$ in $X_a$.*

REMARK. We can even add the condition $|\hat{\mu}_A(n_k)| \longrightarrow 1$ along a subsequence $\{n_k\}$, where $\mu_A$ is the maximal spectral type of $A$.

https://doi.org/10.4153/CMB-1998-021-4 Published online by Cambridge University Press

## REFERENCES

**1.** J. R. Choksi and M. G. Nadkarni [CN], *Baire category in spaces of measures, unitary operators and transformations*. Invariant Subspaces and Allied Topics, (eds. H. Helson and B. S. Yadav), Narosa Publ. Co., New Delhi, 1990, 147—163.

**2.** R del Rio, S. Jitomirskaya, N. Makarov and B. Simon, *Singular continuous spectrum is generic*. Bull. Amer. Math. Soc. **31**(1994), 208–212.

**3.** B. Simon, *Operators with singular continuous spectrum: I. General operators*. Ann. of Math. **141**(1995), 131–145.

*Department of Mathematics & Statistics*
*McGill University*
*Montréal, Québec*
*H3A 2K6*
*e-mail: choksi@math.mcgill.ca*

*Department of Mathematics*
*University of Mumbai*
*Vidyanagri*
*Mumbai, 400 098*
*India*
*e-mail: nadkarni@mathbu.ernet.in*

https://doi.org/10.4153/CMB-1998-021-4 Published online by Cambridge University Press