SINGULAR INTEGRALS ARE PERRON INTEGRALS OF A CERTAIN TYPE

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Introduction. In [**7**] a Perron-like integral was defined in an arbitrary topological space and many of its basic properties were established. In this paper we shall show (the theorem in § 2) that in a suitable setting the integral from [**7**] includes a class of so-called singular integrals, i.e., generalized forms of the Cauchy principal value of an integral. Thus, the powerful machinery of Perron integration, e.g., the monotone and dominant convergence theorems, can be automatically applied to these singular integrals.

1. Preliminaries. In this section we shall recall some definitions and facts given in [**7**] and establish our notation.

Throughout, P is a topological space and $P^{\sim} = P \cup (\infty)$ is a one-point compactification of P. If $A \subset P$, A^{-} and $A^{\sim}$ denote the closure of A in P and $P^{\sim}$, respectively. For $x \in P^{\sim}$, Γ_x is a local base at x in $P^{\sim}$ (see [**4**, p. 50]).

Let σ be a system of subsets of P. A *convergence* (sometimes called a derivation basis; see [**3**, 1.1]) on σ is a function κ which to every $x \in P^{\sim}$ associates a family κ_x of nets $\{B_\alpha, \alpha \in D, >\} \subset \sigma$. If $A \in \sigma$, then $\sigma_A = \{B \in \sigma : B \subset A\}$. For $\delta \subset \sigma$ and $x \in P^{\sim}$, $\kappa_x(\delta) = \{\{B_\alpha\} \in \kappa_x\colon \{B_\alpha\} \subset \delta\}$.

Definition. A triple $\mathfrak{J} = \langle \sigma, \kappa, G\rangle$, where σ is a pre-ring (see [**2**]) of subsets of P, κ is a convergence on σ, and G is a non-negative finitely additive function on σ, is called an *integration basis* if and only if the following conditions are satisfied:

(i) For every $x \in P$, $\Gamma_x \subset \sigma$ and $\{U \cap P : U \in \Gamma_\infty\} \subset \sigma$;

(ii) For every $x \in P^{\sim}$, $\{U \cap P, U \in \Gamma_x, \subset\} \in \kappa_x$;

(iii) If $x \in P^{\sim}$ and $\{B_\alpha\}_{\alpha \in D} \in \kappa_x$, then for no $U \in \Gamma_x$ is the set $\{\alpha \in D\colon B_\alpha - U \neq \emptyset\}$ cofinal with D;

(iv) If $x \in P^{\sim}$, $\{B_\alpha\}_{\alpha \in D} \in \kappa_x$, and D' is a cofinal subset of D, then also $\{B_\alpha\}_{\alpha \in D'} \in \kappa_x$;

(v) If $x \in P^{\sim}$, $\{B_\alpha\} \in \kappa_x$, and $A \in \sigma$, then also $\{B_\alpha \cap A\} \in \kappa_x$;

(vi) If $\delta \subset \sigma$ is a non-empty semihereditary, stable system (see [**7**, 4.1 and 4.2]), then the set $\{x \in P^{\sim}\colon \kappa_x(\delta) \neq \emptyset\}$ is uncountable;

(vii) For every $U \in \bigcup\{\Gamma_x\colon x \in P\}$, $G(U) < +\infty$.

It was shown in [**7**] that integration bases exist and that for each of them we can define a non-absolutely convergent integral which is closely related to the

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Lebesgue integral. For the definition of this integral and its basic properties we refer the reader to [**7**].

It should be mentioned that this setting for integration slightly varies from that used in [**7**]; e.g., we use pre-rings instead of pre-algebras (see [**7**, 1.1]). However, the reader can easily see that the changes, which were adopted mainly for the sake of brevity, are inessential.

2. Main theorem. Let $\mathfrak{J} = \langle \sigma, \kappa, G \rangle$ be an integration basis and let $x_0 \in P^\sim$. We define the triple $\mathfrak{J}' = \langle \sigma', \kappa', G' \rangle$ as follows:

(i) $A \in \sigma$ belongs to σ' if and only if either $x_0 \notin A^\sim$ or $A = U \cap P$ for some $U \in \Gamma_{x_0}$;

(ii) $\kappa_x' = \kappa_x$ if $x \neq x_0$, and κ_{x_0}' consists of all nets $\{U \cap P, U \in \Gamma, \subset\}$ and $\{U \cap \emptyset, U \in \Gamma, \subset\}$, where Γ is a cofinal subset of Γ_{x_0};

(iii) G' is the restriction of G on σ'.

It can be readily verified that $\mathfrak{J}'$ is also an integration basis. It is called the *singularization* of $\mathfrak{J}$ at x_0. We shall denote by I and I' the integrals associated with the bases $\mathfrak{J}$ and $\mathfrak{J}'$, respectively. These integrals are understood to be defined on algebras $\sigma^\wedge$ and $\sigma'^\wedge$ generated by σ and σ', respectively (see [**8**]).

Let $A \in \sigma'^\wedge$ and let f be an extended real-valued function on A^-. It follows from [**7**, 6.17] that if $I(f, A)$ exists so does $I'(f, A)$ and they are equal. Moreover, if $I'(f, A)$ exists and either $x_0 \notin A^-$ or $\lim_{U \in \Gamma_{x_0}} G'(U \cap A) = 0$, then f is I-integrable over $A - U$ for every $U \in \Gamma_{x_0}$ and

$$\lim_{U \in \Gamma_{x_0}} I(f, A - U) = I'(f, A)$$

(see [**7**, 6.15 and 6.6]). The following theorem shows that under some additional hypotheses about the point x_0 this necessary condition for the existence of $I'(f, A)$ is also sufficient.

THEOREM. *Suppose that $P^\sim$ is first-countable and Hausdorff at x_0.*† *Let f be an extended real-valued function defined on A^-, where $A \in \sigma'^\wedge$, and let either $x_0 \notin A^-$ or $\lim_{U \in \Gamma_{x_0}} G'(U \cap A) = 0$. If f is I-integrable over $A - U$ for every $U \in \Gamma_{x_0}$ and if*

$$\lim_{U \in \Gamma_{x_0}} I(f, A - U) = c \neq \pm\infty,$$

then f is I'-integrable over A and $I'(f, A) = c$.

Proof. If $x_0 \notin A^\sim$, the theorem holds trivially. The set A being in $\sigma'^\wedge$ is a disjoint union of sets from σ' (see [**6**, (1.1)]). According to the definition of σ', at most one of these sets contains x_0 in its closure. Thus, without loss of generality, we may assume that A itself belongs to σ'.

By [**1**, Chapter 1, § 1, sec. 3, Theorem 2_x], we can choose a countable decreasing local base $\{U_n\}_{n=1}^\infty \subset \Gamma_{x_0}$ at x_0 in $P^\sim$. Let $A_1 = A - U_1$,

†This means that $P^\sim$ has a countable local base at x_0 (which need not coincide with Γ_{x_0}) and that to every $x \in P^\sim - (x_0)$ there exist disjoint sets $U \in \Gamma_{x_0}$ and $V \in \Gamma_x$.

$A_{n+1} = A \cap (U_n - U_{n+1})$, and let A_n^k, $k = 1, 2, \ldots, p_n$, be disjoint sets from σ' for which $A_n = \bigcup_{k=1}^{p_n} A_n^k$, $n = 1, 2, \ldots$. Given $\epsilon > 0$ we can find majorants $M_n^k \in \mathfrak{M}(f, A_n^k)$ and $m_n^k \in \mathfrak{M}(-f, A_n^k)$ (see [**7**, 3.2]) such that

$$M_n^k(A_n^k) < I(f, A_n^k) + \epsilon/p_n 2^{n+1}$$

and

$$m_n^k(A_n^k) < I(-f, A_n^k) + \epsilon/p_n 2^{n+1}.$$

If $B \in \sigma_A'$, then for some $n_0 \geqq 1$ either $B \cap U_{n_0} = \emptyset$ and the sum $\sum_{n=1}^{\infty} \sum_{k=1}^{p_n} M_n^k(B \cap A_n^k)$ is finite, or $U_{n_0} \subset B \cup (x_0)$ and the series $\sum_{n=1}^{\infty} \sum_{k=1}^{p_n} M_n^k(B \cap A_n^k)$ is convergent; for we have:

$$\begin{aligned}\lim_{r,s} \left| \sum_{n=r}^{s} \sum_{k=1}^{p_n} M_n^k(B \cap A_n^k) \right| &\leqq \lim_{r,s} \sum_{n=r}^{s} \sum_{k=1}^{p_n} [M_n^k(A_n^k) - I(f, A_n^k)] \\ &\quad + \lim_{r,s} \left| \sum_{n=r}^{s} I(f, A_n) \right| \leqq \lim_{r,s} \sum_{n=r}^{s} \epsilon/2^{n+1} \\ &\quad + \lim_{r,s} |I(f, A - U_{r-1}) - I(f, A - U_s)| = 0.\end{aligned}$$

Hence for $B \in \sigma_A'$ we can define

$$M'(B) = \sum_{n=1}^{\infty} \sum_{k=1}^{p_n} M_n^k(B \cap A_n^k)$$

and by a similar argument also

$$m'(B) = \sum_{n=1}^{\infty} \sum_{k=1}^{p_n} m_n^k(B \cap A_n^k).$$

Obviously, M' and m' are finite superadditive functions on σ_A'. Since $P^\sim$ is Hausdorff at x_0, it follows rather easily from [**7**, 5.3] that $M' \in \mathfrak{M}'(f, B)$ for every $B \in \sigma_A'$ for which $x_0 \notin B^\sim$.

Let $\{B_\alpha\} \in \kappa_{x_0}'(\sigma_A')$. Since $B_\alpha \cup (x_0) \in \Gamma_{x_0}$, there is an integer $n_\alpha \geqq 1$ such that $U_{n_\alpha} \subset B_\alpha \cup (x_0)$. By the definition of κ_{x_0}', $\lim n_\alpha = +\infty$ and thus

$$\begin{aligned}\liminf M'(B_\alpha) &\geqq \liminf \sum_{n=1}^{n_\alpha} \sum_{k=1}^{p_n} M_n^k(B_\alpha \cap A_n^k) \\ &\quad + \liminf \sum_{n=n_\alpha+1}^{\infty} \sum_{k=1}^{p_n} M_n^k(A_n^k) \geqq \liminf \sum_{n=1}^{n_\alpha} I(f, B_\alpha \cap A_n) \\ &= \liminf[I(f, A - U_{n_\alpha}) - I(f, A - B_\alpha)] = 0.\end{aligned}$$

Therefore ${}_{\#}M'(x_0, A) \geqq 0$ (see [**7**, 3.1]), and consequently $M' \in \mathfrak{M}'(f, A)$. Similarly we can prove that $m' \in \mathfrak{M}'(-f, A)$. From [**7**, 6.1] we obtain

$$\begin{aligned}0 &\leqq I_u'(f, A) + I_u'(-f, A) \leqq M'(A) + m'(A) \\ &= \sum_{n=1}^{\infty} \sum_{k=1}^{p_n} [M_n^k(A_n^k) + m_n^k(A_n^k)] \leqq 2 \sum_{n=1}^{\infty} \epsilon/2^{n+1} = \epsilon.\end{aligned}$$

Since ϵ is arbitrary, f is I'-integrable over A and it follows from [**7**, 6.16] that $I'(f, A) = c$.

The previous theorem justifies the term "singularization". Indeed, the integral I' is just a singular integral of the integral I at the point x_0 (see [**5**, p. 37, § 5, sec. 1]).

3. Application. As an application of the theorem from § 2 we shall give a simple proof of dominant convergence theorem for Cauchy principal value.

Let R be the set of all real numbers and let $R^+ = R \cup \{+\infty, -\infty\}$. By $(pv)\int_{-\infty}^{+\infty} f(x)\,dx$ we denote the *Cauchy principal value* of a function $f: R \to R^+$ which has a singularity at zero, i.e., $(pv)\int_{-\infty}^{+\infty} f(x)\,dx$ exists if and only if f is Lebesgue integrable over every closed set $A \subset R - (0)$ and a finite limit

$$\lim_{\epsilon \to 0+}\left[\int_{-\infty}^{-\epsilon} f(x)\,dx + \int_{\epsilon}^{+\infty} f(x)\,dx\right]$$

exists (see [**9**, p. 18]).

PROPOSITION. *Let* $g \leqq f_n \leqq h$ *and let* $(pv)\int_{-\infty}^{+\infty} g(x)\,dx$, $(pv)\int_{-\infty}^{+\infty} h(x)\,dx$, *and* $(pv)\int_{-\infty}^{+\infty} f_n(x)\,dx$, $n = 1, 2, \ldots$, *exist. If* $\lim f_n = f$, *then also* $(pv)\int_{-\infty}^{+\infty} f(x)\,dx$ *exists and*

$$(pv)\int_{-\infty}^{+\infty} f(x)\,dx = \lim\,(pv)\int_{-\infty}^{+\infty} f_n(x)\,dx.$$

Proof. Let $P = R$, $\Gamma_x = \{(x - \epsilon, x + \epsilon): \epsilon > 0\}$ for $x \in P$, and let $\Gamma_\infty = \{P^\sim - [-\epsilon, \epsilon]: \epsilon > 0\}$. We define the integration basis $\mathfrak{J} = \langle \sigma, \kappa^\circ, G\rangle$ as follows: σ is the algebra generated by all kinds of intervals, κ° is the natural convergence on σ (see [**7**, § 2 and Proposition 4.3]), and G is the Lebesgue measure in R restricted to σ. Then, according to [**7**, 8.4], the integral I coincides with the Lebesgue integral. If $\mathfrak{J}'$ is the singularization of $\mathfrak{J}$ at zero, then by the theorem in § 2, $I'(f, P) = (pv)\int_{-\infty}^{+\infty} f(x)\,dx$ for every function $f: R \to R^+$ for which either side has meaning. The proposition follows from [**7**, 6.13 or **8**, Theorem 3].

It is obvious that the previous proposition extends immediately to the more general type of Cauchy principal value defined in [**5**, § 5, sec. 1, p. 37]. It also extends rather easily to singular integrals involving not only one singularity but any discrete set of singularities.

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