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CLOSED SUBALGEBRAS OF HOMOGENEOUS BANACH ALGEBRAS

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Abstract

Rudin's synthesis method for investigating closed subalgebras of $L^1(G)$, where G is an infinite compact abelian group, is extended to the study of closed subalgebras in homogeneous Banach algebras and Segal algebras. Necessary and sufficient conditions are given for the synthesis to hold in certain classes of homogeneous Banach algebras and it is proved that in the $A^p(G)$ algebras the synthesis holds for $1 \leqq p \leqq 2$ but fails for $A^p(T), 2 < p < \infty$.

1. Introduction

Rudin (1962; Chapter 9) started to investigate the structure theorems for the closed subalgebras of $L^1(G)$ by the synthesis method: let A be a closed subalgebra of $L^1(G)$ where G is an infinite compact abelian group with character group Γ and define $\gamma_1 R \gamma_2$ by $\hat{f}(\gamma_1) = \hat{f}(\gamma_2)$ for all $f \in A$. Then R forms an equivalence relation on Γ, which induces the equivalence classes (Δ_λ), say. We follow Kahane in calling each such Δ_λ a Rudin class of A. By the Riemann-Lebesgue lemma, all Rudin classes Δ_λ will be finite sets, with one possible exception, Δ_0, say, where $\hat{f}(\gamma) = 0$ for all $f \in A$ and $\gamma \in \Delta_0$. Rudin asked if every closed subalgebra A of $L^1(G)$ could be synthsised from its Rudin classes. It is possible to generalize Rudin's synthesis method to a more general setting, where $L^1(G)$ is replaced by a homogeneous Banach algebra, which is a group algebra admitting a translation invariant norm, such as the algebras $L^p(G)$, $C^k(T)$ and $A^p(G)$. (See Wang, 1972)). A natural conjecture is that every closed subalgebra A of a homogeneous Banach algebra $B(G)$ is fully determined by its Rudin classes. When $B(G) = L^1(T)$, and $L^p(T)$, $1 < p < 2$ the conjecture is known to be false by examples due to Kahane (1965) and Rider (1969), respectively. When $B(G) = L^2(T)$, the conjecture is shown to be true by Edwards (1967; page 15).

For convenience, if $B(G)$ is a homogeneous Banach algebra, then we say the R-synthesis holds for $B(G)$ if every closed subalgebra A of $B(G)$ is fully determined by its Rudin classes. Otherwise the R-synthesis fails for $B(G)$. With this terminology, the R-synthesis holds for $L^2(T)$, but fails for $L^p(T)$, $1 \leqq p < 2$ In the present article, we prove that in the A^p-algebras, the R-synthesis holds for $1 \leqq p \leqq 2$, but fails for $2 < p < \infty$. Furthermore, we develop the synthesis theory for arbitrary homogeneous Banach algebras. In particular, we give necessary and sufficient conditions for the R-synthesis to hold in some classes of homogeneous Banach algebras.

2. Closed Subalgebras of Homogeneous Banach Algebras

We begin with some definitions.

2.1. DEFINITION. *Let G be an infinite compact abelian group with character group Γ. A subalgebra $B(G)$ of $L^1(G)$ is called a homogeneous Banach algebra on G if $B(G)$ is itself a Banach algebra under some norm $\| \ \|_B \geqq \| \ \|_{L^1}$ and satisfies the following properties*:

(i) *If $f \in B(G)$, $a \in G$ then $L_a f \in B(G)$ and $\| L_a f \|_B = \| f \|_B$, where $L_a f(x) = f_a(x) = f(x-a)$.*

(ii) *For each f in $B(G)$, $x \to L_x f$ is a continuous map of G into $(B(G), \| \ \|_B)$.*

A homogeneous Banach algebra $B(G)$ is called a Segal algebra if $B(G)$ is dense in $(L^1(G), \| \ \|_{L^1})$.

Applying the vector-valued integral, Reiter (1968; page 128) proved that a homogeneous Banach algebra $B(G)$ is a Banach $L^1(G)$-module (hence an ideal in $L^1(G)$) under the convolution $*$ in which the following equality holds.

$$f * g = \int f(x) g_x dx \quad \text{for } f \in L^1(G),\, g \in B(G).$$

Making use of Hewitt-Curtis-Figà-Talamanca Factorization Theorem, (see Hewitt and Ross (1970; page 268)), we have

$$L^1(G) * B(G) = B(G).$$

Moreover, if (K_λ) is a bounded approximate identity in $L^1(G)$ with compactly supported Fourier transforms, then

$$K_\lambda * f \to f \quad (f \in B(G))$$

in the $B(G)$-norm. (see Hewitt and Ross (1970; page 273)). For each λ, if $K_\lambda = \sum_{i=1}^{n} c_i \gamma_i$, where c_i are complex numbers and $\gamma_i \in \Gamma$, then let $K'_\lambda = \sum_{i=1}^{n'} c_i \gamma_i$ where $\sum_{i=1}^{n'}$ is the sum of all $c_i \gamma_i$, $i = 1, 2, \cdots, n$ in which γ_i are outside the zero set $Z(B(G))$ of $B(G)$. By the Localization Lemma (see Reiter (1968; pages 19–20)), $K'_\lambda \in B(G)$. We conclude that

$$\| K'_\lambda * f - f \|_B = \| K_\lambda * f - f \|_B$$
$$\to 0 \quad \text{as } \lambda \to \infty.$$

Consequently,

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2.2. THEOREM. *A homogeneous Banach algebra $B(G)$ admits a possibly unbounded approximate identity.*

For the reader's convenience, we describe briefly the spectral synthesis theory of closed ideals in a homogeneous Banach algebra $B(G)$, where G will denote an *infinite compact abelian group* with character group Γ. We say that the spectral synthesis holds for $B(G)$ if one of the following equivalent statements is satisfied:

(i) If I and J are two closed ideals in $B(G)$ with $Z(I) = Z(J)$, then $I = J$.

(ii) If I is a closed ideal in $B(G)$, and f is a function in $B(G)$ such that $Z(I) \subset Z(f)$, then $f \in I$.

Where

$$Z(I) = \{\gamma \in \Gamma : \hat{f}(\gamma) = 0 \text{ for all } f \in I\},$$

$$Z(f) = \{\gamma \in \Gamma ; \hat{f}(\gamma) = 0\}.$$

Recall that the algebra $L^1(G)$ satisfies the two equivalent conditions. In other words, the spectral synthesis holds for $L^1(G)$. Moreover,

2.3. THEOREM. *The spectral synthesis holds for any homogeneous Banach algebra on a compact abelian group G.*

PROOF. Let $B(G)$ be a homogeneous Banach algebra and I a closed ideal in $B(G)$. We claim that if $\gamma \in \Gamma$ and $\gamma \notin Z(I)$, then $\gamma \in I$. In fact, $\gamma \notin Z(I)$ implies there is $g \in I$ such that $\hat{g}(\gamma) \neq 0$. Recall that $g * \gamma = \hat{g}(\gamma)\gamma$. Thus $\gamma = 1/\hat{g}(\gamma) g * \gamma \in B(G)$ since $g \in B(G)$ and $\gamma \in L^1(G)$. Therefore $g * \gamma \in I$, $\gamma \in I$. Suppose $f \in B(G)$ such that $Z(I) \subset Z(f)$. By Theorem 2.2, $B(G)$ admits an approximate identity with compactly supported Fourier transforms. Consequently, for $\varepsilon > 0$, take $g = \sum_{i=1}^{n} c_i \gamma_i$, where c_i are complex numbers and $\gamma_i \in \Gamma$ such that

$$\| g * f - f \|_B < \varepsilon .$$

Since

$$g * f = \sum_{i=1}^{n} c_i \hat{f}(\gamma_i) \gamma_i ,$$

and since $\hat{f}(\gamma_i) \neq 0$ implies $\gamma_i \in I$, we have $g * f \in I$. We conclude that $f \in I$.

2.4. REMARK. Let $B(G)$ be a homogeneous Banach algebra and I a closed ideal in $B(G)$. The preceding proof tells us that if $\gamma \notin Z(I)$, then $\gamma \in I$. Consequently, if $\gamma_1 \neq \gamma_2$ with $\gamma_1, \gamma_2 \notin Z(I)$, then taking $f = \gamma_1$, it can be seen that there exists $f \in I$ such that $\hat{f}(\gamma_1) \neq \hat{f}(\gamma_2)$. Thus, if R is an equivalence relation on Γ defined by $\gamma_1 R \gamma_2$ if $\hat{f}(\gamma_1) = \hat{f}(\gamma_2)$ for all $f \in I$, each equivalence class Δ_λ other than $\Delta_0 = Z(I)$ is a singleton. Therefore the spectral synthesis hold for $B(G)$ if and only if, for every closed ideal I, and for $f \in B(G)$ with $Z(I) \subset Z(f)$, f can be approximated in the $B(G)$-norm by the trigonometric polynomials P in I such that $\hat{P}$ is constant on each Δ_α, $\alpha \neq 0$.

Similarly, for any not necessarily closed subalgebra A of a homogeneous Banach algebra $B(G)$, write $\gamma_1 R \gamma_2$ if $\hat{f}(\gamma_1) = \hat{f}(\gamma_2)$ for $f \in A$. The relation R is an equivalence relation on Γ, induced by A. One distinguished equivalence class is $\Delta_0 = Z(A)$; Δ_0 may be infinite. The other equivalence classes, denoted by (Δ_λ), where λ runs through a suitable index set Λ, must be finite, since $\hat{f} \in C_0(\Gamma)$ for $f \in A$ and Γ is discrete. We follow Kahane (1965) in calling each such Δ_λ ($\lambda = 0$ or not) a Rudin class of A. It is possible to carry the synthesis theory of closed ideals over to the closed subalgebras. This assertion will follow from the following theorem.

2.5. THEOREM. *Let $B(G)$ be a homogeneous Banach algebra and A a closed subalgebra of $B(G)$. Suppose (Δ_λ) is the set of Rudin classes induced by A, and P_λ, for each $\lambda \neq 0$, is a trigonometric polynomial such that $\hat{P}_\lambda = \chi_{\Delta_\lambda}$, the characteristic function of Δ_λ, then it follows that $P_\lambda \in A$.*

For the proof refer to Edwards (1967; page 12).

Suppose that $B(G)$ is a homogeneous Banach algebra and that A is a subalgebra of $B(G)$ which induces the Rudin classes Δ_λ. Let $P(A)$ denote the subalgebra generated by the trigonometric polynomials P_λ such that $\hat{P}_\lambda = \chi_{\Delta_\lambda}$, $\lambda \neq 0$. (Hence $P(B(G))$ consists of all trigonometric polynomials in $B(G)$) and let $A^{B(G)}$ denote the closed subalgebra of all f in $B(G)$ such that $\hat{f}(\Delta_0) = 0$, and $\hat{f}$ is constant on each Δ_λ, $\lambda \neq 0$. Then $\overline{P(A)}^B$ and $A^{B(G)}$ induce the same Rudin classes Δ_λ) and, if A_1 is any closed subalgebra of $B(G)$ which induces (Δ_λ), then $\overline{P(A)}^B \subset A_1 \subset A^{B(G)}$. Furthermore, both A and $\bar{A}^B$ evidently induce the same Rudin classes and therefore $P(A) = P(\bar{A}^B)$ and $A^{B(G)} = (\bar{A}^B)^{B(G)}$

2.6. DEFINITIONS. *Let $B(G)$ be a homogeneous Banach algebra and A a closed subalgebra of $B(G)$. A is called an R-subalgebra if $\overline{P(A)}^B = A = A^{B(G)}$.*

We say that the R-synthesis holds for $B(G)$ if every closed subalgebra of $B(G)$ is an R-subalgebra. Otherwise, the R-synthesis fails for $B(G)$.

Kahane (1965) proved that the R-synthesis fails for $L^1(T)$ and, Rider (1969) proved that the R-synthesis fails for $L^p(T)$, $1 < p < 2$. As a matter of fact, Rider proved:

2.7. THEOREM. *There is a closed subalgebra A of $L^1(T)$ and a function f in $L^1(T)$ such that*

(i) $f \in \bigcap_{1 \leq p < 2} L^p(T)$.

(ii) $f \in A^{L^1(T)}$ *but* $f \notin \overline{P(A)}^{L^1}$.

The R-synthesis for many other homogeneous Banach algebras will be investigated. Recall that the $A^p(G)$ algebra $(1 \leqq p < \infty)$ is a group algebra which consists of all f in $L^1(G)$ with $\hat{f}$ in $L^p(\Gamma)$.

2.8. THEOREM. *The R-synthesis holds for $A^p(T)$, $1 \leqq p \leqq 2$, but fails for $A^p(T)$, $2 < p < \infty$.*

PROOF. (i) $1 \leqq p \leqq 2$.

Let D be a closed subalgebra of $A^p(T)$ which induces the Rudin classes (Δ_λ), and $f \in A^p(T)$ such that $\hat{f}(\Delta_0) = 0$, $\hat{f}(\Delta_\lambda) = z_\lambda \neq 0$ for $\lambda \neq 0$. Suppose that the cardinality of Δ_λ is n_λ, $\lambda \neq 0$. We conclude that

$$\sum_{\lambda = 0} n_\lambda |z_\lambda|^p = \sum_{n \in z} |\hat{f}(n)|^p < \infty .$$

Consequently, there exists, for $\varepsilon > 0$, a finite set F_0 of indices such that $(\sum_{\beta \notin F} n_\beta |z_\beta|^p) < (\varepsilon/2)^p$ whenever F is a finite set of indices with $F \supset F_0$.

Let P_β, $\beta \neq 0$, be the trigonometric polynomial such that $\hat{P}_\beta = \chi_{\Delta_\beta}$, and let $g = f - \sum_{\beta \in F} z_\beta P_\beta$. Then, since $1 \leqq p \leqq 2$, by the Hausdorff-Young theorem,

$$\|g\|_{L^1} \leqq \|g\|_{L^q} \left(\frac{1}{p} + \frac{1}{q} = 1\right)$$

$$\leqq \|\hat{g}\|_{l^p}$$

But

$$\|g\|_{A^p} = \|g\|_{L^1} + \|\hat{g}\|_{l_p},$$

so

$$\|g\|_{A^p} \leqq 2\|\hat{g}\|_{l_p}$$

$$= 2\left(\sum_{n \in z} |\hat{f}(n) - \sum_{\beta \in F} z_\beta \hat{P}_\beta(n)|^p\right)^{1/p}$$

$$= 2\left(\sum_{\beta \notin F} n_\beta |z_\beta|^p\right)^{1/p}$$

$$< \varepsilon .$$

Or,

$$\|f - \sum_{\beta \in F} z_\beta P_\beta\|_{A^p} < \varepsilon .$$

Since ε is arbitrary and $\sum_{\beta \in F} z_\beta P_\beta \in \overline{P(D)}^{A^p}$, $f \in \overline{P(D)}^{A^p}$. The R-synthesis therefore holds for $A^p(T)$, $1 \leqq p \leqq 2$.

(ii) $2 < p < \infty$.

Let $q = p/(p-1)$. By Theorem 2.7, there is a closed subalgebra E of $L^1(T)$ and an element f of $L^q(T)$ such that $f \in E^{L^1}$ but $f \notin \overline{P(E)}^{L^1}$. By the Hausdorff-Young theorem, $f \in A^p(T)$, and clearly $f \notin \overline{P(E)}^{A^p}$. However, $A^p(T)$, being a Segal algebra, contains $P(E)$ and so $\overline{P(E)}^{A^p}$ and $E^{L^1} \cap A^p(T)$ are distinct closed subalgebras of $A^p(T)$ inducing the same Rudin classes. This proves part (ii).

An adaptation of the proof of Theorem 2.8. (i) gives the following more general result.

2.9. THEOREM. *The R-synthesis holds for $A^p(G)$, $1 \leqq p \leqq 2$.*

We try to derive some conditions under which the R-synthesis holds for a homogeneous Banach algebra.

2.10. THEOREM. *Let $B(G)$ and $D(G)$ be two homogeneous Banach algebras with $B(G) \subset D(G)$ and A an R-subalgebra of $B(G)$. Then $A = \bar{A}^D \cap B(G)$. In particular, $I = \bar{I}^{L^1} \cap B(G)$ for any closed ideal I in $B(G)$.*

PROOF. Since there is a constant $k > 0$ such that $\| \ \|_D \leqq k \| \ \|_B$, $J \cap B(G)$ is a closed subalgebra of $B(G)$ whenever J is a closed subalgebra of $D(G)$. In particular, $\bar{A}^D \cap B(G)$ is a closed subalgebra of $B(G)$. Furthermore, plainly,

$$\hat{f}(\gamma) = \hat{f}(\beta) \text{ for } f \in \bar{A}^D \cap B(G)$$

$$\Leftrightarrow \hat{g}(\gamma) = \hat{g}(\beta) \quad \text{for} \quad g \in A.$$

We conclude that both $\bar{A}^D \cap B(G)$ and A induce the same Rudin classes. Thus $A = \bar{A}^D \cap B(G)$.

Finally, for any closed ideal I in $B(G)$, $I = \bar{I}^{L^1} \cap B(G)$, since, by Theorem 2.3, I is an R-subalgebra.

2.11. THEOREM. *Let $B(G)$ and $D(G)$ be two homogeneous Banach algebras with $B(G) \subset D(G)$, and suppose that the R-synthesis holds for $D(G)$. Then the R-synthesis holds for $B(G)$ if and only if $A = \bar{A}^D \cap B(G)$ for any closed subalgebra A of $B(G)$.*

PROOF. The only-if part follows from Theorem 2.10. Suppose $A = \bar{A}^D \cap B(G)$ for any closed subalgebra A of $B(G)$. Let A be a closed subalgebra of $B(G)$ inducing the Rudin classes (Δ_λ). Then, plainly, we have the following equalities:

$$\begin{aligned} A^B &= \{f \in B(G) : \hat{f}(\Delta_0) = 0, \hat{f}(\Delta_\lambda) = \text{constant}, \ \lambda \neq 0\} \\ &= \{f \in D(G) : \hat{f}(\Delta_0) = 0, \hat{f}(\Delta_\lambda) = \text{constant}, \ \lambda \neq 0\} \cap B(G) \\ &= \overline{P(A)}^D \cap B(G) \text{ since the } R\text{-synthesis holds for } D(G) \\ &= \bar{W}^D \cap B, \text{ where } W = \overline{P(A)}^B \\ &= \overline{P(A)}^B \text{ by the hypothesis.} \end{aligned}$$

This asserts that A is an R-subalgebra and the theorem then follows.

The R-synthesis for special homogeneous Banach algebras will be investigated. For any set Ω in Γ, let

$$L^1_\Omega(G) = \{f \in L^1(G) : \hat{f}(\gamma) = 0 \text{ for } \gamma \notin \Omega\}.$$

Then $(L^1_\Omega(G), \| \ \|_{L^1})$ forms a homogeneous Banach algebra.

2.12. THEOREM. *If A is not an R-subalgebra of $L^1(T)$, and if Ω is the complement of the zero set of A, then the R-synthesis fails for the homogeneous Banach algebra $L^1_\Omega(T)$.*

PROOF. $\overline{P(A)}^{L^1}$ and $A^{L^1(T)}$ are closed subalgebras of L^1_Ω which induce the same Rudin classes but $\overline{P(A)}^{L^1} \subsetneq A^{L^1(T)}$.

It is interesting to note that as a homogeneous Banach algebra, L^1_Ω may lack the R-synthesis property, while, as a subalgebra, L^1_Ω is a R-subalgebra of $L^1(T)$ since it is a closed ideal in $L^1(T)$ (see Thesrem 2.3). A set Δ in Γ is a Λ_p-set, $1 \leqq p < \infty$, if for every f in $L^1(G)$ such that $\hat{f}(\gamma) = 0$ for $\gamma \notin \Delta$, then $f \in L^p(G)$ (see Hewitt and Ross (1970; page 420)).

2.13. THEOREM. *Let Δ be a Λ_p-set in Γ, $2 \leqq p < \infty$. Then the R-synthesis holds for the homogeneous Banach algebra L^1_Δ.*

PROOF. Since $L^2(T) = A^2(T)$, the R-synthesis holds for $L^2(T)$ by Theorem 2.8. We begin with the remark that if Δ is a Λ_p-set, then $L^1_\Delta \subset L^2(T)$, and there is a constant C such that

$$(*) \qquad \| f \|_{L^2} \leqq C \| \quad \|_{L^1}$$

for $f \in L^1_\Delta$. Suppose that A is a closed subalgebra of L^1_Δ, then $\overline{P(A)}^{L^1}$, A, and $A^{L^1(T)}$ induce the same Rudin classes. But, using $(*)$, $\overline{P(A)}^{L^1} = \overline{P(A)}^{L^2}$ and $A^{L^1(T)} = A^{L^2(T)}$. We conclude that $\overline{P(A)}^{L^1}$ and $A^{L^1(T)}$ are two closed subalgebras of $L^2(T)$ which induce the same Rudin classes. Hence $\overline{P(A)}^{L^1} = A^{L^1(T)}$, or A is a R-subalgebra.

3. Closed Subalgebras of Segal Algebras.

We now characterize some Segal algebras in which the R-synthesis holds, beginning with a couple of interesting theorems.

3.1. THEOREM. *Let $R(G)$ and $S(G)$ be two Segal algebras with $R(G) \subset S(G)$. Suppose that the R-synthesis holds for both $R(G)$ abd $S(G)$. Then $A \to \bar{A}^S$ is a 1-1 correspondence between the family of all closed subalgebras of $R(G)$ and that of $S(G)$. More precisely,*

(i) *For any two closed subalgebras J_1 and J_2 of $S(G)$, if $J_1 \cap R(G) = J_2 \cap R(G)$ then $J_1 = J_2$,*

(ii) *For any two closed subalgebras A_1 and A_2 of $R(G)$ if $\overline{A_1}^S = \overline{A_2}^S$, then $A_1 = A_2$,*

(iii) *For any closed subalgebra J of $S(G)$, we have $J = \overline{J \cap R(G)}^S$.*

PROOF. (i) Let J_1 and J_2 be two closed subalgebras of $S(G)$ with $J_1 \cap R(G) = J_2 \cap R(G)$. Then, $P(J_1) \subset R(G)$ (see Reiter (1968, page 128)). Consequently,

$$P(J_1) \subset J_1 \cap R(G)$$

Or,

$$\begin{aligned} J_1 &= \overline{P(J_1)}^S \subset \overline{J_1 \cap R(G)}^S \\ &= \overline{J_2 \cap R(G)}^S \subset \overline{J_2}^S \\ &= J_2 . \end{aligned}$$

Similarly it can be shown that $J_2 \subset J_1$. Hence $J_1 = J_2$.

(ii) It suffices to show that $A = \bar{A}^S \cap R(G)$, for any closed subalgebra A of $R(G)$. However, this assertion follows from Theorem 2.10.

(iii) Suppose that J is a closed subalgebra of $S(G)$. Then $J \cap R(G)$ is a closed subalgebra of $R(G)$. By the proof in (ii),

$$J \cap R(G) = \overline{J \cap R(G)}^S \cap R(G).$$

Applying (i), we get

$$J = \overline{J \cap R(G)}^S.$$

Combining (i), (ii) and (iii), it is easy to see that $A \to \bar{A}^S$ is 1–1 map of the family of all closed subalgebras of $R(G)$ onto that of $S(G)$.

3.2. THEOREM. *Let $R(G)$ and $S(G)$ be two Segal algebras with $R(G) \subset S(G)$. Suppose that the R-synthesis holds for $R(G)$. Then the R-synethsis holds for $S(G)$ if and only if $J = \overline{J \cap R(G)}^S$ for any closed subalgebra J of $S(G)$.*

PROOF. Suppose that the R-synthesis holds for $S(G)$, then, by Theorem 3.1 (iii), $J = \overline{J \cap R(G)}^S$ for any closed subalgebra J of $S(G)$. On the other hand, suppose $J = \overline{J \cap R(G)}^S$ for any closed subalgebra J of $S(G)$. Then we claim that $P(J) = P(J \cap R(G))$. In fact, if $\gamma, \beta \in \Gamma$ with $\hat{f}(\gamma) = \hat{f}(\beta)$ for $f \in J$, then, plainly, $\hat{g}(\gamma) = \hat{g}(\beta)$ for $g \in J \cap R(G)$. Conversely, let $\gamma, \beta \in \Gamma$ with $\hat{g}(\gamma) = \hat{g}(\beta)$ for $g \in J \cap R(G)$. Suppose $f \in J$. Since $J = \overline{J \cap R(G)}^S$, there is a sequence (g_n) in $J \cap R(G)$ with $g_n \to f$ in the S-norm and so with $\hat{g}_n \to \hat{f}$ uniformly. Consequently, $\hat{f}(\gamma) = \hat{f}(\beta)$ for $f \in J$. Now, since the R-synthesis holds for $R(G)$,

$$J \cap R(G) = \overline{P(J \cap R(G))}^R.$$

Therefore

$$\begin{aligned} J &= \overline{J \cap R(G)}^S \\ &= \overline{P(J \cap R(G))}^S \\ &= \overline{P(J)}^S \end{aligned}$$

Thus the R-synthesis holds for $S(G)$.

3.3. DEFINITION *Two Segal algebras $(R(G), \| \ \|_R)$ and $(S(G, \| \ \|_S)$ are said to be comparable if there exists a constant $k > 0$ such that*

$$\| \ \|_R \leqq k \| \ \|_S \text{ or } \| \ \|_S \leqq k \| \ \|_R .$$

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Note that two Segal algebras with one containing the other are comparable.

3.4. THEOREM. *Suppose $R(G)$ and $S(G)$ are two comparable Segal algebras. If the R-synthesis holds for both of them, then the R-synthesis holds for the Segal algebra $R \cap S(G)$.*

PROOF. Take a constant $k > 0$ such that $\| \ \|_R \leqq k \| \ \|_S$, say. Let A be a closed subalgebra of $R \cap S(G)$ and let $f \in \bar{A}^S \cap (R \cap S(G))$. Pick a sequence (f_n) in A such that $\|f_n - f\|_S \to 0$ as $n \to \infty$. Then, plainly, $\|f_n - f\|_R \to 0$ as $n \to \infty$. Consequently,

$$\|f_n - f\|_{R\cap S} \to 0 \text{ as } n \to \infty$$

or $f \in \bar{A}^{R\cap S} = A$. This asserts that $A = \bar{A}^S \cap (R \cap S(G))$ for any closed subalgebra A of $R \cap S(G)$. The proof then follows from Theorem 2.11.

Together with Theorems 3.1 and 3.4, we have

3.5. COROLLARY. *If $R(G)$ and $S(G)$ are two comparable Segal algebras in which the R-synthesis holds, then there is a 1-1 correspondence between the family of all closed subalgebras of $R(G)$ and that of $S(G)$.*

3.6. COROLLARY. *The theory of the closed subalgebras of the $A^p(G)$-algebras, $1 \leqq p \leqq 2$, is independent of p.*

In the final result of this section, we shall study the algebras whose Rudin classes have bounded lengths. The result generalizes a result of Kahane (1965) for $L^1(T)$ and uses the same proof. As Kahane's proof has not been published in detail, we give a full proof here.

3.7. THEOREM. *Let $S(T)$ be a Segal algebra containing $C(T)$. Suppose A is a closed subalgebra of $S(T)$ which induces the Rudin classes (Δ_λ). If there is a constant $k > 0$ such that $|n_1 - n_2| < k$ whenever $n_1, n_2 \in \Delta_\lambda$ for all $\lambda \neq 0$, then A is an R-subalgebra.*

PROOF. Note that $C(T) \subset S(T)$ implies there is a constant $d > 0$ with $\| \ \|_S \leqq d \| \ \|_\infty$. Take the Fejér Kernel $(K_n)_{n=1}^\infty$ on T (see Katznelson (1968; page 12)), and consider a function f in $A^{S(T)}$. Since $\|K_n * f - f\|_S \to 0$ (see Katznelson (1968; page 15)), it suffices to prove that $K_n * f \in \overline{P(A)}^S$ for sufficiently large n.

$$\begin{aligned}(K_n * f)(t) &= \sum_{-n}^{n} \left(1 - \frac{|j|}{n+1}\right)\hat{f}(j)e^{ijt} \\ &= \sum_{j\in P} \left(1 - \frac{|j|}{n+1}\right)\hat{f}(j)e^{ijt} - \sum_{j\in Q}\left(1 - \frac{|j|}{n+1}\right)\hat{f}(j)e^{ijt}\end{aligned}$$

where $P = \bigcup_{\lambda\neq 0}\{\Delta_\lambda : \Delta_\lambda \cap [-n, n] \neq \phi\}$ and $Q = P \cap \{j : n < |j| \leqq k + n\}$. For $j \in \Delta_\lambda \cap P$, pick m_λ such that $m_\lambda R j$ and $|m_\lambda| \leqq n$. Then

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$$\sum_{j\in P}\left(1-\frac{|m_\lambda|}{n+1}\right)\hat{f}(m_\lambda)e^{ijt} = g(t) \text{ for } g\in P(A).$$

We have

$$\|K_n * f-g\|_S = \left\|\sum_{j\in P}\frac{|m_\lambda|-|j|}{n+1}\hat{f}(m_\lambda)e^{ijt}\right\|_S + \left\|\sum_{j\in Q}\left(1-\frac{|j|}{n+1}\right)\hat{f}(j)e^{ijt}\right\|_S$$

$$\leqq \frac{kd}{n+1}\sum_{j\in P}|\hat{f}(m_j)| + d\sum_{j\ Q}|\hat{f}(j)|.$$

For $\varepsilon > 0$ there exists $N_1 > k$ such that

$$|j| \geqq N_1 \Rightarrow |\hat{f}(j)| \leqq \frac{\varepsilon}{8kd}.$$

Then take a positive integer N_2 such that

$$\sum_{j=-N_1}^{N_1}\frac{|\hat{f}(j)|}{N_2+1} < \frac{\varepsilon}{8kd}.$$

For $n \geqq N_1 + N_2$, we get

$$kd\sum_{j\in P}\frac{|\hat{f}(j)|}{n+1} = kd\sum_{|j|\leqq N_1}\frac{|\hat{f}(j)|}{n+1} + kd\sum_{N_1\leqq|j|\leqq n+k}\frac{|\hat{f}(j)|}{n+1}$$

$$\leqq kd\sum_{|j|\leqq N_1}\frac{|\hat{f}(j)|}{N_2+1} + \frac{\varepsilon}{8}\sum_{N_1\leqq|j|\leqq n+k}\frac{1}{n+1}$$

$$< \frac{\varepsilon}{8}+\frac{\varepsilon}{4}$$

$$< \frac{\varepsilon}{2}$$

and

$$d\sum_{j\in Q}|\hat{f}(j)| \leqq d\frac{\varepsilon}{8kd}\cdot 2k$$

$$= \frac{\varepsilon}{4}.$$

Therefore

$$\|K_n * f-g\|_S < \varepsilon \text{ as } n \geqq N_1 + N_2.$$

This completes the proof.

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