7

A Companion to Quantum Groups

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7.1 Introduction

Note that the category of modules of a bialgebra has a tensor product structure. Given two modules V, W, in general it is not clear that $V \otimes W$ is isomorphic to $W \otimes V$ as modules. If the bialgebra possesses a quasitriangular structure, i.e. a universal R-matrix, then there exists a natural isomorphism which is compatible with the tensor structure. Quantum groups form a rich family of (non-commutative and non-cocommutative) quasitriangular bialgebras which in some sense deform associative algebras naturally associated with certain groups and Lie algebras.

Quantum groups were discovered in the 1980s in the context of quantum integrability (simultaneous diagonalizability of commuting Hamiltonians via solutions of the Yang–Baxter equation), initially in [27]. Their theory is a vast topic that has developed immensely in the last four decades. There are various reasons why quantum groups are interesting: there are connections with low-dimensional topology (e.g. representations of braid groups and construction of quasi-invariants for knots, links, etc.), q-deformed harmonic analysis (closely connected to the older theory of special functions depending on a deformation parameter) and non-commutative geometry (the study of deformed algebras of functions on algebraic groups).

It is difficult to give a precise definition of a quantum group that encompasses the various classes of examples which are known as such. Focusing on deformations of *co*commutative bialgebras, one may propose the following "soft" definition, which will be our guide in these notes:

A quantum group is a non-commutative bialgebra depending on a parameter that is

(1) quasitriangular for all values of the parameter and

(2) cocommutative only for special values of the parameter.

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We will showcase a special class of quantum groups: quantized enveloping algebras $U_q\mathfrak{g}$, also known as Drinfeld–Jimbo quantum groups, and explain the construction of the universal R-matrix. The representation theory of $U_q\mathfrak{g}$, if q is not a root of unity, stays close to that of $U\mathfrak{g}$, i.e. that of $\mathfrak{g}$, which is well known. Unfortunately it is beyond the scope of this account to discuss in detail other types of quantum groups (e.g. Yangians and RTT algebras, compact quantum groups, bicrossproduct quantum groups) or delve very deep into particular advanced branches and applications of this theory such as: Lie bialgebra quantization; diagonalization of commuting transfer matrices; the definition of knot (quasi-)invariants; root-of-unity phenomena; canonical bases (crystal bases) and the $q \to 0$ limit; Knizhnik–Zamolodchikov equations. Standard textbook resources focusing on quantum groups are for instance [7, 17, 23, 24, 30, 32]. For an account tailored to the application of quantum groups to quantum integrability. We recommend the lecture notes [36] and the book [19]. Note that quantum groups have applications in mathematical physics beyond integrability; one can in particular point out their role in quantum gravity (see [31]).

We hope that these notes provide the reader with a basic working knowledge of quantum groups and spur them on to a deeper investigation in this rich variety of topics. For most of these lecture notes, we assume fairly little background knowledge beyond abstract linear algebra and basic notions of group theory and representation theory, although some familiarity with Lie theory and category theory is helpful.

7.1.1 Outline

First of all we will review bialgebras, Hopf algebras and their representations (and introduce notation that we will use throughout this chapter) in Section 7.2. We discuss quasitriangular bialgebras and the induced braiding on their categories of representations in Section 7.3. In Section 7.4 we discuss their quantizations, Drinfeld–Jimbo quantum groups $U_q\mathfrak{g}$, and in Section 7.5 the quasitriangular structure of $U_q\mathfrak{g}$; in these sections we pay particular attention to the special case $U_q\mathfrak{sl}_2$. We end the notes with a brief investigation in Section 7.6 into more recent developments involving quasitriangularity for special types of subalgebras of bialgebras, and the associated cylindrically braided structure on the category of modules.

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7.2 Bialgebras

In this section we recall some basic theory surrounding bialgebras and monoidal categories, setting the stage for the next section which deals with quasitriangular bialgebras and braided monoidal categories. For more background on some of this material the reader can consult for instance [7, Sec. 4.1] or the lecture notes [39].

7.2.1 Notation

We fix an arbitrary[1] field k; linear structures will always be with respect to k, which we may suppress from the notation. In this way $\otimes = \otimes_k$ is the tensor product over k, $\mathrm{Hom}(V,W) = \mathrm{Hom}_k(V,W)$ is the vector space of k-linear maps from V to W and $\mathrm{End}(V) = \mathrm{End}_k(V)$ is the algebra of k-linear maps on V.

Let V,W be vector spaces. We denote the identity map on V by id_V. We denote by $\sigma_{V,W}$ the unique linear map from $V \otimes W$ to $W \otimes V$ which sends $v \otimes w$ to $w \otimes v$ for all $v \in V, w \in W$. If there is no cause for confusion, we will simply write id instead of id_V and σ instead of $\sigma_{V,V}$.

7.2.2 Algebras

We consider an algebra A over k, i.e. a vector space that possesses a bilinear multiplication map: $A \times A \to A$ which is compatible with scalar multiplication: $\lambda(ab) = (\lambda a)b = a(\lambda b)$ for all $\lambda \in k$ and $a,b \in A$. Note that since the multiplication map is bilinear we can view it as a linear map $m : A \otimes A \to A$. We will always[2] assume that A is *associative*, i.e.

$$m \circ (m \otimes \mathsf{id}_A) = m \circ (\mathsf{id}_A \otimes m) \in \mathrm{Hom}(A, A \otimes A \otimes A), \qquad (7.2.1)$$

and *unital*, i.e. there is a linear map $\eta : k \to A$ such that

$$m \circ (\eta \otimes \mathsf{id}_A) = \mathsf{id}_A = m \circ (\mathsf{id}_A \otimes \eta) \in \mathrm{End}(A,A), \qquad (7.2.2)$$

where we have used that $k \otimes A \cong A \cong A \otimes k$. Note that such η must be injective and hence we can, and shall, identify k with $\eta(k) \subseteq A$. In particular, there is an

[1] Later we will assume that k is of characteristic zero and algebraically closed.
[2] Note that Lie algebras are not algebras using this restricted definition.

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element $\eta(1) \in A$, which we simply denoted by 1, with the property $1a = a1 = a$ for all $a \in A$ (conversely, given such an element, there is a unique linear map $\eta : k \to A$ sending $1 \in k$ to $1 \in A$).

Let A and B be algebras with multiplications m_A, m_B and unit maps η_A, η_B, respectively. An *algebra morphism* from A to B is a linear map $f : A \to B$ such that

$$m_B \circ (f \otimes f) = f \circ m_A \in \mathrm{Hom}(A \otimes A, B), \quad \eta_B = f \circ \eta_A \in \mathrm{Hom}(k, B).$$

Also, $A \otimes B$ is an algebra in a natural way, with multiplication

$$\begin{aligned} m_{A\otimes B} &:= (m_A \otimes m_B) \circ (\mathsf{id}_A \otimes \sigma_{A,B} \otimes \mathsf{id}_B) \\ &\in \mathrm{Hom}(A \otimes B \otimes A \otimes B, A \otimes B) \end{aligned} \tag{7.2.3}$$

(note that the swap σ is really necessary here) and unit map

$$\eta_{A\otimes B} := \eta_A \otimes \eta_B \in \mathrm{Hom}(k, A \otimes B). \tag{7.2.4}$$

Naturally associated to an algebra A are two groups: the subset $A^\times$ of invertible elements and the set of algebra automorphisms $\mathrm{Aut}_{\mathsf{alg}}(A)$ (invertible algebra morphisms from A to itself). For any $x \in A^\times$ we denote by $\mathsf{Ad}(x)$ the automorphism of A given by conjugation by x: $\mathsf{Ad}(x)(a) = xax^{-1}$ for all $a \in A$; thus we obtain a group morphism Ad from $A^\times$ to $\mathrm{Aut}_{\mathsf{alg}}(A)$.

7.2.3 Algebra Representations

In general, a good way to study (or "test") A is by looking at representations of A. A *representation* of A on V is an algebra morphism $\pi_V : A \to \mathrm{End}(V)$ (more loosely, we also say that V carries a representation of A if such a π_V exists). For all $a \in A$ and $v \in V$ the element $\pi_V(a)(v)$ depends linearly on both a and v and is thus a linear map on the tensor product $A \otimes V$. Accordingly, we say that V has a *(left) A-module structure* consisting of the left action map $\lambda_V : A \otimes V \to V$ defined by $\lambda_V(a \otimes v) = \pi_V(a)(v)$, which is sometimes denoted $a \cdot v$ if the representation or the module structure is clear from the context.

If V, W are left A-modules, then we call a linear map $\varphi : V \to W$ an *A-intertwiner* (or A-module morphism) if φ commutes with the action of A, i.e. if the following diagram commutes for all $a \in A$:

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\pi_V(a)"'] & W \arrow[d, "\pi_W(a)"] \\ V \arrow[r, "\varphi"'] & W. \end{tikzcd} \tag{7.2.5}$$

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7.2.4 Tensor Products of Modules

If V and W are left A-modules, then $V \otimes W$ is not automatically an A-module, but merely an $A \otimes A$-module, with representation map $\pi_V \otimes \pi_W : A \otimes A \to \mathrm{End}(V) \otimes \mathrm{End}(W) \subseteq \mathrm{End}(V \otimes W)$.

Example 7.1 Let G be a group and consider the *group algebra* kG (the k-linear space with basis given by the group elements, which we turn into an algebra by extending the group multiplication linearly). Note that group representations of G are in a natural 1-to-1 correspondence with algebra representations of kG; if $\pi : G \to \mathrm{GL}(V)$ is a group representation then the corresponding algebra representation from kG on V is denoted by the same symbol. If we have two group representations $\pi_V : G \to \mathrm{GL}(V)$ and $\pi_W : G \to \mathrm{GL}(W)$ then $V \otimes W$ automatically carries a representation $\pi_{V\otimes W} : G \to \mathrm{GL}(V \otimes W)$ defined by

$$\pi_{V\otimes W}(g)(v \otimes w) := \pi_V(g)(v) \otimes \pi_W(g)(w) \tag{7.2.6}$$

for all $g \in G$, $v \in V$ and $w \in W$. Viewing π_V, π_W and $\pi_{V\otimes W}$ as algebra representations, note that we have defined $\pi_{V\otimes W} = (\pi_V \otimes \pi_W) \circ \Delta$, where Δ is the algebra morphism form kG to $kG \otimes kG$ uniquely determined by $\Delta(g) = g \otimes g$ for all $g \in G$.

In general, a natural framework for constructing representations of A from tensor products of representations of A arises whenever there is a distinguished algebra morphism $\Delta : A \to A \otimes A$: in this case we immediately see that $\pi_{V\otimes W} := (\pi_V \otimes \pi_W) \circ \Delta$ is an algebra morphism from A to $\mathrm{End}(V \otimes W)$. In this way we have related an additional structure on the representations of A to an additional structure on A itself.

The simplest meaningful representations of A are 1-dimensional representations or *characters*, i.e. algebra morphisms from A to $\mathrm{End}(k) \cong k$. To guarantee the existence of such a representation, it is convenient to further extend the structure on A given by Δ by stipulating that we also have a distinguished algebra morphism $\varepsilon : A \to k$.

Note that the new structure maps Δ and ε are similar to m and η, respectively, but go in the reverse direction. Therefore we call them *coproduct* (or *comultiplication*) and *counit (map)*.

7.2.5 Monoidal Categories

The notion of A-module is naturally bolted onto a more basic notion of vector space, to which is associated a notion of taking tensor products and a special vector space k, which acts as a neutral element when taking tensor products.

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We want to constrain the structure maps Δ, ε so that tensor products of A-modules and the special module k behave as tensor products of vector spaces and the special vector space k. So we need to capture which properties of vector spaces we wish to preserve.

Note that for all vector spaces U, V, W we have[3]

$$(U \otimes V) \otimes W \cong U \otimes (V \otimes W), \qquad k \otimes V \cong V \otimes k \cong V. \tag{7.2.7}$$

These properties are reminiscent of the definition of a monoid, and we call a category $\mathcal{C}$ a *monoidal category* (or *tensor category*) if there exists a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and a special object $1_{\mathcal{C}}$ that are abstractions of the tensor product operation on vector spaces and the special vector space k. To flesh this out precisely requires a bit more work; the precise definition of monoidal category[4] is given for instance in [7, Sec. 5.1B]. In particular, the collection of k-linear spaces together with the k-linear maps between them constitutes a monoidal category, called Vect, with the monoidal structure given by the usual tensor product of vector spaces and the vector space k.

The representations of our algebra A together with their intertwiners also form a category, which we denote by $\mathsf{Rep}(A)$. Note that we have a forgetful function For from $\mathsf{Rep}(A)$ to Vect, mapping each module to the underlying vector space and each intertwiner to the underlying linear map. It is natural to require of A that the isomorphisms in (7.2.7) are preserved when we interpret them as statements about $\mathsf{Rep}(A)$. More precisely, we want For to become a *monoidal functor* (i.e. it maps the tensor product of A-modules to the tensor product of vector spaces). It gives rise to the following definition.

7.2.6 Bialgebras

Definition 7.2 An algebra A is called a *bialgebra* if there exist algebra morphisms $\Delta : A \to A \otimes A$ and $\varepsilon : A \to k$ satisfying *coassociativity* and *counit axioms*:

$$(\Delta \otimes \mathsf{id}_A) \circ \Delta = (\mathsf{id}_A \otimes \Delta) \circ \Delta \qquad \in \mathrm{Hom}(A, A \otimes A \otimes A), \tag{7.2.8}$$

$$(\varepsilon \otimes \mathsf{id}_A) \circ \Delta = \mathsf{id}_A = (\mathsf{id}_A \otimes \varepsilon) \circ \Delta \qquad \in \mathrm{End}(A). \tag{7.2.9}$$

Remark A vector space A which possesses linear maps $\Delta : A \to A \otimes A$ and $\varepsilon : A \to k$ satisfying (7.2.8–7.2.9) is called a *coalgebra*. Bialgebras are at the

[3] Here we are careful in writing isomorphisms instead of identities, since tensor products of vector spaces are only defined up to isomorphism.

[4] Often the terms *monoidal category* and *tensor category* are used interchangeably but in [7] the terminology *tensor category* and *quasitensor category* correspond to what is commonly known as a *symmetric monoidal category* and *braided monoidal category*, respectively.

same time algebras and coalgebras in such a way that the two types of additional structures are compatible: the coalgebra structure maps Δ, ε are algebra morphisms (equivalently, the algebra structure maps m, η are coalgebra morphisms).

Before we study examples of bialgebras, we will develop the basic theory further. If A is a bialgebra, the properties (7.2.8) and (7.2.9) guarantee that the identities in (7.2.7) are identities of A-modules, so $\mathsf{Rep}(A)$ forms a monoidal category. In fact, we have a somewhat stronger statement:

Theorem 7.3 ([e.g. 39, Prop. 1.1]) *Let A be an algebra with multiplication map m. Let $\Delta : A \to A \otimes A$ and $\varepsilon : A \to k$ be algebra maps. Let $\otimes : \mathsf{Rep}(A) \times \mathsf{Rep}(A) \to \mathsf{Rep}(A)$ be the functor which associates to a pair of A-modules (V,W) a module, uniquely defined by stipulating that the underlying vector space is the usual tensor product $V \otimes W$ and the representation is*

$$\pi_{V\otimes W} = (\pi_V \otimes \pi_W) \circ \Delta : A \to \mathrm{End}(V) \otimes \mathrm{End}(W) \subseteq \mathrm{End}(V \otimes W). \quad (7.2.10)$$

Also let k be an A-module with representation $\pi_k = \varepsilon : A \to \mathrm{End}(k) \cong k$. Then $(\mathsf{Rep}(A), \otimes, k)$ is a monoidal category with the same isomorphisms as Vect *in* (7.2.7) *if and only if (Δ, ε) satisfies (7.2.8–7.2.9).*

If $B \subseteq A$ is a subalgebra and $\Delta(B) \subseteq B \otimes B$ then we call B a *subbialgebra of* A (as a consequence, B is a bialgebra in its own right). If A and B are bialgebras with coproducts Δ_A, Δ_B and counits ε_A, ε_B, respectively, then a *bialgebra morphism* is an algebra morphism $f : A \to B$ with the additional property

$$(f \otimes f) \circ \Delta_A = \Delta_B \circ f, \qquad \varepsilon_A = \varepsilon_B \circ f. \quad (7.2.11)$$

Let A be a bialgebra. In order to describe explicitly the A-module structure of a tensor product of any (finite) number of A-modules, we can recursively define *iterated coproducts* $\Delta^{(n)} \in \mathrm{Hom}(A, A^{\otimes n})$ for $n \in \mathbb{Z}_{\geq 0}$ as follows:

$$\Delta^{(0)} = \varepsilon, \qquad \Delta^{(n+1)} = (\Delta^{(n)} \otimes \mathsf{id}) \circ \Delta \quad (7.2.12)$$

so that $\Delta^{(1)} = \mathsf{id}$ by (7.2.9) and hence $\Delta^{(2)} = \Delta$. By virtue of (7.2.8), replacing the recursion in (7.2.12) by $\Delta^{(n+1)} = (\mathsf{id} \otimes \Delta^{(n)}) \circ \Delta$ for any or all n produces the same linear maps $\Delta^{(n)}$.

Remark Since $\Delta^{(n)}$ maps into $A^{\otimes n}$, for $a \in A$ and $n \in \mathbb{Z}_{>0}$ there exist (non-unique) $a_i^{(1)}, a_i^{(2)}, \ldots, a_i^{(n)} \in A$ such that

$$\Delta^{(n)}(a) = \sum_i a_i^{(1)} \otimes a_i^{(2)} \otimes \cdots \otimes a_i^{(n)}. \quad (7.2.13)$$

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This may be abbreviated to *Sweedler notation*:

$$\Delta^{(n)}(a) = \sum a^{(1)} \otimes a^{(2)} \otimes \cdots \otimes a^{(n)}. \tag{7.2.14}$$

For example, we may write (7.2.9) as $\sum \varepsilon(a^{(1)})a^{(2)} = a = \sum a^{(1)}\varepsilon(a^{(2)})$.

7.2.7 Commutativity and Cocommutativity

Let A be a bialgebra. The *opposite bialgebra* A^{op} is the bialgebra obtained from A by replacing the multiplication map, say $m : A \otimes A \to A$ by $m^{\mathsf{op}} := m \circ \sigma$. We call A *commutative* if $A^{\mathsf{op}} = A$, i.e. if $m^{\mathsf{op}} = m$ (in other words the underlying algebra is commutative).

We will be more interested in the *co-opposite bialgebra* A^{cop}. It is the bialgebra obtained from A by replacing Δ by $\Delta^{\mathsf{op}} := \sigma \circ \Delta$. We call A *cocommutative* if $A^{\mathsf{cop}} = A$, i.e. if $\Delta^{\mathsf{op}} = \Delta$. Note that if A is cocommutative then the monoidal category $\mathsf{Rep}(A)$ is *symmetric*: for all $V, W \in \mathsf{Rep}(A)$ there is an A-intertwiner $c_{V,W}$ from the object $V \otimes W$ to the object $W \otimes V$ such that $c_{W,V} c_{V,W} = \mathsf{id}_{V \otimes W}$. It is given by $c_{V,W} = \sigma_{V,W}$. The fact that $\sigma_{V,W}$ is an intertwiner is equivalent to $\Delta = \Delta^{\mathsf{op}}$.

7.2.8 Antipodes and Hopf Algebras

Many bialgebras appearing "in nature" have an additional structure map called *antipode*, which enriches the category of representations of such a bialgebra. To define it, first let A and B be bialgebras with multiplications m_A, m_B, unit maps η_A, η_B, coproducts Δ_A, Δ_B and counit maps ε_A, ε_B. The set of linear maps from A to B, $\mathrm{Hom}(A,B)$, possesses a natural product structure called *convolution product*, sending $f, g : A \to B$ to

$$f * g := m_B \circ (f \otimes g) \circ \Delta_A : A \to B. \tag{7.2.15}$$

It follows from the definition of bialgebra that $(\mathrm{Hom}(A,B), *)$ is a monoid with neutral element $\eta_B \circ \varepsilon_A : A \to B$ called the *convolution monoid*. Setting $B = A$ and suppressing subscripts, an *antipode* is a map $S \in \mathrm{Hom}(A,A)$ which is a $*$-inverse of $\mathsf{id} \in \mathrm{Hom}(A,A)$. In other words, an antipode S satisfies

$$m \circ (S \otimes \mathsf{id}) \circ \Delta = \eta \circ \varepsilon = m \circ (\mathsf{id} \otimes S) \circ \Delta. \tag{7.2.16}$$

One can now combine the convolution monoid construction with uniqueness of inverses to prove a slew of basic properties of antipodes. For proofs of the following we refer for instance to [10, Sec. 4.2].

Lemma 7.4 *Let A be a bialgebra with antipode S. Then*

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1 S is unique;
2 S is a bialgebra morphism from A to $(A^{\mathsf{op}})^{\mathsf{cop}}$ (and hence S^2 is a bialgebra endomorphism of A);
3 if S is invertible (with respect to composition) then S^{-1} is an antipode for the bialgebras A^{op} and A^{cop};
4 if A is commutative or cocommutative then S is involutive;
5 if B is another bialgebra with antipode S' and $f : A \to B$ is a bialgebra morphism, then $S' \circ f = f \circ S$.

Since we are interested in "deformations" of (co)commutative bialgebras, considering Lemma 7.4 (4), it is natural to require that S is invertible (although we have to relinquish involutiveness).

Definition 7.5 We call a bialgebra A a *Hopf algebra* if it has an antipode which is invertible (with respect to composition).

7.2.9 Representations of Bialgebras and Hopf Algebras

We review some more standard terminology of the representation theory of bialgebras. Let A be a bialgebra. Any vector space V automatically becomes a left A-module if we set $a \cdot v = \varepsilon(a)v$ for all $a \in A$, $v \in V$. This is called the *trivial* A-module structure on V.

The *(left) regular representation* of A is the A-module structure on A itself given by left multiplication. If A is additionally a Hopf algebra, the antipode also allows us to define the *(left) adjoint representation* of A on itself. Namely, for all $a, b \in A$ set, in terms of Sweedler notation,

$$\mathsf{ad}(a)(b) := \sum a^{(1)} b S(a^{(2)}). \tag{7.2.17}$$

It is a nice exercise to show that the map $\mathsf{ad} : A \to \mathrm{End}_k(A)$ defined by this assignment is indeed an algebra morphism.

If A is a Hopf algebra then the dual $V^* = \mathrm{Hom}(V, k)$ of $V \in \mathsf{Rep}(A)$ becomes an A-module by setting

$$(a \cdot f)(v) = f(S(a) \cdot v) \qquad \text{for all } a \in A,\ f \in V^*,\ v \in V. \tag{7.2.18}$$

As a consequence of the first equation of (7.2.16), this action of A on V^* implies that the canonical linear map: $V^* \otimes V \to k$ is an A-intertwiner.

Note that S can be replaced by S^{-1} in (7.2.18), requiring us to distinguish between the right-dual V^* and the left-dual ${}^*V$ of A-modules. A natural condition on a Hopf algebra A which implies that $V^* \cong {}^*V$ as A-modules is that the square of the antipode is inner, i.e. if $S^2 = \mathsf{Ad}(u)$ for some $u \in A^\times$. This claim

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follows from the observation that $\varphi_u : V^* \to {}^*V$ defined by $f \mapsto u^{-1} \cdot f$ is an A-intertwiner.

7.2.10 Key Example 1: Group Algebras

We discuss some important families of examples of bialgebras which can be defined in terms of a group G or a Lie algebra $\mathfrak{g}$, both algebraic structures with a well-defined notion of representations.

Let G be a group. Restating the key observation of Example 7.1, the group algebra kG becomes a Hopf algebra if we set

$$\Delta(g) = g \otimes g, \qquad \varepsilon(g) = 1, \qquad S(g) = g^{-1} \tag{7.2.19}$$

for all $g \in G$ and extend linearly. The adjoint representation of kG on itself extends the conjugation action of G on itself given by $g \cdot h = ghg^{-1}$ for all $g, h \in G$. More generally, we can let G be a monoid and the same assignments for Δ and ε define a bialgebra structure on kG, which extends to a Hopf algebra structure if and only if G is a group.

We can "dualize" this example. Consider the commutative algebra k^G of functions $f : G \to k$ (with pointwise addition and multiplication). Note that $k^G \otimes k^G$ naturally embeds into $k^{G \times G}$. If G is finite then this is an algebra isomorphism, so we may identify these algebras, and in that case the following definitions make sense:

$$\Delta(f)(g,h) = f(gh), \qquad \varepsilon(f) = f(1_G), \qquad S(f)(g) = f(g^{-1}) \tag{7.2.20}$$

for all $f \in k^G$ and $g, h \in G$. This endows k^G with a Hopf algebra structure. There are also infinite groups and associated function algebras $F(G)$ where we can make the identification $F(G) \otimes F(G) \cong F(G \times G)$ so that the same construction endows $F(G)$ with a bialgebra structure, for instance:

- let G be an algebraic group over k and replace $F(G)$ by the algebra of regular functions;
- let G be a compact topological group, set $k = \mathbb{R}$ or $k = \mathbb{C}$ and replace $F(G)$ by the algebra generated by the matrix entries of all finite-dimensional representations of G.

7.2.11 Key Example 2: Universal Enveloping Algebras

Let $\mathfrak{g}$ be a Lie algebra. Consider its tensor algebra

$$T\mathfrak{g} := k \oplus \mathfrak{g} \oplus \mathfrak{g}^{\otimes 2} \oplus \cdots, \tag{7.2.21}$$

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a free algebra with multiplication given by the tensor product. The *universal enveloping algebra* is the algebra $U\mathfrak{g} := T\mathfrak{g}/I$, where I is the two-sided ideal of $T\mathfrak{g}$ generated by all elements of the form $x\otimes y - y\otimes x - [x,y]$ for $x,y\in\mathfrak{g}$. Before we give the bialgebra structure, we highlight two key properties of universal enveloping algebras (see e.g. [5]).

1 The *universal property* of $U\mathfrak{g}$ and the canonical embedding $\iota : \mathfrak{g}\to U\mathfrak{g}$ is the following statement. For any Lie algebra morphism $\varphi : \mathfrak{g}\to A$ (where A is any algebra) there exists a unique algebra morphism $\widehat{\varphi} : U\mathfrak{g}\to A$ such that $\varphi = \widehat{\varphi}\circ\iota$. In particular we may take $A = \mathrm{End}(V)$ with V a vector space and obtain that Lie algebra representations of $\mathfrak{g}$ correspond 1-to-1 to algebra representations of $U\mathfrak{g}$.
2 The *Poincaré–Birkhoff–Witt theorem* states that given a totally ordered k-basis X of $\mathfrak{g}$, a k-basis of $U\mathfrak{g}$ is given by the set

$$\bigcup_{n\geq 0}\{\iota(x_1)\cdots\iota(x_n)\,|\,x_1,\dots,x_n\in X, x_1\leqslant\cdots\leqslant x_n\}. \tag{7.2.22}$$

The natural Hopf algebra structure on $U\mathfrak{g}$ is uniquely determined by

$$\Delta(x) = x\otimes 1 + 1\otimes x, \qquad \varepsilon(x) = 0, \qquad S(x) = -x \tag{7.2.23}$$

for all $x\in\mathfrak{g}$. This map Δ corresponds precisely to the standard action of Lie algebras on tensor products of their representations: if $\pi_V : \mathfrak{g}\to\mathrm{End}(V)$ and $\pi_W : \mathfrak{g}\to\mathrm{End}(W)$ are Lie algebra representations then $\pi_{V\otimes W}$ defined by $\pi_{V\otimes W}(x) := \pi_V(x)\otimes\mathsf{id}_W + \mathsf{id}_V\otimes\pi_W(x)$ for all $x\in\mathfrak{g}$ is a representation of $\mathfrak{g}$ on $V\otimes W$. Also, in this case the adjoint representation of $U\mathfrak{g}$ on itself corresponds to the usual adjoint representation of $\mathfrak{g}$ given by $x\cdot y = [x,y]$ for all $x,y\in\mathfrak{g}$.

7.2.12 Grouplike and Skew-primitive Elements

Let A be a bialgebra. Inspired by the examples above we highlight two important types of elements of A. We call an element $a\in A$ *grouplike* if

$$\Delta(a) = a\otimes a, \qquad a\neq 0. \tag{7.2.24}$$

The set of grouplike elements is denoted by $\mathrm{Gr}(A)$. From (7.2.9) it follows that $\varepsilon(a) = 1$ for all $a\in A$ and hence $\mathrm{Gr}(A)\cap A^\times$ is a group. If A has an antipode S then $\mathrm{Gr}(A)$ is a subgroup of $A^\times$ and S acts on $\mathrm{Gr}(A)$ by inversion.

We call $a\in A$ *skew-primitive* if there exist $g,h\in\mathrm{Gr}(A)$ such that

$$\Delta(a) = a\otimes g + h\otimes a. \tag{7.2.25}$$

The set of such elements, denoted $\mathsf{Pri}_{g,h}(A)$, is a subspace of $\mathsf{Ker}(\varepsilon)$. If A has an antipode S then for all $g,h\in\mathrm{Gr}(A)$, S acts on $\mathsf{Pri}_{g,h}(A)$ as $a\mapsto -h^{-1}ag^{-1}$.

Elements of $\mathsf{Pri}_{1,1}(A)$ are called *primitive*. We have an inclusion of Lie algebras (with Lie bracket given by the commutator) $\mathsf{Pri}_{1,1}(A) \subseteq \mathsf{Ker}(\varepsilon) \subset A$.

Note that $U\mathfrak{g}$ is generated by primitive elements. More precisely, the embedding ι maps $\mathfrak{g}$ into $\mathsf{Pri}_{1,1}(U\mathfrak{g}) \subset U\mathfrak{g}$ and the Poincaré–Birkhoff–Witt theorem can be used to deduce that, if k is of characteristic 0, $\mathsf{Pri}_{1,1}(U\mathfrak{g}) = \mathfrak{g}$. A result due to Kostant [26] gives a wide-ranging converse to this observation: if k is algebraically closed of characteristic zero then any cocommutative Hopf algebra A such that $G(A) = \{1\}$ is isomorphic to the universal enveloping algebra of $\mathsf{Pri}_{1,1}(A)$. One can in fact remove the constraint on $G(A)$ and show that A is isomorphic to a particular type of semidirect product $U(\mathsf{Pri}_{1,1}(A)) \rtimes kG(A)$ known as *Hopf smash product*. The Hopf algebras we will highlight in these notes will be generated by grouplike and skew-primitive elements.

7.3 Quasitriangular Bialgebras and Braided Monoidal Categories

Note that the bialgebras discussed so far are all either cocommutative or commutative. We will see that quantum groups arise in a certain way as non-(co)commutative variations of them. Note especially that in algebraic geometry one studies algebraic groups via their (commutative) algebras of regular functions; it is natural to consider a "modified" or "deformed" algebraic group by making the algebra of regular functions non-commutative. This is the origin of the name "quantum group".

7.3.1 Generalizing Cocommutativity

To provide a context for this, we discuss a generalization of cocommutativity. It starts with the following idea. Let A be a bialgebra with coproduct Δ. Suppose there exists $\mathcal{R} \in (A \otimes A)^\times$ such that

$$\Delta^{\mathsf{op}} = \mathsf{Ad}(\mathcal{R}) \circ \Delta, \tag{7.3.1}$$

i.e. $\mathcal{R}\Delta(a) = \Delta^{\mathsf{op}}(a)\mathcal{R}$ for all $a \in A$. Note that (7.2.9) implies that

$$u^{(1)} := (\varepsilon \otimes \mathsf{id})(\mathcal{R}) \in A^\times, \qquad u^{(2)} := (\mathsf{id} \otimes \varepsilon)(\mathcal{R}) \in A^\times \tag{7.3.2}$$

are central elements of A and hence the element

$$\widetilde{\mathcal{R}} := (\varepsilon \otimes \varepsilon)(\mathcal{R}) \cdot (u^{(2)} \otimes u^{(1)})^{-1} \cdot \mathcal{R} \in (A \otimes A)^\times \tag{7.3.3}$$

satisfies both (7.3.1) and $(\varepsilon \otimes \mathsf{id})(\widetilde{\mathcal{R}}) = (\mathsf{id} \otimes \varepsilon)(\widetilde{\mathcal{R}}) = 1$. Hence without loss of generality we may assume that $\mathcal{R}$ satisfies

$$(\varepsilon \otimes \mathsf{id})(\mathcal{R}) = (\mathsf{id} \otimes \varepsilon)(\mathcal{R}) = 1. \tag{7.3.4}$$

Let us explore how (7.3.1) constrains the element R. Both A and A^{cop} are bialgebras and hence (7.2.8) is satisfied both as-is and with Δ replaced by Δ^{op}. Note that the coassociativity axiom (7.2.8) is a statement about linear maps from A to $A \otimes A \otimes A$. It is therefore convenient to identify the three canonical linear embeddings of $A \otimes A$ into $A \otimes A \otimes A$ and introduce notation for them. For all $a, b \in A$ we will write

$$(a \otimes b)_{12} = a \otimes b \otimes 1, \quad (a \otimes b)_{13} = a \otimes 1 \otimes b, \quad (a \otimes b)_{23} = 1 \otimes a \otimes b \tag{7.3.5}$$

and extend this notation linearly, so that e.g. $\mathcal{R}_{12} = \mathcal{R} \otimes 1$. Now note that $(\Delta^{\mathsf{op}} \otimes \mathsf{id}) \circ \Delta^{\mathsf{op}} = (\mathsf{id} \otimes \Delta^{\mathsf{op}}) \circ \Delta^{\mathsf{op}}$ together with (7.3.1) implies

$$\mathsf{Ad}\big(\mathcal{R}_{12}(\Delta \otimes \mathsf{id})(\mathcal{R})\big) \circ (\Delta \otimes \mathsf{id}) \circ \Delta = \mathsf{Ad}\big(\mathcal{R}_{23}(\mathsf{id} \otimes \Delta)(\mathcal{R})\big) \circ (\mathsf{id} \otimes \Delta) \circ \Delta.$$

Now using (7.2.8) for Δ itself we obtain that the element

$$X := \big(\mathcal{R}_{23} \cdot (\mathsf{id} \otimes \Delta)(\mathcal{R})\big)^{-1} \cdot \mathcal{R}_{12} \cdot (\Delta \otimes \mathsf{id})(\mathcal{R}) \in (A \otimes A \otimes A)^{\times} \tag{7.3.6}$$

centralizes the image of $\Delta^{(3)}$ in $A \otimes A \otimes A$. Owing to (7.3.4) we have

$$(\varepsilon \otimes \mathsf{id} \otimes \mathsf{id})(X) = (\mathsf{id} \otimes \varepsilon \otimes \mathsf{id})(X) = (\mathsf{id} \otimes \mathsf{id} \otimes \varepsilon)(X) = 1 \otimes 1. \tag{7.3.7}$$

The simplest possible X satisfying these constraints is $X = 1 \otimes 1 \otimes 1$. If we assume this, $\mathcal{R}$ satisfies the *cocycle condition*

$$\mathcal{R}_{12}(\Delta \otimes \mathsf{id})(\mathcal{R}) = \mathcal{R}_{23}(\mathsf{id} \otimes \Delta)(\mathcal{R}). \tag{7.3.8}$$

Without loss of generality we may write $(\Delta \otimes \mathsf{id})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}Y$ for some $Y \in (A \otimes A \otimes A)^{\times}$. Taking counits again, we obtain

$$(\varepsilon \otimes \mathsf{id} \otimes \mathsf{id})(Y) = (\mathsf{id} \otimes \varepsilon \otimes \mathsf{id})(Y) = (\mathsf{id} \otimes \mathsf{id} \otimes \varepsilon)(Y) = 1, \tag{7.3.9}$$

and hence again we assume the simplest possible solution: $Y = 1 \otimes 1 \otimes 1$, so that

$$(\Delta \otimes \mathsf{id})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}. \tag{7.3.10}$$

Combining (7.3.8) and (7.3.10) and applying (7.3.1), we obtain

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{12}(\Delta \otimes \mathsf{id})(\mathcal{R}) = \mathcal{R}_{23}(\mathsf{id} \otimes \Delta)(\mathcal{R}) = (\mathsf{id} \otimes \Delta^{\mathsf{op}})(\mathcal{R})\mathcal{R}_{23},$$

so that $(\mathsf{id} \otimes \Delta^{\mathsf{op}})(\mathcal{R}) = \mathcal{R}_{12}\mathcal{R}_{13}$. Left-multiplying by $\mathsf{id} \otimes \sigma$, we obtain

$$(\mathsf{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}. \tag{7.3.11}$$

7.3.2 Quasitriangularity: Definition and Basic Properties

The above analysis motivates the following generalization of cocommutativity based on (7.3.1), originally due to Drinfeld [12].

Definition 7.6 Let A be a bialgebra and $\mathcal{R} \in (A \otimes A)^\times$. The pair $(A, \mathcal{R})$ is called *quasitriangular* and $\mathcal{R}$ a *(universal) R-matrix* for A if (7.3.1) and (7.3.10–7.3.11) hold.

If $(A, \mathcal{R})$ and $(B, \mathcal{S})$ are quasitriangular bialgebras then a bialgebra morphism $\psi : A \to B$ is called a *quasitriangular morphism* if $(\psi \otimes \psi)(\mathcal{R}) = \mathcal{S}$.

Lemma 7.7 *Let $(A, \mathcal{R})$ be a quasitriangular bialgebra.*

1 *The* (universal) Yang–Baxter equation *is satisfied:*

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12} \qquad \in A \otimes A \otimes A. \tag{7.3.12}$$

2 *The bialgebras $(A, \sigma(\mathcal{R})^{-1})$, $(A^{\mathsf{op}}, \sigma(\mathcal{R}))$ and $(A^{\mathsf{cop}}, \sigma(\mathcal{R}))$ are quasitriangular.*
3 *The counit condition* (7.3.4) *is satisfied.*
4 *If A is a Hopf algebra then*

$$S^2 = \mathrm{Ad}(u), \quad \textit{where } u = \big(m \circ (S \otimes \mathrm{id}) \circ \sigma\big)(\mathcal{R}) \in A^\times \tag{7.3.13}$$

$$(S \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}^{-1} = (\mathrm{id} \otimes S^{-1})(\mathcal{R}), \qquad (S \otimes S)(\mathcal{R}) = \mathcal{R}. \tag{7.3.14}$$

For the proofs see e.g. [7, Props. 4.2.3 and 4.2.7]. Here we reproduce the proof of (7.3.12), which relies on (7.3.1) and (7.3.10):

$$\begin{aligned} \mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} &= \mathcal{R}_{12}(\Delta \otimes \mathsf{id})(\mathcal{R}) &&= (\Delta^{\mathsf{op}} \otimes \mathsf{id})(\mathcal{R})\mathcal{R}_{12} \\ &= (\sigma \otimes \mathsf{id})(\mathcal{R}_{13}\mathcal{R}_{23})\mathcal{R}_{12} &&= \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \end{aligned} \tag{7.3.15}$$

Note that (7.3.12) can also be deduced, in a very similar way, from (7.3.1) and the other coproduct formula (7.3.11). Indeed, it is natural for a given quasitriangular bialgebra to possess a symmetry interchanging the two coproduct formulas. In the following lemma we identify such a symmetry.

Lemma 7.8 *Let A be a bialgebra with a bialgebra morphism $\omega : A \to A^{cop}$. If $\mathcal{R} \in (A \otimes A)^\times$ is fixed by $\sigma \circ (\omega \otimes \omega)$ then conditions* (7.3.10) *and* (7.3.11) *are equivalent.*

Proof This follows from $\mathsf{id} \otimes \Delta = (\sigma \otimes \mathsf{id}) \circ (\mathsf{id} \otimes \sigma) \circ (\Delta \otimes \mathsf{id}) \circ \sigma$ and $(\omega \otimes \omega) \circ \sigma = \sigma \circ (\omega \otimes \omega)$. □

Remark Another condition on quasitriangular bialgebras guaranteeing the equivalence of (7.3.10) and (7.3.11) is $\sigma(\mathcal{R}) = \mathcal{R}^{-1}$; such quasitriangular bialgebras are called *triangular*.

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7.3.3 Braided Monoidal Categories

The main point of having a quasitriangular structure on a bialgebra is that the category of (left) A-modules is not just a monoidal category, but that the two possible tensor products of A-modules V and W, namely $V \otimes W$ and $W \otimes V$, are naturally isomorphic as A-modules, thereby preserving a key property of symmetric monoidal categories. Moreover the category of A-modules carries a natural braided structure.

More precisely, let V and W be A-modules with corresponding representations $\pi_V : A \to \mathrm{End}(V)$, $\pi_W : A \to \mathrm{End}(W)$. Denote $R_{V,W} = (\pi_V \otimes \pi_W)(\mathcal{R})$ (the linear map on $V \otimes W$ corresponding to the action of $\mathcal{R}$). Recall the linear map $\sigma_{V,W} : V \otimes W \to W \otimes V$ and define

$$\check{R}_{V,W} := \sigma_{V,W} \circ R_{V,W} \in \mathrm{Hom}(V \otimes W, W \otimes V). \tag{7.3.16}$$

Note that, since $\mathcal{R}$ is invertible, $\check{R}_{V,W}$ is invertible.

Lemma 7.9 *The map $\check{R}_{V,W}$ intertwines the modules $V \otimes W$ and $W \otimes V$:*

$$\check{R}_{V,W}\pi_{V\otimes W}(a) = \pi_{W\otimes V}(a)\check{R}_{V,W} \qquad \textit{for all } a \in A. \tag{7.3.17}$$

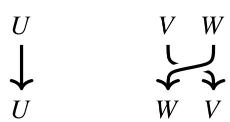
In particular, $V \otimes W$ and $W \otimes V$ are isomorphic as A-modules.

Proof The axiom (7.3.1) implies

$$R_{V,W}(\pi_V \otimes \pi_W)(\Delta(a)) = (\pi_V \otimes \pi_W)(\Delta^{\mathrm{op}}(a))R_{V,W} \tag{7.3.18}$$

for all $a \in A$. Left-multiplying by $\sigma_{V,W}$ we obtain (7.3.17). $\square$

It is possible to represent the category $\mathsf{Rep}(A)$ using a diagrammatical calculus developed in [37]. Here A-intertwiners from $U_1 \otimes \cdots \otimes U_m$ to $V_1 \otimes \cdots \otimes V_n$ correspond to diagrams with m incoming arrows and n outgoing arrows, labelled by the corresponding modules. Furthermore taking tensor products corresponds to horizontal juxtaposition, and composition of intertwiners corresponds to vertical juxtaposition; we use the convention that composition is downward, which is also indicated by arrows. In particular, the intertwiners id_U and $\check{R}_{V,W}$ are represented by a single strand and a braiding:



(7.3.19)

We also represent the action of $a \in A$ on $V \in \mathsf{Rep}(A)$ by a decoration, marked

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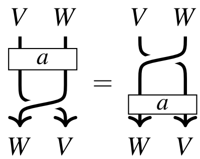
by a, of the strand labelled by V:

(7.3.20)

In particular, (7.3.17) corresponds to



(7.3.21)

Also the coproduct axioms (7.3.10–7.3.11) correspond to natural conditions. Namely, let $U, V, W \in \mathsf{Rep}(A)$. Applying $\pi_U \otimes \pi_V \otimes \pi_W$ to (7.3.10) yields

$$R_{U \otimes V, W} = (R_{U,W})_{13}(R_{V,W})_{23}. \tag{7.3.22}$$

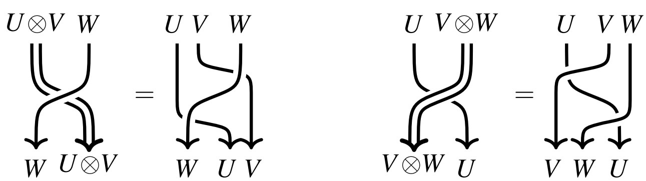
Left-multiplying by $(\sigma_{U,W})_{12}(\sigma_{V,W})_{23} = \sigma_{U \otimes V, W}$, we obtain

$$\check{R}_{U \otimes V, W} = (\check{R}_{U,W} \otimes \mathsf{id}_V)(\mathsf{id}_U \otimes \check{R}_{V,W}), \tag{7.3.23}$$

an equation in $\mathrm{Hom}(U \otimes V \otimes W, W \otimes U \otimes V)$. In the same way, from (7.3.11) we obtain

$$\check{R}_{U, V \otimes W} = (\mathsf{id}_V \otimes \check{R}_{U,W})(\check{R}_{U,V} \otimes \mathsf{id}_W), \tag{7.3.24}$$

an equation in $\mathrm{Hom}(U \otimes V \otimes W, V \otimes W \otimes U)$. In terms of the diagrammatical calculus, (7.3.23–7.3.24) correspond to the topological identities



(7.3.25)

This means that the monoidal category $\mathsf{Rep}(A)$ is *braided*, see [20].

Remark If $(A, \mathcal{R})$ is triangular then, for all $V, W \in \mathsf{Rep}(A)$, we have $\check{R}_{W,V} = \check{R}_{V,W}^{-1}$ so that $\mathsf{Rep}(A)$ is a symmetric monoidal category.

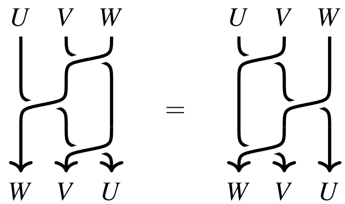
To complete the description of the braided structure on $\mathsf{Rep}(A)$, suppose $U, V, W \in \mathsf{Rep}(A)$ and apply $\pi_U \otimes \pi_V \otimes \pi_W$ to (7.3.12). We obtain the (matrix) Yang–Baxter equation

$$\begin{aligned}(R_{U,V})_{12}(R_{U,W})_{13}(R_{V,W})_{23} \\ = (R_{V,W})_{23}(R_{U,W})_{13}(R_{U,V})_{12} \in \mathrm{End}(U \otimes V \otimes W),\end{aligned} \tag{7.3.26}$$

or, equivalently,

$$\begin{aligned}(\mathsf{id}_W \otimes \check{R}_{U,V})(\check{R}_{U,W} \otimes \mathsf{id}_V)(\mathsf{id}_U \otimes \check{R}_{V,W})\\ = (\check{R}_{V,W} \otimes \mathsf{id}_U)(\mathsf{id}_V \otimes \check{R}_{U,W})(\check{R}_{U,V} \otimes \mathsf{id}_W),\end{aligned} \tag{7.3.27}$$

an equation in $\mathrm{Hom}(U \otimes V \otimes W, W \otimes V \otimes U)$. This corresponds diagrammatically to



(7.3.28)

Finally, for arbitrary $L \in \mathbb{Z}_{>0}$ consider the braid group

$$\mathsf{Br}_L := \left\langle b_1, \ldots, b_{L-1} \,\middle|\, b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}, b_i b_j = b_j b_i \text{ if } |i-j| > 1 \right\rangle$$

(the fundamental group of the L-th unordered configuration space of the disk). For all $V \in \mathsf{Rep}(A)$, we obtain a representation of Br_L on $V^{\otimes L}$, given by

$$b_i \mapsto \mathsf{id}_V^{\otimes(i-1)} \otimes \check{R}_{V,V} \otimes \mathsf{id}_V^{\otimes(L-i-1)}. \tag{7.3.29}$$

7.3.4 Sweedler's Hopf Algebra – A Warm-up Exercise

We discuss a finite-dimensional quasitriangular Hopf algebra with a nontrivial R-matrix found in [41]. It does not depend on a parameter (and so is not a quantum group following our soft definition from the introduction). Consider the algebra A generated by symbols f and g subject to the relations

$$f^2 = 0, \qquad g^2 = 1, \qquad fg = -gf. \tag{7.3.30}$$

Note that $\{1, f, g, fg\}$ is a k-basis for A. Straightforward checks on generators show that the assignments

$$\begin{aligned}&\Delta(f) = f \otimes g + 1 \otimes f, \qquad &&\varepsilon(f) = 0, \qquad &&S(f) = gf,\\ &\Delta(g) = g \otimes g, &&\varepsilon(g) = 1, &&S(g) = g\end{aligned} \tag{7.3.31}$$

define a Hopf algebra structure on A with $g \in \mathrm{Gr}(A)$ and $f \in \mathsf{Pri}_{g,1}(A)$. The algebra A is the smallest noncommutative non-cocommutative Hopf algebra. Some of its properties foreshadow similar properties of Drinfeld–Jimbo quantum groups.

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Now assume that $\mathsf{char}(k) \neq 2$. We will show that, for all $\beta \in k$, the following expression determines a quasitriangular structure on A

$$\mathcal{R}_\beta = \tfrac{1}{2}(1 \otimes 1 + 1 \otimes g + g \otimes 1 - g \otimes g)(1 \otimes 1 + \beta f \otimes gf) \in A \otimes A. \quad (7.3.32)$$

See [7, Sec. 4.2F] for a somewhat different approach. Consider

$$\widetilde{\mathcal{R}}_\beta := \mathcal{R}_0^{-1}\mathcal{R}_\beta = 1 \otimes 1 + \beta f \otimes gf \quad (7.3.33)$$

and note that $\widetilde{\mathcal{R}}_\beta \widetilde{\mathcal{R}}_{-\beta} = 1 \otimes 1$. By writing $\mathcal{R}_0 = 1 \otimes 1 - 2\frac{1-g}{2} \otimes \frac{1-g}{2}$ and noting that $\frac{1-g}{2}$ is an idempotent, we deduce that $\mathcal{R}_0$ is an involution. From the invertibility of $\mathcal{R}_0$ and $\widetilde{\mathcal{R}}_\beta$ we deduce that $\mathcal{R}_\beta$ is invertible. Moreover, by a direct computation we obtain

$$\mathcal{R}_0(f \otimes g) = (f \otimes 1)\mathcal{R}_0, \qquad \mathcal{R}_0(1 \otimes f) = (g \otimes f)\mathcal{R}_0. \quad (7.3.34)$$

Lemma 7.10 *The involutive linear map $\omega : A \to A$ is uniquely determined by $\omega(1) = 1$, $\omega(g) = g$ and $\omega(f) = fg$ is a bialgebra morphism from A to A^{cop}. Furthermore $\mathcal{R}_\beta$ is fixed by $\sigma \circ (\omega \otimes \omega)$.*

Proof The first statement follows directly from (7.3.30–7.3.31). The second statement is a consequence of $\sigma(\mathcal{R}_0) = \mathcal{R}_0 = (\omega \otimes \omega)(\mathcal{R}_0)$ and $\sigma(\widetilde{\mathcal{R}}_\beta) = (\omega \otimes \omega)(\widetilde{\mathcal{R}}_\beta)$. □

Theorem 7.11 *For all $\beta \in k$, $(A, \mathcal{R}_\beta)$ is quasitriangular.*

Proof This is essentially a computation, but it is instructive to highlight some salient points. For the axiom (7.3.1), it suffices[5] to prove

$$\widetilde{\mathcal{R}}_\beta \Delta(a) = \Delta(a)\widetilde{\mathcal{R}}_\beta, \qquad \mathcal{R}_0\Delta(a) = \Delta^{\mathsf{op}}(a)\mathcal{R}_0 \qquad \text{for all } a \in A. \quad (7.3.35)$$

In turn, it suffices to verify these statements for $a \in \{f, g\}$, which is a straightforward consequence of (7.3.34). By Lemma 7.8 it now suffices to prove the axiom (7.3.10). A direct computation shows that

$$(\Delta \otimes \mathsf{id})(\mathcal{R}_0) = (\mathcal{R}_0)_{13}(\mathcal{R}_0)_{23}, \quad (7.3.36)$$

so that it remains to prove that

$$(\Delta \otimes \mathsf{id})(\widetilde{\mathcal{R}}_\beta) = (\mathcal{R}_0)_{23}(\widetilde{\mathcal{R}}_\beta)_{13}(\mathcal{R}_0)_{23}(\widetilde{\mathcal{R}}_\beta)_{23}. \quad (7.3.37)$$

[5] This is the main reason for introducing the factorization $\mathcal{R}_\beta = \mathcal{R}_0\widetilde{\mathcal{R}}_\beta$. We will approach the quasitriangularity of $U_q\mathfrak{g}$ in a similar way.

We expand with respect to powers of β. It now suffices to prove:

$$(\Delta \otimes \mathsf{id})(1 \otimes 1) = (\mathcal{R}_0)_{23}(\mathcal{R}_0)_{23}, \tag{7.3.38}$$
$$(\Delta \otimes \mathsf{id})(f \otimes gf) = (\mathcal{R}_0)_{23}(f \otimes 1 \otimes gf)(\mathcal{R}_0)_{23} + 1 \otimes f \otimes gf, \tag{7.3.39}$$
$$0 = (\mathcal{R}_0)_{23}(f \otimes 1 \otimes gf)(\mathcal{R}_0)_{23}(1 \otimes f \otimes gf). \tag{7.3.40}$$

Note that the first equation is trivial. The second equation follows by combining (7.3.34) with the coproduct formulas for f and g. Finally, the third equation follows by combining (7.3.34) with $f^2 = 0$. □

Remark The quasitriangular bialgebra $(A, \mathcal{R})$ is in fact triangular, since $\sigma(\mathcal{R}_\beta) = \mathcal{R}_\beta^{-1}$. This follows from the identity $\mathcal{R}_0\sigma(\widetilde{\mathcal{R}}_\beta) = \widetilde{\mathcal{R}}_{-\beta}\mathcal{R}_0$, a direct consequence of (7.3.34).

To illustrate how nontrivial solutions of (7.3.26) arise in tensor products of modules over a quasitriangular bialgebra, consider the following two non-isomorphic indecomposable representations of A on a 2-dimensional vector space V. They are defined by:

$$\pi^\pm(f) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \pi^\pm(g) = \begin{pmatrix} \pm 1 & 0 \\ 0 & \mp 1 \end{pmatrix} \tag{7.3.41}$$

with respect to a fixed ordered basis (v_1, v_2). With respect to the ordered basis $(v_1 \otimes v_1, v_1 \otimes v_2, v_2 \otimes v_1, v_2 \otimes v_2)$ of $V \otimes V$, we have

$$(\pi^\pm \otimes \pi^\pm)(\mathcal{R}_\beta) = \begin{pmatrix} \pm 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta & 0 & 0 & \mp 1 \end{pmatrix}, \tag{7.3.42}$$

a nontrivial solution of the Yang–Baxter equation (7.3.26). Unfortunately, the representation theory of A is not very rich:

Theorem 7.12 *If k is algebraically closed, A has exactly four isomorphism classes of indecomposable modules. More precisely, up to isomorphism there are two 1-dimensional modules, given by $\pm\varepsilon$ and the two 2-dimensional modules defined by* (7.3.41).

Proof This follows from the fact that we may assume that g acts as a diagonalizable map on the module, which must therefore split up as a direct sum of ± 1-eigenspaces. For more details, see [7, 4.2F(g)]. □

On the other hand, semisimple Lie algebras $\mathfrak{g}$ have a very rich category of modules. Their enveloping algebras $U\mathfrak{g}$ are naturally cocommutative (and

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hence quasitriangular) Hopf algebras; their quantizations $U_q\mathfrak{g}$ inherit the category of modules and are, up to a technicality, quasitriangular Hopf algebras themselves.

7.4 Drinfeld–Jimbo Quantum Groups

In Section 7.4 we will study a deformation of the universal enveloping algebra of a Lie algebra associated to a (connected, complex, semisimple) Lie group, called quantized universal enveloping algebras or Drinfeld–Jimbo quantum groups. From now on we assume that k is algebraically closed and $\mathsf{char}(k)=0$ (in particular $\mathbb{Q} \subset k$). First we deal with the $\mathfrak{sl}_2$ case.

7.4.1 $\mathfrak{sl}_2$ and $U\mathfrak{sl}_2$

Let us first study the basic case of $\mathfrak{sl}_2$, the Lie algebra of traceless 2×2-matrices over k. It has a basis given by

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7.4.1}$$

In this case it is not hard to see that we have only the following Lie bracket relations between the basis elements:

$$[h,e] = 2e, \qquad [h,f] = -2f, \qquad [e,f] = h. \tag{7.4.2}$$

Hence, it follows immediately that $U\mathfrak{sl}_2$ is obtained from the free algebra over the symbols E,F,H by imposing the relations

$$HE - EH = 2E, \qquad HF - HF = -2F, \qquad EF - FE = H. \tag{7.4.3}$$

The canonical embedding $\iota : \mathfrak{sl}_2 \to U\mathfrak{sl}_2$ is given by $e \mapsto E$, $f \mapsto F$ and $h \mapsto H$. In the quantum deformed version we will “keep” E and F and “replace” H by a well-chosen linear combination of a grouplike element and its inverse.

7.4.2 Quantum $\mathfrak{sl}_2$

Let q be an indeterminate[6] and consider the algebra $U_q\mathfrak{sl}_2$ generated over $k(q)$ by symbols E, F, t and t^{-1} subject to the relations

[6] In a related formalism, q is not an indeterminate but a scalar. Typically it is imposed that q is nonzero and not a root of unity. The study of quantum groups for root-of-unity values of q is very interesting but outside the scope of this introductory course.

$$\begin{aligned} tE &= q^2Et, \qquad tF = q^{-2}Ft, \\ [E,F] &= \frac{t-t^{-1}}{q-q^{-1}}, \qquad tt^{-1} = t^{-1}t = 1. \end{aligned} \tag{7.4.4}$$

We will use the following convention for the additional structure maps:

$$\begin{aligned} \Delta(E) &= E\otimes 1 + t\otimes E, & \varepsilon(E) &= 0, & S(E) &= -t^{-1}E, \\ \Delta(F) &= F\otimes t^{-1} + 1\otimes F, & \varepsilon(F) &= 0, & S(F) &= -Ft, \\ \Delta(t^{\pm 1}) &= t^{\pm 1}\otimes t^{\pm 1}, & \varepsilon(t^{\pm 1}) &= 1, & S(t^{\pm 1}) &= t^{\mp 1}. \end{aligned} \tag{7.4.5}$$

It is a standard check to see that this endows $U_q\mathfrak{sl}_2$ with a Hopf algebra structure. Note that the subalgebras $\langle E,t,t^{-1}\rangle$ and $\langle F,t,t^{-1}\rangle$ are Hopf subalgebras, whereas $\langle E,t\rangle$ and $\langle F,t^{-1}\rangle$ are subbialgebras which are not Hopf subalgebras.

7.4.3 The Topological Quantum Group $U_{[[h]]}\mathfrak{sl}_2$

Morally, sending $q\to 1$ should recover the defining relations and Hopf algebra structure of $U\mathfrak{sl}_2$. By making the formal substitution $t = q^H$ this can indeed be done. For instance, in the right-hand side of the relation $[E,F] = \frac{t-t^{-1}}{q-q^{-1}}$ one may take the formal limit $q\to 1$ and immediately obtain H, as required. Writing $t = q^H$ and $q^2t = q^{H+2}$ as formal power series in $\log(q)$, the relation $tE = q^2Et$ is equivalent to

$$\sum_{r\geq 0}\frac{1}{r!}\log(q)^r H^r E = \sum_{r\geq 0}\frac{1}{r!}\log(q)^r E(H+2)^r. \tag{7.4.6}$$

Since q is an indeterminate, this should be true on the level of the coefficients, yielding the $U\mathfrak{sl}_2$-relations $H^rE = E(H+2)^r$. This suggests a connection between $U_q\mathfrak{sl}_2$ and $U\mathfrak{sl}_2[[\log(q)]]$.

To make this rigorous, choose a new indeterminate h. The h-adic topology on a vector space V over $k[[h]]$ is defined by stipulating that

1 $\{h^nV \mid n\in\mathbb{Z}_{\geq 0}\}$ is a base of the neighbourhoods of 0 in V,

2 translations in V are continuous.

It follows then that $k[[h]]$-linear maps are continuous. A *topological Hopf algebra* over $k[[h]]$ is an h-adic complete $k[[h]]$-module A equipped with $k[[h]]$-linear structure maps η, m, ε, Δ and S satisfying the Hopf algebra axioms discussed in Section 2, but with algebraic tensor products replaced by h-adic completions. We then may study the topological Hopf algebra $U_{[[h]]}\mathfrak{sl}_2$, defined as follows.

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Namely, consider the free algebra $\mathcal{P} := k\langle E,F,H\rangle$ and the algebra of power series $\mathcal{P}[[h]]$. Consider the two-sided ideal I of $\mathcal{P}[[h]]$ generated by

$$[H,E]-2E, \qquad [H,F]+2F, \qquad [E,F]-\frac{\mathrm{e}^{hH}-\mathrm{e}^{-hH}}{\mathrm{e}^{h}-\mathrm{e}^{-h}} \tag{7.4.7}$$

and let I^{cl} be its closure in the h-adic topology. Then we can define $U_{[[h]]}\mathfrak{sl}_2 := \mathcal{P}[[h]]/I^{\mathrm{cl}}$. One then can deduce that $U_{[[h]]}\mathfrak{sl}_2 \cong (U\mathfrak{sl}_2)[[h]]$ as algebras over $k[[h]]$ (see [7, Cor. 6.5.4]).

7.4.4 Some Representations of $U_q\mathfrak{sl}_2$

It is easy to explicitly construct finite-dimensional representations of $U_q\mathfrak{sl}_2$, which simplify to $\mathfrak{sl}_2$-representations if we let q go to 1. We denote, for $m \in \mathbb{Z}$,

$$[m]_q = \frac{q^m - q^{-m}}{q-q^{-1}} \in k(q). \tag{7.4.8}$$

Since $q^{1-m}[m]_q$ is a power series in $q-1$ with constant term m, in the formal limit $q \to 1$, we recover the integer m. Consider, for $n \in \mathbb{Z}_{>0}$, the n-dimensional vector space

$$V^{(n)} = k(q)v_1^{(n)} \oplus \cdots \oplus k(q)v_n^{(n)} \cong k(q)^n. \tag{7.4.9}$$

Consider the assignments

$$\begin{aligned} \pi^{(n)}(E)(v_i^{(n)}) &= [i-1]_q v_{i-1}^{(n)}, \\ \pi^{(n)}(F)(v_i^{(n)}) &= [n-i]_q v_{i+1}^{(n)}, \\ \pi^{(n)}(t^{\pm 1})(v_i^{(n)}) &= q^{\pm(n-2i+1)} v_i^{(n)} \end{aligned} \tag{7.4.10}$$

for $i \in \{1,2,\ldots,n\}$, where we have set $v_i^{(n)} := 0$ if $i<1$ or $i>n$. By straightforward checks it follows that $\pi^{(n)}$ extends to a representation of $U_q(\mathfrak{sl}_2)$ on $V^{(n)}$. Note that $\pi^{(1)} = \varepsilon$. Also note that

$$\pi^{(n)}\left(\frac{t-t^{-1}}{q-q^{-1}}\right)(v_i^{(n)}) = [n-2i+1]_q v_i^{(n)}.$$

Hence, formally letting $q \to 1$, we obtain the representation of $\mathfrak{sl}_2$ on $kv_1^{(n)} \oplus \cdots \oplus kv_n^{(n)}$ given by

$$\begin{aligned} e \cdot v_i^{(n)} &= (i-1)v_{i-1}^{(n)}, \\ f \cdot v_i^{(n)} &= (n-i)v_{i+1}^{(n)}, \\ h \cdot v_i^{(n)} &= (n-2i+1)v_i^{(n)}. \end{aligned} \tag{7.4.11}$$

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7.4.5 Chevalley–Serre Presentation of Finite-dimensional Semisimple Lie Algebras

Likewise we are interested in constructing quantum groups $U_q\mathfrak{g}$ for arbitrary finite-dimensional semisimple Lie algebras $\mathfrak{g}$. This is most conveniently done using the Chevalley–Serre presentation of $\mathfrak{g}$ in terms of its Cartan matrix. More precisely, let $C = (c_{ij})_{i,j\in I}$ be an arbitrary Cartan matrix, i.e. $c_{ii} = 2$, $c_{ij} \in \mathbb{Z}_{\geq 0}$, $c_{ij} = 0$ if and only if $c_{ji} = 0$ and finally all submatrices $(c_{ij})_{i,j\in J}$ with $J \subseteq I$ have positive determinant (we briefly discuss the Kac–Moody generalization in Section 7.4.7). There exist positive setwise-coprime integers d_i such that $d_i c_{ij} = d_j c_{ji}$ for all $i, j \in I$. Then each semisimple finite-dimensional Lie algebra arises as follows. Consider the Lie algebra $\mathfrak{g} = \mathfrak{g}(C)$ generated by the subalgebras

$$\mathfrak{sl}_{2,i} := \langle e_i, f_i, h_i \rangle \tag{7.4.12}$$

for all $i \in I$, subject to the $\mathfrak{sl}_2$-relations (7.4.2) with e, f, h replaced by e_i, f_i, h_i, respectively, and, for $i \neq j$, the cross relations

$$\begin{gathered} [h_i, h_j] = 0, \quad [h_i, e_j] = c_{ij} e_j, \quad [h_i, f_j] = -c_{ij} f_j, \quad [e_i, f_j] = 0, \\ [e_i, [e_i, \ldots, [e_i, e_j] \cdots]] = [f_i, [f_i, \ldots, [f_i, f_j] \cdots]] = 0, \end{gathered} \tag{7.4.13}$$

where there are $1 - c_{ij}$ nested Lie brackets in the last two relations (Serre relations). Consider the subalgebras

$$\mathfrak{n}^+ = \langle e_i \,|\, i \in I \rangle, \qquad \mathfrak{h} = \langle h_i \,|\, i \in I \rangle \qquad \mathfrak{n}^- = \langle f_i \,|\, i \in I \rangle. \tag{7.4.14}$$

Like any other Lie algebra, $\mathfrak{g}$ acts on itself by the adjoint action. It is particularly useful to study the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$, with respect to which we have the *triangular decomposition*

$$\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^- \qquad \text{as } \mathfrak{h}\text{-modules.} \tag{7.4.15}$$

More generally, we are interested in representations $V \in \mathsf{Rep}(\mathfrak{g})$ with a weight decomposition with respect to $\mathfrak{h}$:

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda, \qquad V_\lambda = \{v \in V \,|\, h \cdot v = \lambda(h) v \text{ for all } h \in \mathfrak{h}\}. \tag{7.4.16}$$

Any $\lambda \in \mathfrak{h}^*$ for which V_λ is nontrivial is called a ($\mathfrak{h}$-)*weight* for V. Consider the *weight lattice*

$$P = \{\lambda \in \mathfrak{h}^* \,|\, \lambda(h_i) \in \mathbb{Z} \text{ for all } i \in I\}. \tag{7.4.17}$$

A weight for the adjoint action is called a *root* and the *root system* Φ is the set of nonzero roots. Then $\mathfrak{n}^+ = \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$ for some $\Phi^+ \subset \Phi$. For $j \in I$, define the simple root $\alpha_j \in \mathfrak{h}^*$ by $\alpha_j(h_i) = c_{ij}$ so that $\mathfrak{g}_{\alpha_j} = \mathbb{C} e_j$ and hence

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$\alpha_j \in \Phi^+$. Now define a symmetric bilinear form $(\,,\,)$ on $\mathfrak{h}^*$ by $(\alpha_i, \alpha_j) = d_i c_{ij}$ for all $i, j \in I$; it satisfies $(\lambda, \alpha_i) = \lambda(d_i h_i)$ for all $i \in I$.

The category $\mathcal{O}$ is the full subcategory of $\mathsf{Rep}(\mathfrak{g}) \cong \mathsf{Rep}(U\mathfrak{g})$ whose objects are $\mathfrak{g}$-modules V with the decomposition (7.4.16) with all V_λ finite-dimensional, such that $U\mathfrak{n}^+$ acts locally finitely, i.e. for all $v \in V$ the $U\mathfrak{n}^+$-module generated by v is finite-dimensional. The category $\mathcal{O}$ is monoidal[7].

A subcategory called $\mathcal{O}_{\mathsf{int}}$ is obtained by additionally assuming that for each $i \in I$ the subalgebra $U\mathfrak{sl}_{2,i}$ acts locally finitely; by the triangular decomposition for this subalgebra, this is equivalent to $E_i = \iota(e_i)$ and $F_i = \iota(f_i)$ acting locally nilpotently: for all $v \in V$ there exists $m \in \mathbb{Z}_{\geq 0}$ such that $E_i^m \cdot v = F_i^m \cdot v = 0$. Then $\mathcal{O}_{\mathsf{int}}$ is a monoidal category and a semisimple category, with the simple objects given by irreducible highest-weight representations (more precisely, the associated highest weight λ is dominant and integral: $\lambda(h_i) \geq 0$ for all $i \in I$ and $\lambda \in P$). In fact, $\mathcal{O}_{\mathsf{int}}$ corresponds to the category of finite-dimensional $\mathfrak{g}$-representations, with, after a suitable choice of basis, each e_i acting as a strict upper triangular matrix and f_i as its transpose.

7.4.6 Drinfeld–Jimbo Quantum Groups

The definition of the quantum deformation of $U\mathfrak{g}$, independently due to Drinfeld [12] and Jimbo [18], is as follows. For a given Cartan matrix C, let q be an indeterminate and set $q_i := q^{d_i}$. The *Drinfeld–Jimbo quantum group* is the algebra $U_q\mathfrak{g}$ generated over $k(q)$ by subalgebras

$$U_q\mathfrak{sl}_{2,i} := \langle E_i, F_i, t_i, t_i^{-1} \rangle \tag{7.4.18}$$

for $i \in I$, subject to the $U_q\mathfrak{sl}_2$ relations (7.4.4) with $E, F, t^{\pm}, q$ replaced by $E_i, F_i, t_i^{\pm}, q_i$, respectively,[8] and, for $i \neq j$, the cross relations

$$\begin{gathered} t_i E_j = q_i^{c_{ij}} E_j t_i, \quad t_i F_j = q_i^{-c_{ij}} F_j t_i, \quad [t_i, t_j] = 0, \quad [E_i, F_j] = 0, \\ [E_i, [E_i, \ldots, [E_i, E_j]_{q_i^{c_{ij}}} \cdots]_{q_i^{-c_{ij}-2}}]_{q_i^{-c_{ij}}} = 0, \\ [F_i, [F_i, \ldots, [F_i, F_j]_{q_i^{-c_{ij}}} \cdots]_{q_i^{c_{ij}+2}}]_{q_i^{c_{ij}}} = 0, \end{gathered} \tag{7.4.19}$$

where we have used the notation $[x, y]_p := xy - pyx$ for the deformed commutator. Denote the root lattice by $Q = \sum_{i \in I} \mathbb{Z}\alpha_i$. For $\mu = \sum_{i \in I} m_i \alpha_i \in Q$ we denote $t_\mu = \prod_{i \in I} t_i^{m_i}$. The *(quantum) Cartan subalgebra* is the commutative subalgebra

$$U^0 = \langle t_i, t_i^{-1} \,|\, i \in I \rangle = \mathrm{Sp}_k\{t_\mu \,|\, \mu \in Q\}. \tag{7.4.20}$$

[7] Note that we are using the version of category O favoured by Kac, see [21]. The category $\mathcal{O}$ as originally defined in [4] is not a monoidal category.

[8] In the spirit of Section 7.4.3, we may think of t_i as $q_i^{H_i} = q^{d_i H_i}$.

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Remark To be precise, we have given the so-called *adjoint form* of $U_q\mathfrak{g}$, where the Cartan subalgebra is defined in terms of the root lattice $\mathbb{Z}\Phi \subseteq P$. More generally, we may take any sublattice Λ of the weight lattice P, yielding a larger Cartan subalgebra generated by t_λ for $\lambda \in \Lambda$ satisfying e.g. $t_\lambda E_j = q^{(\lambda,\alpha_j)} E_j t_\lambda$. The *simply connected form* of $U_q\mathfrak{g}$ is obtained when we choose $\Lambda = P$. In the case $\mathfrak{g} = \mathfrak{sl}_2$, $P = \frac{1}{2}Q$ and the simply connected and adjoint forms are the only relevant forms of $U_q\mathfrak{g}$, with the simply connected form obtained from the adjoint form by adjoining square roots of the generators t_i and t_i^{-1}.

By a straightforward check on generators (see e.g. [17, Lem. 4.8]), one has the following result.

Lemma 7.13 *$U_q\mathfrak{g}$ is a (non-cocommutative) Hopf algebra with the additional structure maps given by*

$$\begin{aligned} \Delta(E_i) &= E_i \otimes 1 + t_i \otimes E_i, & \varepsilon(E_i) &= 0, & S(E_i) &= -t_i^{-1}E_i, \\ \Delta(F_i) &= F_i \otimes t_i^{-1} + 1 \otimes F_i, & \varepsilon(F_i) &= 0, & S(F_i) &= -F_i t_i, \\ \Delta(t_i^{\pm 1}) &= t_i^{\pm 1} \otimes t_i^{\pm 1}, & \varepsilon(t_i^{\pm 1}) &= 1, & S(t_i^{\pm 1}) &= t_i^{\mp 1}. \end{aligned} \tag{7.4.21}$$

For later convenience we record the explicit formulas for S^{-1}:

$$S^{-1}(E_i) = -E_i t_i^{-1}, \quad S^{-1}(F_i) = -t_i F_i, \quad S^{-1}(t_i^{\pm 1}) = t_i^{\mp 1}. \tag{7.4.22}$$

The triangular decomposition induced on $U\mathfrak{g}$ by the multiplication map, namely $U\mathfrak{g} \cong U\mathfrak{n}^+ \otimes U\mathfrak{h} \otimes U\mathfrak{n}^-$, lifts directly to

$$U_q\mathfrak{g} \cong U^+ \otimes U^0 \otimes U^-, \tag{7.4.23}$$

where we have introduced the subalgebras

$$U^+ = \langle E_i \,|\, i \in I \rangle, \qquad U^- = \langle F_i \,|\, i \in I \rangle. \tag{7.4.24}$$

For $V \in \mathsf{Rep}(U_q\mathfrak{g})$ and $\lambda \in P$, denote the (quantum) weight space

$$V_\lambda = \{ v \in V \,|\, t_i \cdot v = q_i^{\lambda(h_i)} v = q^{(\alpha_i,\lambda)} v \text{ for all } i \in I \}. \tag{7.4.25}$$

In particular, as part of the adjoint action of $U_q\mathfrak{g}$ on itself, the t_i act by conjugation, and we have the root space decompositions

$$U^\pm = \bigoplus_{\lambda \in Q^+} U^\pm_{\pm\lambda}, \qquad \text{where } Q^+ := \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i. \tag{7.4.26}$$

The category $\mathcal{O}_q$ is defined as the full subcategory of $\mathsf{Rep}(U_q\mathfrak{g})$ whose objects are modules V such that

$$V = \bigoplus_{\lambda \in P} V_\lambda \tag{7.4.27}$$

with all V_λ finite-dimensional and such that U^+ acts locally finitely. As before, $\mathcal{O}_q$ is a monoidal category. Note that the E_i-action and F_i-action on $V \in \mathcal{O}_q$ satisfy

$$E_i(V_\lambda) \subseteq V_{\lambda+\alpha_i}, \qquad F_i(V_\lambda) \subseteq V_{\lambda-\alpha_i}. \tag{7.4.28}$$

The subcategory $\mathcal{O}_{q,\mathsf{int}}$ is obtained by additionally assuming that each subalgebra $U_q\mathfrak{sl}_{2,i}$ acts locally finitely. Then $\mathcal{O}_{q,\mathsf{int}}$ is the category of finite-dimensional representations such that each t_i acts diagonalizably with integer powers of q_i as eigenvalues (so-called type-**1** representations). As in the $(q \to 1)$-limit, $\mathcal{O}_{q,\mathsf{int}}$ is a monoidal category and a semisimple category, whose simple objects are irreducible highest-weight representations with dominant integral highest weight, see e.g. [30, Cor. 6.2.3] or [7, Sec. 10.1].

In the case $\mathfrak{g} = \mathfrak{sl}_2$, the weight lattice is $P = \frac{\mathbb{Z}}{2}\alpha$, where α is the unique simple root and the representation $\pi^{(n)}$ defined in (7.4.10) defines a simple object in $\mathcal{O}_{q,\mathsf{int}}$. It is an irreducible highest-weight representation with highest weight vector $v_1^{(n)}$ and highest weight $\frac{n-1}{2}\alpha$.

7.4.7 Kac–Moody Generalization

The definition of the Drinfeld–Jimbo quantum group can straightforwardly be extended to the case where $C = (c_{ij})_{i,j\in I}$ is a symmetrizable generalized Cartan matrix, thereby quantum-deforming universal enveloping algebras of Kac–Moody Lie algebras [21]. This means we require $c_{ii} = 2$, $c_{ij} \in \mathbb{Z}_{\geq 0}$, $c_{ij} = 0$ if and only if $c_{ji} = 0$ and the existence of a set of positive setwise-coprime integers d_i such that $d_i c_{ij} = d_j c_{ji}$ for all $i, j \in I$. As in the classical $(q \to 1)$ case, the Cartan subalgebra is larger: U^0 is defined in terms of a lattice which as a free abelian group has rank $|I| + \mathsf{cork}(C)$. The category $\mathcal{O}_q$ and the subcategory $\mathcal{O}_{q,\mathsf{int}}$ can be defined as above and are still monoidal categories. Moreover $\mathcal{O}_{q,\mathsf{int}}$ is still semisimple with simple objects given by irreducible highest-weight representations with dominant integral highest weight. However, neither category contains nontrivial finite-dimensional representations.

We say that C is of *affine type* if $\det(C) = 0$ and all *proper* submatrices $(c_{ij})_{i,j\in J}$ with $J \subset I$ have positive determinant, see e.g. [6, 15]. If C is of affine type, then $\mathfrak{g}' := \langle e_i, f_i, h_i \,|\, i \in I \rangle$ and similarly $U_q\mathfrak{g}' := \langle E_i, F_i, t_i^{\pm 1} \,|\, i \in I \rangle$ (but not $\mathfrak{g}$ and $U_q\mathfrak{g}$ themselves) have finite-dimensional representations that arise from the identification of $\mathfrak{g}'$ as a central extension of a loop algebra $\mathfrak{g}_0 \otimes k[z, z^{-1}]$ of a finite-dimensional simple Lie algebra $\mathfrak{g}_0$ (or a twisted loop algebra), see e.g. [6, 15] for details. These affine quantum groups are the most relevant in quantum integrability.

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7.5 Quasitriangularity for $U_q\mathfrak{g}$

Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over k. We will construct the universal R-matrix for $U_q\mathfrak{g}$, roughly following the approach from [17, Ch. 6 and 7] which is based on the approaches by [30, Ch. 4] and [42]. A closely related construction is that via the quantum double construction due to Drinfeld, see [13] and cf. [7, Sec. 4.2D]. We complement the fairly technical arguments by explicit formulas for the special case $\mathfrak{g} = \mathfrak{sl}_2$.

From now on we work over the larger field $k(q^{1/d})$ for a suitable positive integer d, since we want to allow linear maps acting on objects in $\mathcal{O}_{q,\mathsf{int}}$ by multiplication by such scalars in certain weight spaces. Let us set the stage.

7.5.1 The Bar Involution

The *bar involution* is an involutive algebra automorphism of $U_q\mathfrak{g}$ denoted by $\overline{\cdot}$ which acts nontrivially on the base field $k(q^{1/d})$: it sends $q^{1/d}$ to $q^{-1/d}$. On the generators it is defined as follows:

$$\overline{E_i} = E_i, \qquad \overline{F_i} = F_i, \qquad \overline{t_i^{\pm 1}} = t_i^{\mp 1}. \tag{7.5.1}$$

It is straightforward to check that these assignments preserve the defining relations of $U_q\mathfrak{g}$, as required.

We will give a construction of the universal R-matrix by considering, in addition to Δ and Δ^{op}, a third coproduct $\overline{\Delta} := (\overline{\cdot} \otimes \overline{\cdot}) \circ \Delta \circ \overline{\cdot}$. Explicitly, we have

$$\begin{gathered} \overline{\Delta}(E_i) = E_i \otimes 1 + t_i^{-1} \otimes E_i, \qquad \overline{\Delta}(F_i) = F_i \otimes t_i + 1 \otimes F_i, \\ \overline{\Delta}(t_i^{\pm 1}) = t_i^{\pm 1} \otimes t_i^{\pm 1}. \end{gathered} \tag{7.5.2}$$

Our plan is to construct an invertible element $\mathcal{R}$ that intertwines Δ with Δ^{op} as follows:

$$\mathcal{R}\Delta(u) = \Delta^{\mathsf{op}}(u)\mathcal{R} \qquad \text{for all } u \in U_q\mathfrak{g}. \tag{7.5.3}$$

It turns out that $\overline{\Delta}$ is convenient in an intermediate stage of the proof of this. Namely, we will establish (7.5.3) by constructing two elements $\widetilde{\mathcal{R}}$ and κ that intertwine Δ with $\overline{\Delta}$, and $\overline{\Delta}$ with Δ^{op}, respectively:

$$\widetilde{\mathcal{R}}\Delta(x) = \overline{\Delta}(x)\widetilde{\mathcal{R}}, \qquad \kappa\overline{\Delta}(x) = \Delta^{\mathsf{op}}(x)\kappa \qquad \text{for all } x \in U_q\mathfrak{g}. \tag{7.5.4}$$

From these two equations (7.5.3) readily follows if we set $\mathcal{R} = \kappa\widetilde{\mathcal{R}}$. Compare this with the proof of Theorem 7.11 for Sweedler's Hopf algebra.

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7.5.2 The Chevalley Involution

Define the Chevalley involution of $U_q\mathfrak{g}$ on its generators as follows:

$$\omega(E_i) = -F_i, \qquad \omega(F_i) = -E_i, \qquad \omega(t_i^{\pm 1}) = t_i^{\mp 1}. \tag{7.5.5}$$

It corresponds to the matrix Lie algebra automorphism $x \mapsto -x^{\mathsf{t}}$ in finite-dimensional representations of $\mathfrak{g}$. By straightforward checks we obtain the following result:

Lemma 7.14 *The map ω is a bialgebra morphism from $U_q\mathfrak{g}$ to $U_q\mathfrak{g}^{\mathsf{cop}}$.*

7.5.3 The Completions $\widehat{U}$ and $\widehat{U}^{(2)}$

In order to construct the universal R-matrix $\mathcal{R}$ for $U_q\mathfrak{g}$, we will consider an algebra properly containing $U_q\mathfrak{g} \otimes U_q\mathfrak{g}$. The fact that $\mathcal{R}$ does not lie in $U_q\mathfrak{g} \otimes U_q\mathfrak{g}$ is the only obstacle for $U_q\mathfrak{g}$ being quasitriangular, so we say that $U_q\mathfrak{g}$ is quasitriangular "up to completion". It means that $\mathcal{R}$ has a well-defined action on a proper subcategory of $\mathsf{Rep}(U_q\mathfrak{g})$, namely $\mathcal{O}_{q,\mathsf{int}}$.

We discuss now one possible definition of the completion, closely following [2, Sec. 3.1], but also see [e.g. 38, Sec. 1.3]. Since $\mathcal{O}_{q,\mathsf{int}}$ is a subcategory of $\mathsf{Rep}(U_q\mathfrak{g})$, we have a forgetful functor $\mathsf{For} : \mathcal{O}_{q,\mathsf{int}} \to \mathsf{Vect}$, which is a monoidal functor (it preserves tensor products). Consider now the algebra $\widehat{U}$ of all natural transformations from For to itself. A *natural transformation* of For is a tuple (φ_V), where V runs through $\mathcal{O}_{q,\mathsf{int}}$, consisting of linear maps $\varphi_V : \mathsf{For}(V) \to \mathsf{For}(V)$ such that the following diagram in Vect commutes:

$$\begin{tikzcd} \mathsf{For}(V) \arrow[r, "\varphi_V"] \arrow[d, "\mathsf{For}(f)"'] & \mathsf{For}(V) \arrow[d, "\mathsf{For}(f)"] \\ \mathsf{For}(W) \arrow[r, "\varphi_W"'] & \mathsf{For}(W) \end{tikzcd} \tag{7.5.6}$$

for all $V, W \in \mathcal{O}_{q,\mathsf{int}}$ and for all $U_q\mathfrak{g}$-intertwiners $f : V \to W$. Note that $\widehat{U}$ naturally has the structure of an algebra over $k(q^{1/d})$ since we can add, scalar-multiply and compose such tuples entrywise. Furthermore, the $U_q\mathfrak{g}$-action on objects of $\mathcal{O}_{q,\mathsf{int}}$ produces an algebra morphism $U_q\mathfrak{g} \to \widehat{U}$. Indeed, compare the definition of natural transformation with the definition of intertwiner (defined by means of another commuting diagram of linear maps (7.2.5)). This morphism is injective, see [Lu94, Prop. 3.5.4], and henceforth we will view $U_q\mathfrak{g}$ as a subalgebra of $\widehat{U}$.

We also have a functor $\mathsf{For}^{(2)} : \mathcal{O}_{q,\mathsf{int}} \times \mathcal{O}_{q,\mathsf{int}} \to \mathsf{Vect}$, sending pairs of modules (V, W) to $\mathsf{For}(V \otimes W)$ and pairs of intertwiners (f, g) to $\mathsf{For}(f \otimes g)$. We define $\widehat{U}^{(2)} = \mathrm{End}(\mathsf{For}^{(2)})$, which is an algebra for the same reasons as $\widehat{U}$, and we may view $\widehat{U} \otimes \widehat{U} \subset \widehat{U}^{(2)}$ via $(\varphi_V)_V \otimes (\psi_W)_W \mapsto (\varphi_V \otimes \psi_W)_{(V,W)}$. Analogously we can define a completion $\widehat{U}^{(n)}$ for any $n \in \mathbb{Z}_{\geq 0}$ with natural algebra embeddings $\widehat{U}^{(m)} \otimes \widehat{U}^{(n)} \to \widehat{U}^{(m+n)}$.

Any $\varphi \in \widehat{U}$ can be restricted to $\mathsf{For}(V \otimes W)$ for all $V, W \in \mathcal{O}_{q,\mathsf{int}}$; since restriction is compatible with composition and linearity of natural transformations, we obtain an algebra morphism

$$\Delta : \widehat{U} \to \widehat{U}^{(2)}, \qquad (\varphi_V) \mapsto (\varphi_{V \otimes W}). \tag{7.5.7}$$

It restricts to the usual coproduct of the embedded subalgebra $U_q\mathfrak{g} \subset \widehat{U}$, motivating the notation. The algebra maps $\Delta \otimes \mathsf{id}, \mathsf{id} \otimes \Delta$ from $\widehat{U} \otimes \widehat{U}$ to $\widehat{U}^{(3)}$ extend to algebra maps from $\widehat{U}^{(2)}$ to $\widehat{U}^{(3)}$.

7.5.4 The Element κ

For $V, W \in \mathcal{O}_{q,\mathsf{int}}$ a linear map $\kappa_{V,W} \in \mathrm{End}(V \otimes W)$ is uniquely determined by the condition

$$\kappa_{V,W}(v \otimes w) = q^{(\mu,\nu)} v \otimes w \text{ for all } \mu, \nu \in P,\ v \in V_\mu,\ w \in W_\nu. \tag{7.5.8}$$

The tuple $\kappa := (\kappa_{V,W})$ lies in $\widehat{U}^{(2)}$ (but not in $\widehat{U} \otimes \widehat{U}$).

Lemma 7.15 *The map $\mathsf{Ad}(\kappa)$ preserves $U_q\mathfrak{g} \otimes U_q\mathfrak{g}$; more precisely*

$$\begin{aligned} \mathsf{Ad}(\kappa)(E_i \otimes 1) &= E_i \otimes t_i, & \mathsf{Ad}(\kappa)(1 \otimes E_i) &= t_i \otimes E_i, \\ \mathsf{Ad}(\kappa)(F_i \otimes 1) &= F_i \otimes t_i^{-1}, & \mathsf{Ad}(\kappa)(1 \otimes F_i) &= t_i^{-1} \otimes F_i, \\ \mathsf{Ad}(\kappa)(t_i^{\pm 1} \otimes 1) &= t_i^{\pm 1} \otimes 1, & \mathsf{Ad}(\kappa)(1 \otimes t_i^{\pm 1}) &= 1 \otimes t_i^{\pm 1}. \end{aligned} \tag{7.5.9}$$

Proof Note that $U_q\mathfrak{g} \otimes U_q\mathfrak{g}$ is generated by $E_i \otimes 1$, $F_i \otimes 1$, $t_i^{\pm 1} \otimes 1$, $1 \otimes E_i$, $1 \otimes F_i$ and $1 \otimes t_i^{\pm 1}$. Let $V, W \in \mathcal{O}_{q,\mathsf{int}}$ be arbitrary and let $\mu, \nu \in P$. Owing to (7.4.28), we have

$$\begin{aligned} \mathsf{Ad}(\kappa)(E_i \otimes 1)|_{V_\mu \otimes W_\nu} &= \kappa(E_i \otimes 1)\kappa^{-1}|_{V_\mu \otimes W_\nu} \\ &= q^{-(\mu,\nu)} \kappa(E_i \otimes 1)|_{V_\mu \otimes W_\nu} \\ &= q^{(\mu+\alpha_i,\nu)-(\mu,\nu)}(E_i \otimes 1)|_{V_\mu \otimes W_\nu} \\ &= q^{(\alpha_i,\nu)}(E_i \otimes 1)|_{V_\mu \otimes W_\nu} \\ &= (E_i \otimes t_i)|_{V_\mu \otimes W_\nu}, \end{aligned}$$

as required. The computations for $F_i \otimes 1$, $1 \otimes E_i$ and $1 \otimes F_i$ are entirely similar. Finally, since $t_i^{\pm 1} \otimes 1$ and $1 \otimes t_i^{\pm 1}$ preserve the weight summands of objects in $\mathcal{O}_{q,\mathsf{int}}$, they are fixed by conjugation by κ. □

From (7.5.9) we obtain the desired intertwining property of κ:

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Lemma 7.16 *We have* $\mathrm{Ad}(\kappa) \circ \overline{\Delta} = \Delta^{\mathrm{op}}$.

We continue our study of the element κ with the following result:

Lemma 7.17 *We have* $(\Delta \otimes \mathrm{id})(\kappa) = \kappa_{13}\kappa_{23}$.

Proof Let $U, V, W \in \mathcal{O}_{q,\mathrm{int}}$ and $\lambda, \mu, \nu \in P$. From the definition of the coproduct map $\widehat{U} \to \widehat{U}^{(2)}$ and the embedding $\widehat{U}^{(2)} \to \widehat{U}^{(3)}$, we obtain

$$
\begin{aligned}
(\Delta \otimes \mathrm{id})(\kappa)_{U,V,W}|_{U_\lambda \otimes V_\mu \otimes W_\nu} &= \kappa_{U \otimes V, W}|_{U_\lambda \otimes V_\mu \otimes W_\nu} \\
&= \text{multiplication by } q^{(\lambda+\mu,\nu)}|_{U_\lambda \otimes V_\mu \otimes W_\nu} \\
&= \text{multiplication by } q^{(\lambda,\nu)} q^{(\mu,\nu)}|_{U_\lambda \otimes V_\mu \otimes W_\nu} \\
&= (\kappa_{U,W})_{13}(\kappa_{V,W})_{23}|_{U_\lambda \otimes V_\mu \otimes W_\nu},
\end{aligned}
$$

as required. Here we have used that $U_\lambda \otimes V_\mu \subseteq (U \otimes V)_{\lambda+\mu}$, which follows directly from the definition of weight space. □

7.5.5 The Algebra $\widehat{U}^{-+}$

We also consider the algebra $\widehat{U}^+ := \prod_{\mu \in Q^+} U^+_\mu$. Let $(x_\mu)_{\mu \in Q^+} \in \widehat{U}^+$ be arbitrary. Note that for all $V \in \mathcal{O}_{q,\mathrm{int}}$ and all $v \in V$, there are only finitely many $\mu \in Q^+$ such that $x_\mu \cdot v$ is nonzero. Hence the expression $\sum_{\mu \in Q^+} x_\mu \cdot v$ is a well-defined element of V. It can be checked that $(x_\mu)_{\mu \in Q^+}$ defines an element of $\widehat{U}$, so that we may consider $\widehat{U}^+$ as a subalgebra of $\widehat{U}$. Considering the inclusion $U^+ \subseteq \widehat{U}^+$, it is safe to write elements of $\widehat{U}^+$ additively as $x = \sum_{\mu \in Q^+} x_\mu$.

Owing again to the finiteness of the U^+-action, elements of the form $\sum_{\mu,\nu \in Q^+} y_\nu \otimes x_\mu$ with $x_\mu \in U^+_\mu$, $y_\nu \in U^-_{-\nu}$ have a well-defined action on $V \otimes W$ for all $V, W \in \mathcal{O}^+_q$, and lie in $\widehat{U}^{(2)}$. The subalgebra of $\widehat{U}^{\otimes 2}$ generated by such elements is denoted $U^- \widehat{\otimes} U^+$.

Remark We can define subalgebras $U^+ \widehat{\otimes} U^+, U^+ \widehat{\otimes} U^- \subset \widehat{U}^{(2)}$ in a similar way, but not $U^- \widehat{\otimes} U^-$: its putative elements do not have a well-defined action on objects in $\mathcal{O}_{q,\mathrm{int}}$.

We now claim that the desired element $\mathcal{R}$ can be chosen in the subalgebra

$$\widehat{U}^{-+} := \langle U_q\mathfrak{g} \otimes U_q\mathfrak{g}, U^- \widehat{\otimes} U^+, \kappa \rangle \subset \widehat{U}^{(2)}. \qquad (7.5.10)$$

We can extend the composition $\omega^{(2)} := \sigma \circ (\omega \otimes \omega)$ from an involutive algebra automorphism of $U_q\mathfrak{g} \otimes U_q\mathfrak{g}$ to an involutive algebra automorphism of $\widehat{U}^{-+}$; we simply stipulate that the extension fix κ and act on $U^- \widehat{\otimes} U^+$ as follows:

$$\sum_{\mu,\nu \in Q^+} c_{\mu,\nu} y_\nu \otimes x_\mu \leftrightarrow \sum_{\mu,\nu \in Q^+} c_{\mu,\nu} \omega(x_\mu) \otimes \omega(y_\nu). \qquad (7.5.11)$$

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This is consistent with the relations (7.5.9) and the natural relations involving series, and hence defines an algebra automorphism of $\widehat{U}^{-+}$.

7.5.6 Bialgebra Pairings

We now start the construction of the desired element $\widetilde{\mathcal{R}} \in U^- \widehat{\otimes} U^+$. Suppose A, B are two bialgebras (with coproducts Δ_A, Δ_B and counits ε_A, ε_B, respectively). A *bialgebra pairing* between A and B (see [7, 4.1D]), is a k-linear map $\langle \cdot, \cdot \rangle : A \otimes B \to k$ with the properties

$$\begin{aligned} \langle \Delta_A(a), b \otimes b' \rangle &= \langle a, bb' \rangle, & \langle a \otimes a', \Delta_B(b) \rangle &= \langle aa', b \rangle, \\ \varepsilon_A(a) &= \langle a, 1 \rangle, & \varepsilon_B(b) &= \langle 1, b \rangle \end{aligned} \tag{7.5.12}$$

for all $a, a' \in A$ and $b, b' \in B$. Here we denote by the same symbol the canonical extension of the pairing of A and B to a k-linear map: $(A \otimes A) \otimes (B \otimes B) \to k$ defined by $\langle a \otimes a', b \otimes b' \rangle = \langle a, b \rangle \langle a', b' \rangle$ for all $a, a' \in A$ and $b, b' \in B$. In particular, $\langle a \otimes a', b \otimes b' \rangle = \langle a' \otimes a, b' \otimes b \rangle$ and we automatically obtain a bialgebra pairing between A^{cop} and B^{op} and between A^{op} and B^{cop}.

The quantum analogues of the standard Borel subalgebras, viz.

$$U_q\mathfrak{b}^+ = \langle E_i, t_i, t_i^{-1} \,|\, i \in I \rangle, \qquad U_q\mathfrak{b}^- = \langle F_i, t_i, t_i^{-1} \,|\, i \in I \rangle \tag{7.5.13}$$

are subbialgebras of $U_q\mathfrak{g}$ over $k(q^{1/d})$. Then the following assignments define a unique bialgebra pairing between $U_q\mathfrak{b}^{-,\mathsf{cop}}$ and $U_q\mathfrak{b}^+$:

$$\begin{aligned} \langle t_\lambda, t_\mu \rangle &= q^{-(\lambda,\mu)}, & \langle F_i, E_j \rangle &= \frac{\delta_{ij}}{q_i^{-1} - q_i}, \\ \langle t_\lambda, E_j \rangle &= 0, & \langle F_i, t_\mu \rangle &= 0 \end{aligned} \tag{7.5.14}$$

for all $i, j \in I$ and $\lambda, \mu \in Q$ (see e.g. [17, 6.12]), where it is presented as a pairing between $U_q\mathfrak{b}^-$ and $U_q\mathfrak{b}^{+,\mathsf{op}}$. This is a nondegenerate pairing (see [17, 6.21]); moreover its restriction to $U^-_{-\nu} \times U^+_\mu$ vanishes if $\mu \neq \nu$ and is nondegenerate if $\mu = \nu$: if for some $x \in U^+_\mu$ we have $\langle y, x \rangle = 0$ for all $y \in U^-_{-\mu}$ then $x = 0$ (and vice versa).

7.5.7 Skew Derivations

In order to construct the desired element $\widetilde{\mathcal{R}}$ and establish its key properties we introduce linear maps on U^+ called *right and left skew derivations* due to Lusztig (see [30, Sec. 1.2 and 3.1]). More precisely, for each $i \in I$ there exist $D_i^{(r,\ell)} \in \mathrm{End}_{k(q^{1/d})}(U^+)$ uniquely determined by stipulating that $D_i^{(r,\ell)}(E_j) = \delta_{ij}$ for all $j \in I$ and

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$$D_i^{(r)}(xx') = D_i^{(r)}(x)\mathsf{Ad}(t_i)(x') + xD_i^{(r)}(x'),$$
$$D_i^{(\ell)}(xx') = D_i^{(\ell)}(x)x' + \mathsf{Ad}(t_i)(x)D_i^{(\ell)}(x') \tag{7.5.15}$$

and $x, x' \in U^+$. Clearly, the two maps $D_i^{(r,\ell)}$ send U^+_μ to $U^+_{\mu-\alpha_i}$ for all $\mu \in Q^+$. By [17, 6.14], for all $\mu \in Q^+$, $x \in U^+_\mu$, we have

$$\Delta(x) - x \otimes 1 - \sum_{i \in I} D_i^{(r)}(x) t_i \otimes E_i \in \sum_{\substack{\nu \in Q^+ \\ \nu \neq \alpha_j, \nu \neq 0}} U^+_{\mu-\nu} t_\nu \otimes U^+_\nu,$$
$$\Delta(x) - t_\mu \otimes x - \sum_{i \in I} E_i t_{\mu-\nu_i} \otimes D_i^{(\ell)}(x) \in \sum_{\substack{\nu \in Q^+ \\ \nu \neq \alpha_j, \nu \neq 0}} U^+_\nu t_{\mu-\nu} \otimes U^+_{\mu-\nu}. \tag{7.5.16}$$

By [17, 6.15 (5)], for all $x \in U^+$, $y \in U^-$ and $i \in I$, we have

$$\langle F_i y, x\rangle = \langle F_i, E_i\rangle \langle y, D_i^{(\ell)}(x)\rangle, \quad \langle y F_i, x\rangle = \langle F_i, E_i\rangle \langle y, D_i^{(r)}(x)\rangle. \tag{7.5.17}$$

By [17, 6.17], for all $x \in U^+$ and $i \in I$, we have

$$[x, F_i] = (q_i - q_i^{-1})^{-1}\big(D_i^{(r)}(x) t_i - t_i^{-1} D_i^{(\ell)}(x)\big). \tag{7.5.18}$$

In fact, each of (7.5.16–7.5.18) can be used to define the linear maps $D_i^{(r,\ell)}$ uniquely. By [30, Lem. 1.2.15 (a)] we have

$$\forall i \in I\, D_i^{(r)}(x) = 0 \quad \Leftrightarrow \quad \forall i \in I\, D_i^{(\ell)}(x) = 0 \quad \Leftrightarrow \quad x = 0 \tag{7.5.19}$$

for all $x \in U^+$.

Note that ω restricts to a bialgebra morphism from $U_q\mathfrak{b}^+$ to $U_q\mathfrak{b}^{-,\mathsf{cop}}$ interchanging U^+_μ and $U^-_{-\mu}$ for all $\mu \in Q^+$. One can similarly define skew derivations for U^- in the natural way, namely via the compositions $\omega \circ D_i^{(r,\ell)} \circ \omega$. As a consequence we have $\langle \omega(x), \omega(y)\rangle = \langle y, x\rangle$ for all $x \in U_q\mathfrak{b}^+$, $y \in U_q\mathfrak{b}^-$, see [17, 6.16].

7.5.8 The Element Θ

For arbitrary $\mu \in Q^+$, choose a basis $(x_{\mu,r})_r$ for the finite-dimensional $k(q^{1/d})$-vector space U^+_μ and let $(y_{\mu,r})_r$ be the dual basis of $U^-_{-\mu}$ with respect to the bilinear pairing $\langle\,,\,\rangle$. Consider the element defined by

$$\Theta = \sum_{\mu \in Q^+} \Theta_\mu \in \widehat{U}^{(2)}, \quad \Theta_\mu = \sum_r y_{\mu,r} \otimes x_{\mu,r} \in U^-_{-\mu} \otimes U^+_\mu. \tag{7.5.20}$$

In other words, Θ is the "canonical element" of the restriction of $\langle\,,\,\rangle$ to $U^- \times U^+$. Note that $U^-_0 = U^+_0 = k$ so that $\Theta_0 = 1 \otimes 1$.

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Since Θ_μ is independent of the choice of basis for U^+_μ and ω is invertible, each Θ_μ is fixed by $\omega^{(2)}$. Straightaway we obtain

Lemma 7.18 *The element* Θ *satisfies* $\omega^{(2)}(\Theta) = \Theta$.

We stay close to the approach in [17, Ch. 7] in establishing key properties of Θ.

Theorem 7.19 *The linear space*

$$\{X \in U^- \widehat{\otimes} U^+ \mid \mathsf{Ad}(X) \circ \overline{\Delta} = \Delta\} \tag{7.5.21}$$

is 1-dimensional and spanned by Θ.

Proof Suppose $X \in U^- \widehat{\otimes} U^+$ is such that $X\overline{\Delta}(u) = \Delta(u)X$ for all $u \in U_q\mathfrak{g}$. Since $\Delta(t_i^{\pm 1}) = \overline{\Delta}(t_i^{\pm 1}) = t_i^{\pm 1} \otimes t_i^{\pm 1}$ for all $i \in I$, it follows that $X = \sum_{\mu \in Q^+} X_\mu$ with $X_\mu \in U^-_{-\mu} \otimes U^+_\mu$. Assuming X is of this form, the condition $\mathsf{Ad}(X) \circ \overline{\Delta} = \Delta$ is equivalent to the following identities:

$$[X_\mu, E_i \otimes 1] = (t_i \otimes E_i) X_{\mu-\alpha_i} - X_{\mu-\alpha_i}(t_i^{-1} \otimes E_i), \tag{7.5.22}$$

$$[X_\mu, 1 \otimes F_i] = (F_i \otimes t_i^{-1}) X_{\mu-\alpha_i} - X_{\mu-\alpha_i}(F_i \otimes t_i) \tag{7.5.23}$$

for all $i \in I$, where $X_\mu := 0$ if $\mu \notin Q^+$. By applying $\sigma \circ (\omega \otimes \omega)$ one sees that these identities are equivalent. By (7.5.18) and linear independence, (7.5.23) is equivalent to the system

$$\begin{aligned} (F_i \otimes 1) X_{\mu-\alpha_i} &= (q_i^{-1} - q_i)^{-1} (\mathsf{id} \otimes D_i^{(\ell)})(X_\mu), \\ X_{\mu-\alpha_i}(F_i \otimes 1) &= (q_i^{-1} - q_i)^{-1} (\mathsf{id} \otimes D_i^{(r)})(X_\mu) \end{aligned} \tag{7.5.24}$$

for all $i \in I$. It suffices to prove that the solution set of (7.5.24) are precisely the scalar multiples of Θ. The computation in [17, 7.1], which relies on (7.5.17), shows that $X = \Theta$ satisfies these conditions. To show uniqueness up to scalar multiples, we may follow the proof of [30, Thm. 4.1.2], which relies on (7.5.19). □

Theorem 7.20 *We have*

$$(\Delta \otimes \mathsf{id})(\Theta) = \Theta_{23} \mathsf{Ad}(\kappa_{23}^{-1})(\Theta_{13}). \tag{7.5.25}$$

Proof Considering (7.5.9), it suffices to prove

$$(\Delta \otimes \mathsf{id})(\Theta_\mu) = \sum_{\nu \in Q^+} (\Theta_{\mu-\nu})_{23} (1 \otimes t_\nu^{-1} \otimes 1)(\Theta_\nu)_{13}. \tag{7.5.26}$$

Fix $y \in U^-_{-\mu}$ with $\mu = \sum_{i\in I} m_i \alpha_i$ with $m_i \in \mathbb{Z}_{\geq 0}$. By induction with respect to the height of μ, viz. $\sum_{i\in I} m_i$, we obtain from the coproduct formula for F_i in (7.4.21) that, for all

$$\Delta(y) \in \sum_{\nu \in Q^+} U^-_{-\nu} \otimes U^-_{\nu-\mu} t_\nu^{-1} \tag{7.5.27}$$

and hence

$$\Delta(y) = \sum_{\nu\in Q^+} \sum_{r,s} c^\nu_{r,s} y_{\nu,r} \otimes y_{\mu-\nu,s} t_\nu^{-1} \tag{7.5.28}$$

for some $c^\nu_{r,s} \in k(q^{1/d})$. Since the basis $\{y_{\mu,r}\}_r$ is dual to $\{x_{\mu,r}\}_r$ with respect to $\langle , \rangle$, we have

$$c^\nu_{r,s} = \langle \Delta(y), x_{s,\nu} \otimes x_{r,\mu-\nu} \rangle = \langle \Delta^{\mathsf{op}}(y), x_{r,\mu-\nu} \otimes x_{s,\nu} \rangle = \langle y, x_{r,\mu-\nu} x_{s,\nu} \rangle.$$

Now (7.5.26) follows by recalling the definition of Θ_μ in terms of the basis elements $y_{\mu,r}$ and $x_{\mu,r}$. We refer to the proof of [17, Lem. 7.4] for the remaining computation. $\square$

7.5.9 The Quasi R-matrix $\widetilde{\mathcal{R}}$ and the Universal R-matrix $\mathcal{R}$

We now simply define $\widetilde{\mathcal{R}} = \Theta^{-1}$. Immediately we obtain from Lemma 7.18, Theorem 7.19 and Theorem 7.20 the following result for $\widetilde{\mathcal{R}}$:

Theorem 7.21 *The element $\widetilde{\mathcal{R}}$ is fixed by $\omega^{(2)}$, has the intertwining property $\mathsf{Ad}(\widetilde{\mathcal{R}}) \circ \Delta = \overline{\Delta}$ and the coproduct formula $(\Delta \otimes \mathsf{id})(\widetilde{\mathcal{R}}) = \mathsf{Ad}(\kappa_{23}^{-1})(\widetilde{\mathcal{R}}_{13})\widetilde{\mathcal{R}}_{23}$.*

The desired element $\mathcal{R}$ is now given by

$$\mathcal{R} = \kappa \widetilde{\mathcal{R}} \in \widehat{U}^{-+}. \tag{7.5.29}$$

As a consequence of Theorem 7.21 and the various properties of κ from Section 7.5.4 we obtain that $\mathcal{R}$ satisfies the properties of a universal R-matrix:

Theorem 7.22 *The element $\mathcal{R}$ is fixed by $\omega^{(2)}$, has the intertwining property $\mathsf{Ad}(\mathcal{R}) \circ \Delta = \Delta^{\mathsf{op}}$ and the coproduct formula $(\Delta \otimes \mathsf{id})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}$.*

Recall that the other coproduct formula $(\mathsf{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}$ follows from Lemma 7.8.

7.5.10 The R-matrix for $U_q\mathfrak{sl}_2$

For the quantum group $U_q\mathfrak{sl}_2$ it is possible to make the formula for $\widetilde{\mathcal{R}}$ rather explicit. It leads to the following result.

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Theorem 7.23 *The subspace of elements $\widetilde{\mathcal{R}} \in U^- \widehat{\otimes} U^+$ satisfying*

$$\mathsf{Ad}(\widetilde{\mathcal{R}}) \circ \Delta = \overline{\Delta} \tag{7.5.30}$$

is 1-dimensional. The unique such element with $(\varepsilon \otimes \varepsilon)(\widetilde{\mathcal{R}}) = 1$ is

$$\widetilde{\mathcal{R}} = \sum_{r=0}^{\infty} c_r (F \otimes E)^r, \tag{7.5.31}$$

where for $r \in \mathbb{Z}_{\geq 0}$ we have

$$c_r := \frac{(q-q^{-1})^r}{[r]_q!} q^{r(r-1)/2}, \quad \text{with } [r]_q! := [r]_q [r-1]_q \cdots [2]_q [1]_q. \tag{7.5.32}$$

Additionally, it satisfies

$$(\Delta \otimes \mathsf{id})(\widetilde{\mathcal{R}}) = \mathsf{Ad}(\kappa_{23}^{-1})(\widetilde{\mathcal{R}}_{13})\widetilde{\mathcal{R}}_{23}. \tag{7.5.33}$$

It can be proven directly from the relations and the coproduct formulas for the generators of $U_q\mathfrak{g}$. We refer to [17, Ch. 3] for details, but if tempted the reader might want to take it on as a useful exercise. As a hint towards the solution, it is helpful to prove the following relation

$$[E, F^{r+1}] = [r+1]_q \frac{q^r t - q^{-r} t^{-1}}{q - q^{-1}} F^r, \tag{7.5.34}$$

and the following formula for the coproduct

$$\Delta(F^r) = \sum_{s=0}^{r} q^{s(s-r)} \binom{r}{s}_q F^{r-s} \otimes t^{s-r} F^s, \tag{7.5.35}$$

where for $r \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}$, we have

$$\binom{r}{s}_q := \begin{cases} \frac{[r]_q!}{[s]_q![r-s]_q!} & \text{if } 0 \leq s \leq r, \\ 0 & \text{otherwise.} \end{cases} \tag{7.5.36}$$

Note, in the formal limit $q \to 1$, both κ and $\widetilde{\mathcal{R}}$, and hence also $\mathcal{R} = \kappa\widetilde{\mathcal{R}}$, go to $1 \otimes 1$.

A large range of matrix solutions to the Yang–Baxter equation (7.3.26) in representations of $U_q(\mathfrak{sl}_2)$ now arises naturally. Recall the n-dimensional representation $(\pi^{(n)}, V^{(n)})$ of $U_q(\mathfrak{sl}_2)$ defined in (7.4.10). By evaluating $(\pi^{(m)} \otimes \pi^{(n)})(\mathcal{R})$ for various m, n, we obtain linear maps on $V^{(m)} \otimes V^{(n)}$ which satisfy (7.3.26) in $V^{(l)} \otimes V^{(m)} \otimes V^{(n)}$ for various l, m, n.

To make this explicit as well, with respect to the basis $(v_1^{(2)}, v_2^{(2)})$, the 2-dimensional representation $\pi^{(2)}$ can be written as

$$\pi^{(2)}(E) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; \pi^{(2)}(F) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; \pi^{(2)}(t) = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}. \tag{7.5.37}$$

With respect to the basis $(v_1^{(2)} \otimes v_1^{(2)}, v_1^{(2)} \otimes v_2^{(2)}, v_2^{(2)} \otimes v_1^{(2)}, v_2^{(2)} \otimes v_2^{(2)})$ of $V^{(2)} \otimes V^{(2)}$, we obtain

$$\begin{aligned}(\pi^{(2)} \otimes \pi^{(2)})(\kappa) &= \begin{pmatrix} q^{1/2} & 0 & 0 & 0 \\ 0 & q^{-1/2} & 0 & 0 \\ 0 & 0 & q^{-1/2} & 0 \\ 0 & 0 & 0 & q^{1/2} \end{pmatrix}, \\ (\pi^{(2)} \otimes \pi^{(2)})(\widetilde{\mathcal{R}}) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\end{aligned} \tag{7.5.38}$$

and hence the following nontrivial solution of the Yang–Baxter equation:

$$R := (\pi^{(2)} \otimes \pi^{(2)})(\mathcal{R}) = q^{-1/2} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}. \tag{7.5.39}$$

7.5.11 The Dual Quantum Group $F_q(\mathsf{SL}(2))$

We mention here also the dual object $F_q(\mathsf{SL}(2))$, the quantized algebra of scalar-valued functions on $\mathsf{SL}(2)$. We refer to [7, Ch. 7] for a more in-depth discussion. The algebra $F_q(\mathsf{SL}(2))$ is generated over $k(q)$ by elements a,b,c,d subject to

$$\begin{gathered} ab = qba, \quad bd = qdb, \quad ac = qca, \quad cd = qdc, \quad bc = cb, \\ ad - qbc = 1 = da - q^{-1}cb. \end{gathered} \tag{7.5.40}$$

The Hopf algebra structure on $F_q(\mathsf{SL}(2))$ is as follows:

$$\begin{aligned} \Delta(a) &= a \otimes a + b \otimes c, & \varepsilon(a) &= 1, & S(a) &= d, \\ \Delta(b) &= a \otimes b + b \otimes d, & \varepsilon(b) &= 0, & S(b) &= -q^{-1}b, \\ \Delta(c) &= c \otimes a + d \otimes c, & \varepsilon(c) &= 0, & S(c) &= -qc, \\ \Delta(d) &= c \otimes b + d \otimes d, & \varepsilon(d) &= 1, & S(d) &= a. \end{aligned} \tag{7.5.41}$$

As $q \to 1$, we formally recover the commutative algebra of functions on $\mathsf{SL}(2)$, where a corresponds to the function returning the $(1,1)$-entry, b to the function returning the $(1,2)$-entry, etc., with the standard Hopf algebra structure on k-valued functions on $\mathsf{SL}(2)$, given by (7.2.20).

Since the generators correspond to matrix entries, it is natural to form the matrix

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{End}(k(q)^2) \otimes F_q(\mathsf{SL}(2)). \tag{7.5.42}$$

Then the Hopf algebra structure maps are given simply as

$$\Delta(T) = T \otimes T, \quad \varepsilon(T) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S(T) = T^{-1}, \tag{7.5.43}$$

where the $\otimes$ in $T \otimes T$ means: use ordinary matrix multiplication and take tensor products at the level of the matrix entries. Also, recall the matrix $R \in \mathrm{End}(k(q)^2 \otimes k(q)^2)$ from (7.5.39); then the *RTT-relation*

$$R_{12}T_{13}T_{23} = T_{23}T_{13}R_{12} \in \mathrm{End}(k(q)^2 \otimes k(q)^2) \otimes F_q(\mathsf{SL}(2)), \tag{7.5.44}$$

together with the *q-determinant* condition $ad - qbc = 1$, is equivalent to the relations (7.5.40). Here we see a nice aspect of duality at play: the object $\mathcal{R}$ controls the failure of cocommutativity for the algebra $U_q\mathfrak{sl}_2$ and the failure of commutativity for the algebra $F_q(\mathsf{SL}(2))$.

7.6 Coideal Subalgebras and Cylinder Braiding

The material in this supplementary section deals with more recent developments in the field of braided monoidal categories with cylinder twists [1, 2, 16, 43], for which the algebraic counterpart is a coideal subalgebra of the quasitriangular bialgebra, which is itself endowed with a quasitriangular structure. Various aspects of quantum group theory have been extended to this setting such as q-deformed harmonic analysis [29, 33, 34] and canonical bases and q-analogues of Schur–Weyl duality [3, 14].

7.6.1 Coideal Subalgebras

First, we return to the setting where k is any field. Let A be a bialgebra and $B \subseteq A$ a subalgebra. We call B a *two-sided coideal subalgebra*, *right coideal subalgebra* or *left coideal subalgebra* if

$$\begin{gathered} \Delta(B) \subseteq B \otimes A + A \otimes B, \\ \Delta(B) \subseteq B \otimes A, \qquad\qquad \Delta(B) \subseteq A \otimes B, \end{gathered} \tag{7.6.1}$$

respectively. Given a bialgebra A, all subbialgebras of A are right and left coideal subalgebras of A. Also, all right coideal subalgebras of A and all left coideal subalgebras of A are two-sided coideal subalgebras. If A is cocommutative then these four concepts coincide.

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Example 7.24 Consider a bialgebra A, an element $g \in \mathrm{Gr}(A)$ and an element $b \in \mathsf{Pri}_{g,1}(A)$. Consider the subalgebra B generated by b (its elements are polynomial expressions in b). Then B is a right coideal, since $\Delta(b) = b \otimes g + 1 \otimes b \in B \otimes A$. Also, B is graded by the degree in b and $B \otimes B$ is graded by the sum of the degrees. Using this it is straightforward to see that $B \cap \mathrm{Gr}(A) = \{1\}$, so that B is not a subbialgebra unless $g = 1$. The assumptions on A are indeed met if, for instance, A is equal to Sweedler's Hopf algebra or $U_q\mathfrak{sl}_2$.

7.6.2 Cylinder Braiding

We saw in Section 7.3.3 that the monoidal category $\mathsf{Rep}(A)$, if A is a quasitriangular bialgebra, possesses a braided structure. In particular, there is an action of the braid group Br_L on $V^{\otimes L}$, for any $V \in \mathsf{Rep}(A)$, given by (7.3.29). Let us adjoin a generator b_0 to Br_L to obtain a larger group Br_L^0, subject to the relations

$$b_0 b_1 b_0 b_1 = b_1 b_0 b_1 b_0, \qquad b_0 b_i = b_i b_0 \text{ if } i > 1. \tag{7.6.2}$$

This is known as the Artin–Tits braid group of type B_L (the subgroup Br_L is the Artin–Tits braid group of type A_{L-1}) and is the fundamental group of the L-th unordered configuration space of the *punctured* disk. Given the representation of Br_L in terms of $\check{R}_{V,V}$, see (7.3.29), it is natural to require that b_0 acts as follows:

$$b_0 \mapsto K_V \otimes \mathsf{id}_V^{\otimes(L-1)} \tag{7.6.3}$$

for some invertible $K_V \in \mathrm{End}(V)$ since then automatically the relations $b_0 b_i = b_i b_0$ for $i > 1$ are preserved. In order to preserve the quartic relation, we must have

$$(K_V \otimes \mathsf{id}_V)\check{R}_{V,V}(K_V \otimes \mathsf{id}_V)\check{R}_{V,V} = \check{R}_{V,V}(K_V \otimes \mathsf{id}_V)\check{R}_{V,V}(K_V \otimes \mathsf{id}_V),$$

or, equivalently,

$$\begin{aligned}(K_V \otimes \mathsf{id}_V)\sigma(R_{V,V})(\mathsf{id}_V \otimes K_V)R_{V,V} \\ = \sigma(R_{V,V})(\mathsf{id}_V \otimes K_V)R_{V,V}(K_V \otimes \mathsf{id}_V),\end{aligned} \tag{7.6.4}$$

which is also known as the *(constant) reflection equation*.

Remark The study of this tensorial version of the quartic braid relation originated in mathematical physics, more precisely quantum integrability in the presence of a boundary, see [8, 40] and for a more general type of reflection equation cf. [9]. Note, however, that in this application the objects $R_{V,V}$ and K_V depend on an additional parameter, called the *spectral parameter*, which varies in the equation:

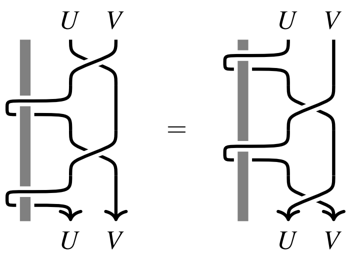
$$
\begin{aligned}
&(K_V(y) \otimes \mathrm{id}_V)\sigma(R_{V,V}(yz))(\mathrm{id}_V \otimes K_V(z))R_{V,V}(z/y) \\
&\quad = \sigma(R_{V,V}(z/y))(\mathrm{id}_V \otimes K_V(z))R_{V,V}(yz)(K_V(y) \otimes \mathrm{id}_V).
\end{aligned}
\tag{7.6.5}
$$

This spectral parameter roughly corresponds to the loop parameter appearing in the definition of a loop algebra.

It is now natural to ask what the additional structure on the braided monoidal category $\mathsf{Rep}(A)$ is, which allows for the action of Br_L to extend to an action of Br_L^0. Because of the embedding $\mathsf{Br}_L < \mathsf{Br}_L^0$, in terms of the graphical calculus from Section 7.3.3 topologically we are adding an obstacle, requiring us to interpret the generator b_0 as the interaction of one of the L strands with the obstacle. This extended calculus was given in [43] and discussed further in [16]. Let us indicate the obstacle by a vertical grey bar to the left of the strands and parallel to their direction of travel and the linear map K_V by a winding around this obstacle:

(7.6.6)

The pictorial version of (7.6.4) (or rather its generalization to the case where the two modules are distinct) is as follows:



(7.6.7)

Let us identify further natural conditions on the objects K_V. Given an A-module V, let $\pi_V : A \to \mathrm{End}(V)$ be the corresponding algebra morphism. Let us take inspiration from the situation for $\check{R}_{U,V}$. By construction, K_V is an intertwiner for the subalgebra

$$
B = \{b \in A \,|\, K_V \pi_V(b) = \pi_V(b) K_V\},
\tag{7.6.8}
$$

which typically will be a proper subalgebra of A. Note furthermore that, as a consequence of the intertwining property of $\check{R}^{V,V}$, the action of $\Delta^{(L)}(A)$ and the action of Br_L mutually commute in $\mathrm{End}(V^{\otimes L})$, for all $L \in \mathbb{Z}_{>0}$. It is natural to impose that the action of $\Delta^{(L)}(B)$ and the action of Br_L^0 mutually commute for all $L \in \mathbb{Z}_{>0}$, since it is already satisfied for $L = 1$. The most general assumption

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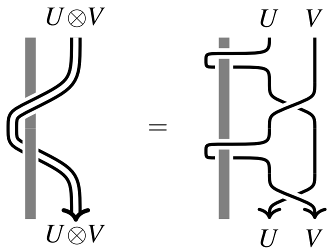
on B that allows us to perform recursion and obtain the mutual commutativity for all $L \in \mathbb{Z}_{>0}$ is that $\Delta(b) \in B \otimes A$ for all $b \in B$, in other words that B is a right coideal subalgebra; in particular, we can define B to be the largest right coideal subalgebra of the algebra defined in (7.6.8).

We also need a rule for assigning a value to $K_{U\otimes V}$. A topologically natural condition is for instance given by

$$K_{U\otimes V} = \check{R}_{U,V}^{-1}(K_V \otimes \mathsf{id}_U)\check{R}_{U,V}(K_U \otimes \mathsf{id}_V) \tag{7.6.9}$$

or, diagrammatically,



(7.6.10)

Such a collection of linear maps $(K_V)_{V\in\mathsf{Rep}(A)}$ is called a *cylinder twist* on the category $\mathsf{Rep}(A)$.

7.6.3 Cylindrical Quasitriangularity

We now consider an additional structure on a quasitriangular bialgebra $(A,\mathcal{R})$ such that its category of A-modules is braided with a cylinder twist. We allow a generalization, which appeared in [2], namely twisting by a quasitriangular automorphism. Given a quasitriangular bialgebra $(A,\mathcal{R})$ with a quasitriangular automorphism ψ, we denote $\mathcal{R}^{\psi} := (\psi \otimes \mathsf{id})(\mathcal{R})$. Recall that $\mathcal{R}^{\psi\psi} := (\psi \otimes \psi)(\mathcal{R})$ equals $\mathcal{R}$.

Definition 7.25 ([2, 43]) Let $(A,\mathcal{R})$ be a quasitriangular bialgebra, ψ a quasitriangular automorphism of A and $\mathcal{K} \in A^{\times}$. We call $(A,\mathcal{R},\psi,\mathcal{K})$ *cylindrically quasitriangular* and $\mathcal{K}$ a *ψ-twisted universal K-matrix* for A if

$$\Delta(\mathcal{K}) = \mathcal{R}^{-1}(1\otimes\mathcal{K})\mathcal{R}^{\psi}(\mathcal{K}\otimes 1). \tag{7.6.11}$$

Furthermore, we call a subalgebra $B \subseteq A$ *ψ-cylindrically invariant* if

$$\mathsf{Ad}(\mathcal{K})|_B = \psi|_B. \tag{7.6.12}$$

Theorem 7.26 *If $(A,\mathcal{R},\psi,\mathcal{K})$ is cylindrically quasitriangular then $\mathcal{K}$ satisfies the (universal) ψ-twisted reflection equation:*

$$(\mathcal{K}\otimes 1)\sigma(\mathcal{R}^{\psi})(1\otimes\mathcal{K})\mathcal{R} = \sigma(\mathcal{R})(1\otimes\mathcal{K})\mathcal{R}^{\psi}(\mathcal{K}\otimes 1). \tag{7.6.13}$$

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Proof This follows from (7.3.1) and (7.6.11):

$$\begin{aligned}(\mathcal{K}\otimes 1)\sigma(\mathcal{R}^{\psi})(1\otimes\mathcal{K})\mathcal{R} &= \sigma(\mathcal{R})\Delta^{\mathsf{op}}(\mathcal{K})\mathcal{R}\\ &= \sigma(\mathcal{R})\mathcal{R}\Delta(\mathcal{K})\\ &= \sigma(\mathcal{R})(1\otimes\mathcal{K})\mathcal{R}^{\psi}(\mathcal{K}\otimes 1)\\ &= \sigma(\mathcal{R})(1\otimes\mathcal{K})\mathcal{R}^{\psi}(\mathcal{K}\otimes 1).\end{aligned}$$ □

7.6.4 The Construction of a Universal K-matrix for $U_q\mathfrak{sl}_2$

Given a finite-dimensional semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ and an involutive Lie algebra automorphism $\theta : \mathfrak{g}\to\mathfrak{g}$, Letzter gave a general theory of the quantization of symmetric pairs $(\mathfrak{g},\mathfrak{s})$, where $\mathfrak{s}=\mathfrak{g}^{\theta}$ is the fixed-point Lie subalgebra, see [29] and cf. [33, 34]. In this theory $U_q\mathfrak{g}$ is the usual Drinfeld–Jimbo quantum group and $U_q\mathfrak{s}$ is a coideal subalgebra of $U_q\mathfrak{g}$. In [2] a construction is given of a universal K-matrix associated to such a pair $(U_q\mathfrak{g},U_q\mathfrak{s})$, generalizing a construction first given in [3]. This can be seen as a coideal version of the construction of the universal R-matrix due to Lusztig [30].

Here we follow the approach of [1], which gives a further generalization of the above construction. We discuss this by illustrating a special case, which can be seen in parallel to the discussion of the universal R-matrix for $U_q\mathfrak{sl}_2$ in Section 7.5.10. Essentially the same object appeared previously in a more ad-hoc setting in [28, 43].

Recall the Hopf algebra $U_q\mathfrak{sl}_2$ with generators E,F,t,t^{-1} subject to relations (7.4.4) and with the additional structure maps given by (7.4.5). For $\gamma\in k(q^{1/d})^{\times}$, let $H_\gamma\in\widehat{U}$ be the element acting on objects V of $\mathcal{O}_{q,\mathsf{int}}$ as

$$H_\gamma(v)=q^{-(\mu,\mu)/2}\chi_\gamma(\mu)^{-1}v,\qquad v\in V_\mu,\,\mu\in P, \tag{7.6.14}$$

where $\chi_\gamma : P\to k(q^{1/d})$ is any group homomorphism with the property $\chi_\gamma(\alpha)=\gamma$. Then the algebra automorphism $\theta_\gamma=\omega\circ\mathsf{Ad}(H_\gamma)$ of $U_q\mathfrak{sl}_2$ satisfies

$$\theta_\gamma(E)=-q\gamma^{-1}tF,\quad \theta_\gamma(F)=-q^{-1}\gamma Et^{-1},\quad \theta_\gamma(t^{\pm1})=t^{\mp1}. \tag{7.6.15}$$

By a direct check, we see that the subalgebra B_γ of $U_q\mathfrak{sl}_2$ generated by $F+\theta_\gamma(F)=F-q^{-1}\gamma Et^{-1}$ is a right coideal subalgebra. It is the Letzter quantization of the fixed-point subalgebra of the Chevalley involution of $\mathfrak{sl}_2$ with an extra parameter dependence. We are interested in an element $\mathcal{K}\in\widehat{U}$ satisfying the coproduct formula (7.6.11) and the intertwining property (7.6.12) with $\psi=\mathsf{id}$.

There exists (see [22]) an element $\mathcal{T}\in\widehat{U}$ which resolves the R-matrix in the sense that $\mathcal{R}=(\mathcal{T}^{-1}\otimes\mathcal{T}^{-1})\Delta(\mathcal{T})$ and satisfies $\mathsf{Ad}(\mathcal{T})=\omega$; such an element can be constructed as a Cartan modification of an element $\widetilde{\mathcal{T}}\in\widehat{U}$ which satisfies

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$\widetilde{\mathcal{R}} = (\widetilde{\mathcal{T}}^{-1} \otimes \widetilde{\mathcal{T}}^{-1})\Delta(\widetilde{\mathcal{T}})$ (see [30, 5.2.1]). Hence it suffices to construct an element $\mathcal{K}' = \mathcal{T}\mathcal{K}$ such that

$$\begin{aligned} \mathsf{Ad}(\mathcal{K}')|_{B_\gamma} &= \omega|_{B_\gamma}, \\ \Delta(\mathcal{K}') &= (1 \otimes \mathcal{K}')(\omega \otimes \mathsf{id})(\mathcal{R})(\mathcal{K}' \otimes 1) \end{aligned} \tag{7.6.16}$$

(note that the coproduct formula is now an expression of three factors). An algebra morphism $f : B_\gamma \to \overline{B_{\overline{\gamma}}}$ is uniquely defined by

$$f(F - q^{-1}\gamma E t^{-1}) = \overline{F - q^{-1}\overline{\gamma} E t^{-1}} = F - q\gamma E t. \tag{7.6.17}$$

It can be checked by a direct computation that

$$\mathsf{Ad}(H_\gamma) \circ f|_{B_\gamma} = \omega|_{B_\gamma}, \qquad \Delta(H_\gamma) = H_\gamma \otimes H_\gamma \kappa^{-1}, \tag{7.6.18}$$

in terms of the element κ defined by (7.5.8). Now from (7.6.16) and (7.6.18) one deduces that it suffices to construct an element $\widetilde{K} = H_\gamma^{-1} K'$ such that

$$\begin{aligned} &\mathsf{Ad}(\widetilde{\mathcal{K}})|_{B_\gamma} = f|_{B_\gamma}, \\ &\Delta(\widetilde{\mathcal{K}}) = \mathsf{Ad}(\kappa)(1 \otimes \widetilde{\mathcal{K}}) \cdot (\mathsf{Ad}(H_\gamma^{-1})\omega \otimes \mathsf{id})(\widetilde{\mathcal{R}}) \cdot (\widetilde{\mathcal{K}} \otimes \mathsf{id}). \end{aligned} \tag{7.6.19}$$

In [2, Sec. 6] and, without constraints on γ, in [1, Sec. 7] it is shown that the element $\widetilde{\mathcal{K}}$ exists in $\widehat{U}^+$ and is unique if we impose $\varepsilon(\widetilde{\mathcal{K}}) = 1$. We can give an explicit formula.

Lemma 7.27 ([11]) *The system* (7.6.19) *is satisfied by the following element* $\widetilde{\mathcal{K}} \in \widehat{U}^+$:

$$\widetilde{\mathcal{K}} = \sum_{r \in \mathbb{Z}_{\geq 0}} q^{2r(r-1)} [2r]_q!! \big(\gamma(q - q^{-1})E^2\big)^r \tag{7.6.20}$$

where $[2r]_q!! := [2r]_q[2(r-1)]_q \cdots [2]_q[0]_q$.

We obtain that the element $\mathcal{K} = \mathcal{T}^{-1} H_\gamma \widetilde{\mathcal{K}}$ satisfies (7.6.12) and (7.6.11) and hence the reflection equation (7.6.13) (all with $\psi = \mathsf{id}$).

The representations $\pi^{(n)}$ are objects in category $\mathcal{O}_{q,\mathsf{int}}$ and hence we can evaluate $\mathcal{K}$ for instance by applying $\pi^{(2)}$, obtaining an element of $\mathrm{End}(V^{(2)})$. We recall the matrices defined in (7.5.37). The action on $V^{(2)}$ of the three constituent factors of $\mathcal{K}$ is as follows:

$$\begin{aligned} \pi^{(2)}(\mathcal{T}^{-1}) &= q^{-3/4} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\ \pi^{(2)}(H_\gamma) &= q^{3/4} \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix}, \\ \pi^{(2)}(\widetilde{\mathcal{K}}) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{7.6.21}$$

There is an indeterminacy in H_γ (more precisely, in χ_γ) corresponding to the choice of an overall scalar. We choose the scalar so that H_γ acts in the way above. It follows that

$$K := \pi^{(2)}(\mathcal{K}) = \begin{pmatrix} 0 & -\gamma \\ 1 & 0 \end{pmatrix} \tag{7.6.22}$$

is a solution to the matrix reflection equation

$$(K \otimes \mathsf{id}_{V(2)})\check{R}(K \otimes \mathsf{id}_{V(2)})\check{R} = \check{R}(K \otimes \mathsf{id}_{V(2)})\check{R}(K \otimes \mathsf{id}_{V(2)}), \tag{7.6.23}$$

where $\check{R} = \sigma \circ R$ with R given by (7.5.39).

7.6.5 Cylinder Quasitriangularity Twisted by an Algebra Automorphism

The setup outlined in Section 7.6.3 can be generalized as follows, which allows us to apply it in the context of Drinfeld–Jimbo quantum groups of non-finite type, see [1]. It can be directly motivated from the setup in Definition 7.25 by noting that also the coproduct formula $\Delta(\mathcal{K}) = \sigma(\mathcal{R})(1 \otimes \mathcal{K})\mathcal{R}^\psi(\mathcal{K} \otimes 1)$, which is the one actually used in [2, 25], also leads to the reflection equation (7.6.13), and so a more general ansatz $\Delta(\mathcal{K}) = \mathcal{F}^{-1}(1 \otimes \mathcal{K})\mathcal{R}^\psi(\mathcal{K} \otimes 1)$ with $\mathcal{F} \in (A \otimes A)^\times$ is tempting.

To make it precise, assume that $\psi : A \to A$ is merely an algebra automorphism (so not necessarily a quasitriangular or even a bialgebra automorphism) and recall the notations $\mathcal{R}^\psi := (\psi \otimes \mathsf{id})(\mathcal{R})$, $\mathcal{R}^{\psi\psi} := (\psi \otimes \psi)(\mathcal{R})$. We can generalize Definition 7.25 and call $(A, \mathcal{R}, \psi, \mathcal{K})$ *cylindrically quasitriangular* and $\mathcal{K}$ a *ψ-twisted universal K-matrix* for A if there exists $\mathcal{F} \in (A \otimes A)^\times$ such that

$$\Delta(\mathcal{K}) = \mathcal{F}^{-1}(1 \otimes \mathcal{K})\mathcal{R}^\psi(\mathcal{K} \otimes 1), \tag{7.6.24}$$

$$\sigma(\mathcal{R}^{\psi\psi}) = \sigma(\mathcal{F})\mathcal{R}\mathcal{F}^{-1}. \tag{7.6.25}$$

(The notion of a ψ-cylindrically invariant subalgebra remains the same as in Definition 7.25.)

Note that the element $\mathcal{F}$ appearing in (7.6.11) has to satisfy certain constraints. By applying coassociativity (7.2.8) and the counit axiom (7.2.9), we obtain

$$\begin{aligned} &\big(\mathcal{F}_{12}(\Delta \otimes \mathsf{id})(\mathcal{F})\big)^{-1}(1 \otimes 1 \otimes \mathcal{K})\big((\mathsf{Ad}(\mathcal{F}) \circ \Delta \circ \psi) \otimes \mathsf{id}\big)(\mathcal{R}) \\ &= \big(\mathcal{F}_{23}(\mathsf{id} \otimes \Delta)(\mathcal{F})\big)^{-1}(1 \otimes 1 \otimes \mathcal{K})\big(((\psi \otimes \psi) \circ \Delta^{\mathsf{op}}) \otimes \mathsf{id}\big)(\mathcal{R}), \\ &\qquad \varepsilon(\mathcal{K}) = (\mathsf{id} \otimes \varepsilon)(\mathcal{F}), \\ &\mathcal{K}\big((\varepsilon \circ \psi) \otimes \mathsf{id}\big)(\mathcal{R})\varepsilon(\mathcal{K}) = (\varepsilon \otimes \mathsf{id})(\mathcal{F})\mathcal{K}, \end{aligned}$$

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which is most easily satisfied by making two assumptions. First of all, recalling (7.3.8), we assume that $\mathcal{F}$ satisfies the cocycle condition

$$\mathcal{F}_{12}(\Delta \otimes \mathsf{id})(\mathcal{F}) = \mathcal{F}_{23}(\mathsf{id} \otimes \Delta)(\mathcal{F}), \tag{7.6.26}$$

so that by (7.2.9) we obtain

$$(\mathsf{id} \otimes \varepsilon)(\mathcal{F}) = (\varepsilon \otimes \mathsf{id})(\mathcal{F}) = \varepsilon(\mathcal{K}) \in k^\times, \tag{7.6.27}$$

which we may as well set equal to 1 (by rescaling $\mathcal{F}$ and $\mathcal{K}$ in a compatible manner). Such $\mathcal{F}$ are called *Drinfeld twists* (for bialgebras), see [13, 35]. Secondly, we assume that ψ and $\mathcal{F}$ twist the bialgebra structure on A in related ways:

$$(\psi \otimes \psi) \circ \Delta^{\mathsf{op}} \circ \psi^{-1} = \mathsf{Ad}(\mathcal{F}) \circ \Delta, \qquad \varepsilon \circ \psi^{-1} = \varepsilon. \tag{7.6.28}$$

This extends the assumption made on $\mathcal{F}$ in (7.6.25), yielding related twists of quasitriangular structures on A.

The proof of Theorem 7.26 now produces the following generalization of the ψ-twisted reflection equation:

$$(\mathcal{K} \otimes 1)\sigma(\mathcal{R}^\psi)(1 \otimes \mathcal{K})\mathcal{R} = \sigma(\mathcal{R}^{\psi\psi})(1 \otimes \mathcal{K})\mathcal{R}^\psi(\mathcal{K} \otimes 1). \tag{7.6.29}$$

It is argued in [1, Sec. 9] that, if $A = U_q(\mathfrak{g})$ with $\mathfrak{g}$ a Kac–Moody algebra of affine type then, for suitable choices of ψ, the image of (7.6.29) in finite-dimensional modules recovers the matrix equation (7.6.5), or rather a generalized version of it, see [9, Eq. (4.15)].

Bibliography

[1] Appel, A., and Vlaar, B. 2020. Universal k-matrices for quantum Kac-Moody algebras. *arXiv preprint arXiv:2007.09218*.

[2] Balagović, M., and Kolb, S. 2019. Universal K-matrix for quantum symmetric pairs. *Journal für die reine und angewandte Mathematik*, **2019**(747), 299–353.

[3] Bao, H, and Wang, W. 2018. *A new approach to Kazhdan-Lusztig theory of type B via quantum symmetric pairs*. Société mathématique de France Paris.

[4] Bernstein, I N, Gelfand, I M, and Gelfand, S I. 1975. Differential operators on the base affine space and a study of $\mathfrak{g}$-modules. *Lie groups and their representations (Proc. Summer School, Bolyai János Math. Soc., Budapest, 1971)*, 21–64.

[5] Bourbaki, N. 1994. Lie groups and Lie algebras. Pages 247–267 of: *Elements of the History of Mathematics*. Springer.

[6] Chari, V, and Pressley, A. 1991. Quantum affine algebras. *Communications in Mathematical Physics*, **142**(2), 261–283.

[7] Chari, V, and Pressley, A. 1995. *A guide to quantum groups*. Cambridge University Press.

[8] Cherednik, I. 1984. Factorizing particles on a half-line and root systems. *Theoretical and Mathematical Physics*, **61**(1), 977–983.

[9] Cherednik, I. 1992. Quantum Knizhnik-Zamolodchikov equations and affine root systems. *Communications in Mathematical Physics*, **150**(1), 109–136.

[10] Dascalescu, S, Nastasescu, C, and Raianu, S. 2001. Hopf Algebras: an Introduction. *Marcel Decker Inc., New York*.

[11] Dobson, L, and Kolb, S. 2019. Factorisation of quasi K-matrices for quantum symmetric pairs. *Selecta Mathematica*, **25**(4), 1–55.

[12] Drinfeld, V G. 1985. Hopf algebras and the quantum Yang-Baxter equation. *Soviet Math. Dokl.*, **32**(6), 254–258.

[13] Drinfeld, V G. 1987. Quantum Groups. Pages 798–820 of: Gleason, A M (ed), *Proceedings of the International Congress of Mathematicians*. American Mathematical Society.

[14] Ehrig, M, and Stroppel, C. 2018. Nazarov–Wenzl algebras, coideal subalgebras and categorified skew Howe duality. *Advances in Mathematics*, **331**, 58–142.

[15] Frenkel, I B, and Reshetikhin, N Yu. 1992. Quantum affine algebras and holonomic difference equations. *Communications in Mathematical Physics*, **146**(1), 1–60.

[16] Häring-Oldenburg, R. 2001. Actions of tensor categories, cylinder braids and their Kauffman polynomial. *Topology and its Applications*, **112**(3), 297–314.

[17] Jantzen, J C. 1996. *Lectures on quantum groups*. Vol. 6. American Mathematical Soc.

[18] Jimbo, M. 1986. A q-analogue of $U(\mathfrak{gl}(N+1))$, Hecke algebra, and the Yang-Baxter equation. *Letters in Mathematical Physics*, **11**(3), 247–252.

[19] Jimbo, M, and Miwa, T. 1994. *Algebraic analysis of solvable lattice models*. Vol. 85. Amer. Math. Soc.

[20] Joyal, A, and Street, R. 1993. Braided tensor categories. *Advances in Mathematics*, **102**(1), 20–78.

[21] Kac, V G. 1990. *Infinite-dimensional Lie algebras*. Cambridge University Press.

[22] Kamnitzer, J, and Tingley, P. 2009. The crystal commutor and Drinfeld's unitarized R-matrix. *Journal of Algebraic Combinatorics*, **29**(3), 315–335.

[23] Kassel, C. 1995. Quantum Groups. *Grad. Texts in Math.*

[24] Klimyk, A, and Schmüdgen, K. 1997. *Quantum Groups and their Representations (1997)*. Springer Texts and Monographs in Physics.

[25] Kolb, S. 2020. Braided module categories via quantum symmetric pairs. *Proceedings of the London Mathematical Society*, **121**(1), 1–31.

[26] Kostant, B. 1977. Graded manifolds, graded Lie theory, and prequantization. Pages 177–306 of: *Differential Geometrical Methods in Mathematical Physics*. Springer.

[27] Kulish, P P, and Reshetikhin, N Yu. 1983. Quantum linear problem for the sine-Gordon equation and higher representations. *Journal of Soviet Mathematics*, **23**(4), 2435–2441.

[28] Kulish, P P, Sasaki, R, and Schwiebert, C. 1993. Constant solutions of reflection equations and quantum groups. *Journal of Mathematical Physics*, **34**(1), 286–304.

[29] Letzter, G. 1999. Symmetric pairs for quantized enveloping algebras. *Journal of Algebra*, **220**(2), 729–767.

[30] Lusztig, G. 1994. *Introduction to quantum groups*. Birkhäuser, Boston.

[31] Majid, S. 1988. Hopf algebras for physics at the Planck scale. *Classical and Quantum Gravity*, **5**(12), 1587.
[32] Majid, S. 2002. *A quantum groups primer*. Cambridge University Press.
[33] Noumi, M, and Sugitani, T. 1995. Quantum symmetric spaces and related q-orthogonal polynomials. Pages 28–40 of: Arima, A *et al*. (ed), *Group theoretical methods in physics*. World Scientific.
[34] Noumi, M, Dijkhuizen, M S, and Sugitani, T. 1997. Multivariable Askey-Wilson polynomials and quantum complex Grassmannians. *AMS Field Inst. Commun.*, **14**, 167–177.
[35] Reshetikhin, N. 1990. Multiparameter quantum groups and twisted quasitriangular Hopf algebras. *Letters in Mathematical Physics*, **20**(4), 331–335.
[36] Reshetikhin, N. 2010. Lectures on the integrability of the six-vertex model. *Exact methods in low-dimensional statistical physics and quantum computing*, 197–266.
[37] Reshetikhin, N Yu, and Turaev, V G. 1990. Ribbon graphs and their invariants derived from quantum groups. *Communications in Mathematical Physics*, **127**(1), 1–26.
[38] Saito, Y. 1994. PBW basis of quantized universal enveloping algebras. *Publications of the Research Institute for Mathematical Sciences*, **30**(2), 209–232.
[39] Schneider, H-J. 1995. Lectures on Hopf algebras. *Trabajos de Matemática*, **31**, 95.
[40] Sklyanin, E K. 1988. Boundary conditions for integrable quantum systems. *Journal of Physics A: Mathematical and General*, **21**(10), 2375.
[41] Sweedler, M E. 1969. *Hopf algebras*. Benjamin, New York.
[42] Tanisaki, T. 1992. Killing forms, Harish-Chandra isomorphisms, and universal R-matrices for quantum algebras. *International Journal of Modern Physics A*, **7**(supp01b), 941–961.
[43] tom Dieck, T, and Häring-Oldenburg, R. 1998. Quantum groups and cylinder braiding. Pages 619–639 of: *Forum Math.*, vol. 10.