Canad. J. Math. Vol. **54** (3), 2002 pp. 468–492

Mahler's Measure and the Dilogarithm (I)

David W. Boyd and Fernando Rodriguez-Villegas

Abstract. An explicit formula is derived for the logarithmic Mahler measure $m(P)$ of $P(x, y) = p(x)y - q(x)$, where $p(x)$ and $q(x)$ are cyclotomic. This is used to find many examples of such polynomials for which $m(P)$ is rationally related to the Dedekind zeta value $\zeta_F(2)$ for certain quadratic and quartic fields.

Introduction

The *logarithmic Mahler measure* of a non-zero Laurent polynomial $P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ is defined as

$$m(P) = \int_0^1 \cdots \int_0^1 \log |P(e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_n})| \, d\theta_1 \cdots d\theta_n \tag{1}$$

and its *Mahler measure* as $M(P) = e^{m(P)}$, the geometric mean of $|P|$ on the torus

$$T^n = \left\{ (z_1, \dots, z_n) \in \mathbb{C}^n \;\middle|\; |z_1| = \cdots = |z_n| = 1 \right\}.$$

In the early 80's Smyth [Sm] proved that

$$m(x + y + 1) = L'(\chi, -1) \tag{2}$$

where χ is the Dirichlet character associated with the field $\mathbb{Q}(\sqrt{-3})$.

In this paper we will consider the Mahler measure of polynomials of the form $P(x, y) = p(x)y - q(x)$. We will show that for appropriate choices of p and q, $m(P)$ can be expressed in terms of the values of Bloch-Wigner dilogarithm function at certain algebraic arguments; using a theorem of Borel we will then find that $m(P)$ is related to special values of certain Artin L-functions with Smyth's result (2) being the prototypical example.

We begin by proving the basic formula (8) of Proposition 1 for $m(P) = m\big(p(x)y - q(x)\big)$ where p and q are cyclotomic. It expresses the measure as a sum of Bloch-Wigner dilogarithms evaluated at various algebraic numbers lying on the unit circle. In Section 2, we apply the formula directly to compute $m(y + 1 + x + \cdots + x^n)$.

In Section 3 and Section 4 we summarize the basic theory of the Bloch group and show how it applies to the formula (8) in certain circumstances. This leads to our

Received by the editors October 24, 2000; revised November 5, 2001.

The first author was sponsored in part by a grant from NSERC. The second author was supported in part by grants from the NSF and TARP and by a Sloan Research Fellowship; he would also like to thank the Department of Mathematics at the Université de Bordeaux I for its hospitality while part of this research was conducted.

AMS subject classification: Primary: 11G40; secondary: 11R06, 11Y35.

Theorems 1 and 2 which give conditions under which $m(P)$ is rationally related to $\zeta_F(2)$ for a number field F with a single pair of complex embeddings. Of necessity, the fields to which these theorems apply are of even degree $2n$ and have Galois group a subgroup of the hyperoctahedral group $C_2 \wr S_n$, of order $2^n n!$.

A couple of examples involving more general fields are mentioned in the concluding Section 13. These are derived from the study of A-polynomials of certain hyperbolic manifolds.

Section 6 contains a general discussion of the restrictions on the rational function $r(x) = q(x)/p(x)$ imposed by the conditions of Theorems 1 or 2. These considerations are applied in Section 7–Section 11 to construct the examples summarized in Tables 1 and 2 in which the field F is either imaginary quadratic or quartic. It seems likely that there are only a finite number of different fields to which Theorems 1 and 2 apply, but we do construct an infinite number of distinct examples for some of these fields. For example, Proposition 5 gives a three parameter family of generalizations of Smyth's formula (2) relating $m(P)$ to $L'(\chi, -1)$.

In general, although we know from Section 3 and Section 4 that the ratios s appearing in Tables 1 and 2 are rational, the quantities indicated are those surmised from numerical calculation (to 50 decimal places). We would like to thank Don Zagier for several conversations and for supplying the proof in Section 12. We would also like to thank the referee for several excellent suggestions for improving the exposition.

1 Calculation of $m(P)$

We say a polynomial $p(x) \in \mathbb{Z}[x]$ is *cyclotomic* if it is a non-zero polynomial of the form $p(x) = \pm x^k p_0(x)$, where p_0 is monic and its roots consist of roots of unity only. By a theorem of Kronecker it is equivalent to require that $m(p) = 0$. Note that given any cyclotomic polynomial there exist unique integers k, c_m for $m \in \mathbb{N}$, with $c_m = 0$ except for finitely many m's, such that

$$p(x) = \pm x^k \prod_{m=1}^{\infty} (1 - x^m)^{c_m}. \tag{3}$$

We recall the definition of the *Bloch-Wigner dilogarithm*. Starting with the usual dilogarithm

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}, \quad |z| < 1$$

one defines

$$D(z) = \mathrm{Im}\big(\mathrm{Li}_2(z)\big) + \arg(1 - z) \log |z|$$

and checks that it extends to a real analytic function on $\mathbb{C} \setminus \{0, 1\}$, continuous on $\mathbb{C}$. See [Za1] for an account of its many wonderful properties. It is obvious that

$$D(\bar{z}) = -D(z) \tag{4}$$

and

$$D(e^{i\theta}) = -\int_0^\theta \log|1-e^{it}|\,dt, \tag{5}$$
$$= \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n^2}.$$

Proposition 1 *Let $P(x, y) = p(x)y - q(x)$ with p and q cyclotomic and relatively prime and let*

$$r(x) = \frac{q(x)}{p(x)} = \pm x^k \prod_{m=1}^{\infty} (1-x^m)^{c_m} \quad \text{with } k, c_m \in \mathbb{Z}, \tag{6}$$

with $c_m = 0$ except for finitely many m. Let $\alpha_1, \dots, \alpha_N \in \mathbb{C}$ be the different roots of the equation

$$r(x)r(x^{-1}) = 1, \tag{7}$$

with $|\alpha_k| = 1$ which have odd multiplicity, ordered counterclockwise around the unit circle. Then

$$m(P) = \frac{\epsilon}{2\pi} \sum_{n=1}^{N} (-1)^n \sum_{m=1}^{\infty} \frac{c_m}{m} D(\alpha_n^m), \tag{8}$$

for some $\epsilon = \pm 1$, where $D(z)$ is the Bloch-Wigner dilogarithm.

Proof The proof is essentially the same as Smyth's. Applying Jensen's formula in the variable y to the integral defining $m(P)$ we obtain

$$m(P) = \frac{1}{2\pi i} \int_{|x|=1} \log^+ |r(x)| \frac{dx}{x}, \tag{9}$$

where $\log^+ |z| = \log|z|$ if $|z| \geq 1$ and 0 otherwise.

It is not hard to see that as x moves counter clockwise on the circle $r(x)$ enters or leaves the unit circle precisely at the points $r(\alpha_1), \dots, r(\alpha_N)$ and in that order. Hence, up to possibly a global sign ϵ we have (note that N is even)

$$m(P) = \frac{\epsilon}{2\pi} \sum_{j=1}^{N/2} \int_{\gamma_j} \log|r(x)| \frac{dx}{ix},$$

where γ_j is the arc of circle from α_{2j-1} to α_{2j}. Now we can separate the factors that make up $r(x)$ and, ignoring the factor $\pm x^k$ which does not contribute, we are left to consider the integral

$$\int_{\gamma_j} \log|1-x^m| \frac{dx}{ix},$$

which is easily seen by changing variables in the integral in (5) to equal

$$-\frac{1}{m} D(\alpha_{2j}^m) + \frac{1}{m} D(\alpha_{2j-1}^m)$$

and the claim follows. ∎

https://doi.org/10.4153/CJM-2002-016-9 Published online by Cambridge University Press

2 An Example

For example, to compute $m(y+1+x+\cdots+x^n)$, we need the roots of $(1+x+\cdots+x^n)^2 = x^n$, an equation which simplifies to $(x^{n+2}-1)(x^n-1) = 0$, $x \neq 1$ (a fact pointed out in [MPV]). Let $R(x) = \frac{1}{n+1}D(x^{n+1}) - D(x)$, so if $\alpha^n = 1$ and $\beta^{n+2} = 1$ then $R(\alpha) = -\frac{n}{n+1}D(\alpha)$ and $R(\beta) = -\frac{n+2}{n+1}D(\beta)$. Since the roots of x^n-1 and $x^{n+2}-1$ interlace on the upper half of the unit circle, we obtain the following formula for n, writing $\alpha = \exp(2\pi i/n)$ and $\beta = \exp\big(2\pi i/(n+2)\big)$:

$$(10) \quad m(y+1+x+\cdots+x^n) = \frac{1}{(n+1)\pi}\Big(-(n+2)\sum_{k=1}^{\frac{n+1}{2}} D(\beta^k) + n\sum_{k=1}^{\frac{n-1}{2}} D(\alpha^k)\Big).$$

Since all roots appearing in (10) are roots of unity one can regard the sum that appears there as a finite Fourier transform and thus write the measure as a combination of $L'(\chi,-1)$ for various Dirichlet characters. The functional relation $D(\bar{x}) = -D(x)$ ensures that only odd characters appear in such formulas.

More explicitly, let χ be a primitive odd Dirichlet of conductor N and $L(\chi,s) = \sum_{n=1}^{\infty}\chi(n)n^{-s}$ for $\mathrm{Re}(s) > 1$ be its associated L-function. Then a standard calculation using the series in (5) shows that

$$(11) \quad L(\chi,2) = \tau(\bar{\chi})^{-1}\sum_{k \bmod N}\bar{\chi}(k)D(e^{2\pi ik/N}),$$

where

$$(12) \quad \tau(\bar{\chi}) = \sum_{k \bmod N}\bar{\chi}(k)e^{2\pi ik/N}$$

is a Gauss sum.

For example, using the notation $d_f = L'(\chi_{-f},-1)$, we have $m(y+1+x) = d_3$, $m(y+1+x+x^2) = \frac{2}{3}d_4$, $m(y+1+x+x^2+x^3+x^4) = 2d_3 - \frac{2}{5}d_4$ and $m(y+1+x+x^2+x^3+x^4+x^5+x^6) = \frac{1}{7}(-10d_3+4d_4+2d_8)$.

Let us remark that by results of Boyd and Smyth [Bo1, Sm] we have

$$(13) \quad \lim_{n\to\infty} m(y+1+x+\cdots+x^{n-1}) = m(1+x+y+z) = \frac{7}{2\pi^2}\zeta(3) = 14\zeta'(-2),$$

where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is Riemann's zeta function. Using the Euler-Maclaurin formula, one can derive this directly from the formula (10).

3 The Bloch Group and Borel's Theorem

We summarize in this section the results that we need in order to relate in general the calculation of Section 1 to special values of L-functions. We refer the reader to [Za2] for details.

For any field F we define

$$\mathcal{A}(F) = \Big\{ \sum_i n_i[z_i] \in \mathbb{Z}[F] \;\Big|\; \sum_i n_i\big(z_i \wedge (1-z_i)\big) = 0 \Big\} \tag{14}$$

where the corresponding term in the sum is omitted if $z_i = 0, 1$. We extend the Bloch-Wigner dilogarithm D to $\mathbb{Z}[\mathbb{C}]$ by linearity.

Next we define the group

$$\mathcal{C}(F) = \Big\{ [x] + [y] + \Big[\frac{1-x}{1-xy}\Big] + [1-xy] + \Big[\frac{1-y}{1-xy}\Big] \mid x, y \in F, xy \neq 1 \Big\}. \tag{15}$$

It is not hard to check that $\mathcal{C}(F) \subset \mathcal{A}(F)$. We define the *Bloch group* as the quotient

$$\mathcal{B}(F) = \mathcal{A}(F)/\mathcal{C}(F). \tag{16}$$

Clearly an embedding of fields $\sigma\colon F \to L$ extends by linearity to a map $\sigma\colon \mathcal{B}(F) \to \mathcal{B}(L)$ and in particular $\mathrm{Gal}(F/\mathbb{Q})$ acts on $\mathcal{B}(F)$ if $F/\mathbb{Q}$ is Galois.

One of the wonderful properties of the dilogarithm, the 5-term relation, guarantees that D induces a well defined function on $\mathcal{B}(\mathbb{C})$ (still denoted by D).

The main result we need is the following (due to Bloch, Borel, Suslin, and others).

Theorem *Let F be a number field with r_1 real and r_2 complex embeddings. Then*

1) *The group $\mathcal{B}(F)$ is finitely generated of rank r_2.*
2) *Let $\xi_1, \dots, \xi_{r_2}$ be a $\mathbb{Q}$-basis of $\mathcal{B}(F) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\sigma_1, \dots, \sigma_{r_2}$ a set of pairwise non-conjugate complex embeddings of F into $\mathbb{C}$. Then*

$$\det\Big(D\big(\sigma_i(\xi_j)\big)\Big) \sim_{\mathbb{Q}^*} \frac{|\Delta_F|^{3/2}}{\pi^{2(r_1+r_2)}}\zeta_F(2), \tag{17}$$

where ζ_F is the zeta function of F, Δ_F its discriminant, and $a \sim_{\mathbb{Q}^*} b$ means $a = rb$ for some non-zero rational number r.

In particular, if F is a number field of degree n with only one pair of complex embeddings $\sigma, \bar\sigma$ and $0 \neq \xi \in \mathcal{B}(F) \otimes_{\mathbb{Z}} \mathbb{Q}$ then

$$D\big(\sigma(\xi)\big) \sim_{\mathbb{Q}^*} \frac{|\Delta_F|^{3/2}}{\pi^{2(n-1)}}\zeta_F(2). \tag{18}$$

We will also need the following *Galois descent* property of the Bloch group.

Theorem *Let L/F be a Galois extension of number fields with $G = \mathrm{Gal}(L/F)$. Then*

$$\mathcal{B}(F) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathcal{B}(L)^G \otimes_{\mathbb{Z}} \mathbb{Q}.$$

https://doi.org/10.4153/CJM-2002-016-9 Published online by Cambridge University Press

4 Some Elements of $\mathcal{B}(\bar{\mathbb{Q}})$

We let $\bar{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$.

Proposition 2 *Let $p, q \in \mathbb{Z}[x]$ be cyclotomic and relatively prime. Let*

$$r(x) = \frac{q(x)}{p(x)} = \pm x^k \prod_{m=1}^{\infty} (1 - x^m)^{c_m} \tag{19}$$

as in (6). Then for any $\alpha \in \bar{\mathbb{Q}}$ root of

$$r(x)r(x^{-1}) = 1 \tag{20}$$

we have

$$\sum_{m=1}^{\infty} \frac{c_m}{m}[\alpha^m] \in \mathcal{A}(\bar{\mathbb{Q}}) \otimes_{\mathbb{Z}} \mathbb{Q}. \tag{21}$$

Proof We need to verify that

$$\sum_{m=1}^{\infty} \frac{c_m}{m}\big(\alpha^m \wedge (1-\alpha^m)\big) = 0.$$

We have

$$\sum_{m=1}^{\infty} \frac{c_m}{m}\big(\alpha^m \wedge (1-\alpha^m)\big) = \sum_{m=1}^{\infty} \alpha \wedge (1-\alpha^m)^{c_m} = \alpha \wedge \prod_{m=1}^{\infty} (1-\alpha^m)^{c_m}$$

It is clear that $r(x^{-1}) = \pm x^N r(x)$ for some $N \in \mathbb{Z}$. Hence if α is a root of $r(\alpha)r(\alpha^{-1}) = 1$ then $r(\alpha)^2$ is (up to sign) a power of α, so $\alpha \wedge r(\alpha) = 0$ in $\mathcal{A}(\bar{\mathbb{Q}}) \otimes_{\mathbb{Z}} \mathbb{Q}$ and we are done. ■

The type of element of $\mathcal{B}(\bar{\mathbb{Q}}) \otimes_{\mathbb{Z}} \mathbb{Q}$ given in the proposition is called a "ladder"; see [Za] for more details.

5 Putting Things Together

By Proposition 1, the value of $m\big(yp(x) - q(x)\big)$ is determined by the points on $|x| = 1$ where the graph of $r(x) = q(x)/p(x)$ crosses the unit circle; *i.e.* those α with $|\alpha| = 1$ at which $r(x)r(x^{-1}) = 1$ has a zero of odd multiplicity. Since $r(x)r(x^{-1}) = 1$ is equivalent to

$$q(x)q(x^{-1}) - p(x)p(x^{-1}) = 0, \tag{22}$$

these points are the roots of a square-free polynomial $C(x) \in \mathbb{Z}[z]$ that we call the *crossing polynomial* of $r(x)$. For example, if $p = (x+1)^4$, $q = x^4 + 1$ then (22) reduces to

$$(x^2 + x + 1)^2(2x^2 + 3x + 2) = 0,$$

https://doi.org/10.4153/CJM-2002-016-9 Published online by Cambridge University Press

where both factors have their roots on the unit circle. Here the graph of $|r(x)|$ crosses the circle at the roots of $C(x) = 2x^2 + 3x + 2$, and touches the circle at the primitive 3rd roots of unity.

Combining the Propositions 1 and 2 we obtain the main result of the paper.

Theorem 1 *Let $P(x, y) = p(x)y - q(x)$ with p and q cyclotomic and relatively prime and let $r(x) = q(x)/p(x)$. If the graph of $r(x)$ crosses the circle at exactly one pair of complex conjugate points $\alpha, \bar{\alpha}$, and if α is defined over a number field F with only one pair of complex embeddings then*

$$m(P) \sim_{\mathbb{Q}^\star} \frac{|\Delta_F|^{3/2}}{\pi^{2n-1}} \zeta_F(2). \tag{23}$$

Proof Let

$$\xi = \sum_{m=1}^{\infty} \frac{c_m}{m} [\alpha^m] \in \mathbb{Z}[F],$$

where the $c_m \in \mathbb{Z}$ are defined as in (6).

By Proposition 1 and (4) we have

$$m(P) = \pm \frac{1}{\pi} D(\xi)$$

and by Proposition 2 $\xi \in \mathcal{A}(F)$. Our claim now follows from (19). ■

As an illustration consider Smyth's case where $p = 1, q = -(x+1)$. Here $P(x, y) = 1 + x + y$ and the solutions to (22) are (ζ_3, ζ_3^2), where $\zeta_3 = e^{2\pi i/3}$, and its complex conjugate. Then

$$\xi = -[\zeta_3] + \frac{1}{2}[\zeta_3^2] \in \mathcal{A}(F), \quad F = \mathbb{Q}(\zeta_3)$$

and a direct calculation as in Section 2 yields (2), which is an explicit version of (23) for this case.

Theorem 2 *With the notation of the previous theorem suppose that the crossing polynomial $C(x)$ is an irreducible polynomial in $\mathbb{Q}[x]$, of degree 4 and splitting field L with $G = \mathrm{Gal}(L/\mathbb{Q})$ isomorphic to the Klein group V_4. Order these roots counterclockwise around the unit circle as follows: $\alpha, \beta, \bar{\beta}, \bar{\alpha}$, with $\mathrm{Im}\,\alpha, \mathrm{Im}\,\beta > 0$. Let $\sigma \in G$ be the Galois element that takes α to $\bar{\beta}$ and F the fixed field of σ. Then*

$$m(P) \sim_{\mathbb{Q}^\star} \frac{|\Delta_F|^{3/2}}{\pi^3} \zeta_F(2).$$

Proof The Bloch element $\xi \in \mathcal{B}(L)$ which, by Proposition 1, satisfies $2\pi m(P) = \pm D(\xi)$ is a linear combination of

$$\xi_m = [\alpha^m] - [\beta^m] + [\bar{\beta}^m] - [\bar{\alpha}^m] \in \mathbb{Z}[L].$$

https://doi.org/10.4153/CJM-2002-016-9 Published online by Cambridge University Press

By our choice of σ we verify easily that

$$\xi_m^\sigma = \xi_m$$

and hence σ fixes ξ. By the Galois descent property of the Bloch groups quoted in Section 5, ξ represents an element in $\mathcal{B}(F) \otimes_{\mathbb{Z}} \mathbb{Q}$. Since F is an imaginary quadratic field (σ is not complex conjugation) the desired identity follows from (18). ■

Remark Note that exactly the same argument applies to any $\eta \in \mathcal{B}(L)$ which is a linear combination of

$$\eta_m = [\alpha^m] + [\beta^m] - [\bar{\beta}^m] - [\bar{\alpha}^m].$$

Giving an identity for $D(\eta)$ in which a different imaginary quadratic field appears. This will be useful later.

6 Applications of the Theorem

In order to apply the results of the previous paragraphs, we need to construct rational functions $r(x) = q(x)/p(x)$ with p, q cyclotomic, that cross the circle exactly at the roots on $|x| = 1$ of a squarefree polynomial $C(x)$.

Since $|r(x)| = |r(\bar{x})|$, $r(x)$ cannot cross the circle at $x = \pm 1$ so $C(\pm 1) \neq 0$. Since each irreducible factor of the crossing polynomial $C(x)$ must have a zero $\neq \pm 1$ on the unit circle each of these factors is a reciprocal polynomial of even degree. The field defined by such a factor of degree $2n$ has a Galois group which is a subgroup of the hyperoctahedral group $B_n = Z_2 \wr S_n$ of order $2^n n!$ [La]. So the formulas we obtain from Theorems 1 and 2 are restricted to such fields.

We will denote by $\mathcal{R}$ the set of rational functions $r(x)$ with all poles and zeros at roots of unity. The subset of these with the crossing polynomial $C(x)$ and normalized so that $|r(1)| < 1$ will be denoted by $\mathcal{R}(C)$. (This normalization is harmless since $m^+(r) = m^+(1/r)$.)

In the application of Theorem 1, we are interested in $C(x)$ with exactly one root $\alpha = e^{it_0}$ on the upper half circle. The assumption that $r(x)$ crosses the upper half circle exactly at one point is very restrictive. All the zeros and poles of $r(x)$ are at roots of unity. Between each zero and pole there must be a point where $r(e^{it})$ crosses the circle. If there is only one such point e^{it_0}, then the zeros of $r(x)$ must lie in $|\arg(x)| < t_0$ and the poles in $|\arg(x)| > t_0$. Thus the numerator and denominator of $r(x)$ can only be products of certain restricted sets of cyclotomic polynomials which we call the *admissible numerators* and *admissible denominators* for t_0.

It will be convenient to say that $r(x) \in \mathcal{R}$ is *admissible* for C if its numerator (denominator) is a product of admissible numerators (denominators) and if it crosses the circle at the roots of $C(x)$ (even if it also crosses the circle at other points).

Take for example $C(x) = 2x^2 + 3x + 2$, with root $\alpha = (-3 + \sqrt{-7})/4$, so $t_0 = \arg(\alpha) = 2.418858\cdots = (0.769946\cdots)\pi$. Hence the admissible numerators are F_n with $n = 1, 3, 4, 6, 8, 10, 14$ and the only admissible denominator is $F_2 = x + 1$. (We will write F_n for the n-th order cyclotomic polynomial, *i.e.* the minimal monic

https://doi.org/10.4153/CJM-2002-016-9 Published online by Cambridge University Press

polynomial for the primitive n-th roots of unity.) Observe that Table 1 gives the two examples $r_1(x) = F_3/F_2^2$ and $r_2(x) = F_8/F_2^4$, mentioned above. Since both of these satisfy $|r_k(x)| \le 1$ for $|\arg(x)| < t_0$ and $|r_k(x)| \ge 1$ for $|\arg(x)| > t_0$, the same is true for any $r(x) = r_1(x)^a r_2(x)^b$ for any non-negative integers a, b not both zero, and hence this infinite set of $r(x)$ also has crossing polynomial $2x^2 + 3x + 2$. By Proposition 1, we have $m^+(r_1^a r_2^b) = am^+(r_1) + bm^+(r_2)$. It can be shown that these rational functions and their reciprocals exhaust the set $\mathcal{R}(C)$. (We prove similar results below.)

For $C(x)$ with a single pair of roots $e^{\pm it_0}$ on $|x| = 1$, there is a simple rule to determine the admissible numerators and denominators. If $t_0 < \pi/3$, then the only admissible numerator is $F_1 = x - 1$, and the admissible denominators are those F_n with $n < 2\pi/t_0$. If $\pi/3 < t_0 < \pi/2$ then F_1 and F_6 are the only admissible numerators and F_n for $n \in \{2, 3, 4\}$ are the admissible denominators. If $\pi/2 < t_0 < \pi$ then find the admissible numerators and denominators for $\pi - t_0$, apply the permutation that takes

$$\begin{cases} n \to 2n & \text{if } n \text{ is odd,} \\ n \to n/2 & \text{if } n \equiv 2 \bmod 4, \\ n \to n & \text{if } n \equiv 0 \bmod 4, \end{cases} \tag{24}$$

then interchange numerators and denominators. The proof uses the basic facts that $F_n(-x) = F_{2n}(x)$ if n is odd and $F_n(-x) = F_n(x)$ if $n \equiv 0 \bmod 4$.

For a general crossing polynomial $C(x)$ again one easily sees that there are finite and easily determined sets of admissible numerators and admissible denominators. As in the example above, if $r_1(x), r_2(x) \in \mathcal{R}(C)$ then for any non-negative integers a, b not both zero $r_1^a r_2^b \in \mathcal{R}(C)$.

Let F be a field of discriminant d_F, degree n with $r_2 = 1$. Tables 1 and 2 will express suitable $m\big(yp(x) - q(x)\big) = m^+\big(r(x)\big)$ in the form $s_F Z_F$, where s_F is a non-zero rational number and

$$Z_F = \frac{3|\Delta_F|^{3/2}\zeta_F(2)}{2^{2n-3}\pi^{2n-1}}. \tag{25}$$

The apparently peculiar choice of the rational multiplier $3/2^{2n-3}$ here is so that for the quadratic field of discriminant $-f$, one will have

$$Z_F = L'(\chi_{-f}, -1), \tag{26}$$

where χ_{-f} is the real odd character of conductor f. As in [Bo3], we use the notation $d_f = L'(\chi_{-f}, -1)$.

For our first class of examples, we apply the discussion of the previous section to quadratic $C(x) = ax^2 + bx + a$ with $b^2 - 4a^2 < 0$, with roots α and $1/\alpha$ on the unit circle. Let $F = \mathbb{Q}(\sqrt{b^2 - 4a^2})$ have discriminant $-f$. Then by Theorem 1, an $r(x) = q(x)/p(x) \in \mathcal{R}(C)$ will give us a formula of the form

$$m\big(yp(x) - q(x)\big) = m^+\big(r(x)\big) = s_f d_f. \tag{27}$$

Chinburg [Ch] asked whether one can generalize Smyth's result for the conductor 3 to other conductors, *i.e.* whether for each f there is a polynomial $P_f(x, y)$ with integer coefficients and a non-zero rational s_f so that $m(P_f) = s_f L'(\chi_{-f}, -1)$ where $\chi_{-f} = (\frac{-f}{n})$ is the real odd primitive character of conductor f. He proved that there are *rational functions* P_f with this property. Ray [Ra] constructed examples for the conductors $f = 3, 4, 7, 8, 20$ and 24. Here we construct examples for the conductors $f = 3, 4, 7, 8, 11, 15, 20, 24, 35, 39, 55$ and 84.

7 Quadratic Fields I (d_3 & d_4)

We begin by treating the special cases $C(x) = F_3(x) = x^2 + x + 1$ and $F_4(x) = x^2 + 1$ which are particularly simple since α is a root of unity. As usual, we write $\zeta_n = \exp(2\pi i/n)$. We have the elementary formulas

(28) $$d_3 = \frac{D(\zeta_6)}{\pi} = \frac{3}{2}\frac{D(\zeta_3)}{\pi},$$

and

(29) $$d_4 = \frac{2D(\zeta_4)}{\pi}.$$

Smyth's formula (2)

(30) $$m(y + x + 1) = d_3$$

has already been mentioned, and there is an equally simple formula for d_4 from [Bo2]:

(31) $$m\big(y(x + 1) + (x - 1)\big) = d_4.$$

In these examples the crossing polynomials are $C(x) = F_3(x) = x^2 + x + 1$ and $F_4(x) = x^2 + 1$, respectively. So formulas (28) and (29) give the results (30) and (31) without the intervention of Theorem 1. In fact there is a 3 parameter infinite family of $r(x)$ with crossing polynomial F_3 and a 3 parameter infinite family for F_4 giving infinitely many generalizations of (30) and (31). In each case, we can show that these are all the possible solutions. There is a subtle difference between the two cases as we see below.

Proposition 3 *The rational functions $r(x) \in \mathcal{R}(x^2 + 1)$ are given by*

(32) $$r(x) = \frac{(x-1)^a(x^2 - x + 1)^b}{(x+1)^a(x^2 + x + 1)^c},$$

where a, b, c are non-negative integers, not all 0*. For each such $r(x)$, one has*

(33) $$m\big(y \pm r(x)\big) = m^+\big(r(x)\big) = \Big(a + \frac{2}{3}b + \frac{2}{3}c\Big)\, d_4.$$

https://doi.org/10.4153/CJM-2002-016-9 Published online by Cambridge University Press

Proof The admissible numerators are $F_1 = x - 1$ and $F_6 = x^2 - x + 1$ and the admissible denominators are $F_2 = x + 1$ and $F_3 = x^2 + x + 1$. So $r(x)$ is of the form $F_1^a F_6^b / F_2^d F_3^c$. However, $|F_1(i)| = |F_2(i)| = \sqrt{2}$ and $|F_3(i)| = |F_6(i)| = 1$, so in order that $|r(i)| = 1$, we must have $d = a$ and hence $r(x)$ is of the form (32).

Now consider the size of $f_n(t) = |F_n(e^{it})|$. We clearly have $f_1(t) < f_2(t)$ and $f_6(t) < f_3(t)$ for $0 \le t < \pi/2$ and the reverse inequalities hold in $\pi/2 < t \le \pi$. Hence $r(x)$ of the form (32) crosses the circle exactly at $t = \pi/2$, as required.

The formula (33) follows from Proposition 1 and (29). ■

It should be emphasized that these are not the only $r(x)$ for which $m^+\big(r(x)\big)$ is a rational multiple of d_4, as is evident from Table 1. But other examples must have different crossing polynomials.

A trivial way that this can occur is to consider $r(x^2)$ where $r(x)$ is as in Proposition 3. This has crossing polynomial $F_8 = x^4 + 1$. We do not include such examples in Table 1. However there are non-trivial examples with crossing polynomial F_8, for which $r(x)$ is not even. According to Proposition 1, we can compute $m\big(r(x)\big)$ by evaluating the quantities $D(\zeta_8^k) - D(\zeta_8^{3k})$. For this we need the following known result whose proof is a pleasant exercise in the use of the basic distribution relations for $D(z)$:

$$D(z^n) = n \sum_{j=0}^{n-1} D(\zeta_n^j z), \tag{34}$$

for every positive integer n.

Lemma 1 *For any integer k,*

$$D(\zeta_8^k) - D(\zeta_8^{3k}) = s_k D(\zeta_4), \tag{35}$$

where $s_k = 0, \frac{1}{2}, 2, -\frac{1}{2}, 0, \frac{1}{2}, -2, -\frac{1}{2}$ if $k \equiv 0, 1, \dots, 7$ modulo 8, respectively.

Proposition 4 *The rational functions $r(x) \in \mathcal{R}(x^4 + 1)$ are given by*

$$r(x) = r_1(x)^a r_2(x)^b r_3(x)^c r_4(x)^d r_5(x)^e, \tag{36}$$

where a, b, c, d, e are non-negative integers, not all zero, and where

$$\begin{cases} r_1 = F_4/(F_1F_2), \\ r_2 = F_3F_6, \\ r_3 = 1/F_{12} \\ r_4 = F_3F_4/F_2^2, \\ r_5 = F_4F_6/F_1^2 \end{cases} \tag{37}$$

For such $r(x)$ we have

$$m\big(y - r(x)\big) = m^+\big(r(x)\big) = \Big(a + \frac{2}{3}b + \frac{2}{3}c + \frac{4}{3}d + \frac{4}{3}e\Big)\, d_4. \tag{38}$$

https://doi.org/10.4153/CJM-2002-016-9 Published online by Cambridge University Press

Proof One easily sees that only F_n with $n|12$ can be admissible numerators or denominators. We also verify that each of the 5 functions $r_k(x)$ of (37) has crossing polynomial F_8 and hence so does $r(x)$ of (36).

To see that there are no other possibilities, check that the values of the relevant $|F_n(x)|$ at $x = \zeta_8$ are given by $\sqrt{2-\sqrt{2}}$, $\sqrt{2+\sqrt{2}}$, $1+\sqrt{2}$, $\sqrt{2}$, $-1+\sqrt{2}$ and 1 for $n = 1, 2, 3, 4, 6$ and 12, respectively. Thus if a product $r(x) = \prod_{n|12} F_n(x)^{c_n}$ is to have $|r(\zeta_8)| = 1$, there is no restriction on c_{12}, but $c_1, \dots, c_6$ must satisfy the linear equation

$$\log(2-\sqrt{2})c_1+\log(2+\sqrt{2})c_2+2\log(1+\sqrt{2})c_3+\log 2c_4+2\log(-1+\sqrt{2})c_6 = 0. \tag{39}$$

There is thus at most a 5-dimensional solution for $c_1, \dots, c_{12}$, and since (36) exhibits such a solution, this is the complete solution to the problem.

Finally, the formula (38) follows from Proposition 1, (35) and (29). ∎

Remark Notice that r_1, r_2 and r_3 of Proposition 4 are even functions of x. obtained from the basis $(x-1)/(x+1)$, $F_3(x)$ and $1/F_6(x)$ of the $r(x)$ in Proposition 3 by the substitution $x \to x^2$. The functions r_4 and r_5 satisfy $r_4(-x) = r_5(x)$ and give a basis of the new solutions.

The next result shows that one cannot always expect there to be a finite basis of solutions as in the previous two results.

Proposition 5 *The rational functions $r(x) \in \mathcal{R}(F_3)$ are given by*

$$r(x) = F_2(x)^{-a} F_4(x)^b F_{10}(x)^c, \tag{40}$$

where a is a positive integer, and where b, c are non-negative integers satisfying $b \le a$ and the inequality on c determined by the condition that

$$c \le \min_{t\in[\pi/3,\pi/2]} \frac{a\log(2\cos t/2) - b\log(2\cos t)}{\log(\cos 5t/\cos t)}. \tag{41}$$

A sufficient condition for (41) to hold is that $c \le 2a$ and a necessary condition is that

$$c \le (2.05314185\cdots)a + (3.10628371\cdots)b, \tag{42}$$

where the coefficients in (42) are $\log(5/2)/\log(25/16)$ *and* $\log 2/\log(5/4)$.

For such $r(x)$, we have

$$m\big(y - r(x)\big) = m^+\big(r(x)\big) = \left(a + \frac{1}{2}b + \frac{6}{5}c\right) d_3. \tag{43}$$

Proof The admissible numerators are F_1, F_4, F_6 and F_{10}, and the only admissible denominator is F_2. However, $|F_n(\zeta_3)| = 1$ for $n = 2, 4$ and 10, while $|F_1(\zeta_3)| = \sqrt{3}$ and $|F_6(\zeta_3)| = 2$, so the requirement that $|r(\zeta_3)| = 1$ rules out F_1 and F_6, hence $r(x)$ must be of the form (40).

To determine the restrictions on a, b, c, consider $f_n(t) = |F_n(e^{it})|$ for $n = 2, 4, 10$ and $t \in [0, \pi]$. We have $f_2(t) = 2\cos(t/2)$, $f_4(t) = |2\cos t|$ and $f_{10}(t) =$

$|2\cos 2t - 2\cos t + 1| = |\cos(5t/2)/\cos(t/2)|$. Clearly $f_{10}(t) > f_4(t) > 1 > f_2(t)$ for $t \in (2\pi/3, \pi]$ so $|r(e^{it})| > 1$ in this interval for any nonnegative b, c and $a > 0$.

Observe that $f_2(t) > 1$ for $t < 2\pi/3$, that $f_4(t) < 1$ exactly for $\pi/4 < t < 3\pi/4$, while $f_{10}(t) < 1$ exactly in $(0, \pi/3) \cup (\pi/2, 2\pi/3)$. Since $f_{10}(t) \le 1 < f_4(t) < f_2(t)$ in $[0, \pi/3]$, we will have $|r(e^{it})| \le f_4(t)^b f_2(t)^{-a} \le 1$ if $b \le a$. Furthermore, $|r(1)| = 2^{b-a}$ so $b \le a$ is necessary.

In the interval $[\pi/2, 2\pi/3]$ we will have $|r(e^{it})| \le 1$ with no restriction on a, b, c since all of $f_4(t)$, $f_{10}(t)$ and $1/f_2(t)$ are ≤ 1 here.

This leaves the interval $[\pi/3, \pi/2]$ on which $f_{10}(t) > 1 > f_4(t)$. The condition (41) is equivalent to $|r(e^{it})| \le 1$ for $t \in [\pi/3, \pi/2]$ and hence for r to have a single crossing of the circle at $x = \zeta_3$.

To analyze this condition further, we observe that the maximum of $f_{10}(t)$ on $[\pi/3, \pi/2]$ is attained at $t = t_1 = \cos^{-1}(1/4)$, where $f_{10}(t_1) = 5/4$, $f_2(t_1) = \sqrt{5/2} = 1.581138\cdots$ and $f_4(t_1) = 1/2$. The necessary condition that $r(t_1) \le 1$ is exactly (42). To obtain the sufficient condition $c \le 2a$, we find by Calculus that $f_{10}(t)^2/f_2(t) < .994557\cdots < 1$ on $[\pi/3, \pi/2]$. Hence if $c \le 2a$, we will have $|r(e^{it})| \le f_4(t)^b f_{10}(t)^c/f_2(t)^a \le f_{10}(t)^{2a}/f_2(t)^a < 1$ on this interval, so (41) will hold.

Finally, the formula (43) follows from Proposition 1 and (28). ■

Remarks 1. Let $r_1 = 1/F_1$, $r_2 = F_4/F_2$ and $r_3 = F_{10}/F_2$. Then $r_1^a r_2^b r_3^c \in \mathcal{R}(F_3)$ for all nonnegative integers a, b, c not all zero. However, there are elements of $\mathcal{R}(F_3)$ not of this form.

2. From Proposition 5, by replacing $r(x)$ by $1/r(-x)$, we find that the elements of $\mathcal{R}(F_6)$ are

$$r(x) = F_1(x)^a F_4(x)^{-b} F_5(x)^{-c}, \tag{44}$$

with the same restrictions on a, b, c as in Proposition 5.

As with d_4, we can obtain further $r(x)$ with $m\big(y - r(x)\big)$ a rational multiple of d_3 by changing x to x^2 in (40). These functions will have crossing polynomial F_{12}. As in Proposition 3, there are further non-even elements of $\mathcal{R}(F_{12})$, and the proof involves a similar lemma, again proved by the distribution relations for $D(x)$:

Lemma 2 *For any integer k,*

$$D(\zeta_{12}^k) - D(\zeta_{12}^{5k}) = t_k D(\zeta_3), \tag{45}$$

where $t_k = 0, \frac{3}{4}, 3, 0, 2, -\frac{3}{4}, 0, \frac{3}{4}, -2, 0, -3, -\frac{3}{4}$ *for* $k = 0, 1, \dots, 11$ *modulo* 12, *respectively.*

We leave the proof of Proposition 6 as an exercise:

Proposition 6 *The rational functions* $r(x) \in \mathcal{R}(F_{12})$ *are given by*

$$r(x) = r_1(x)^a r_2(x)^b r_3(x)^c r_4(x)^d r_5(x)^e, \tag{46}$$

where

$$\begin{cases} r_1 = 1/(F_1F_2), \\ r_2 = F_8/(F_1F_2), \\ r_3 = F_5/F_2^2, \\ r_4 = F_{10}/F_1^2, \\ r_5 = F_5F_{10} \end{cases} \tag{47}$$

and where a, b, c, d, e are non-negative integers satisfying further inequalities which are implied by $e \le 2a$.

For such $r(x)$, we have

$$m\big(y - r(x)\big) = m^+\big(r(x)\big) = \Big(a + \frac{3}{2}b + \frac{8}{5}c + \frac{8}{5}d + \frac{6}{5}e\Big)\, d_3. \tag{48}$$

Remarks 1. Note that (46) only formally involves 5 parameters a, b, c, d, e since $r_3r_4 = r_5r_1^2$. However, the inclusion of a 5-th parameter makes the determination of the restrictions on the parameters considerably easier.

2. One can use a result similar to Lemmas 1 and 2 to determine $\mathcal{R}(F_4F_{12})$, where $P = F_4F_{12}$ is a polynomial with 3 pairs of roots on the unit circle. Now the formula for $m^+\big(r(x)\big)$ will be a sum of terms of the form $D(\zeta_{12}^k) - D(i^k) + D(\zeta_{12}^{5k})$, and again the distribution relations show that these are rational multiples of $D(i^k)$. One can show that $\big(D(\zeta_{12}^k) + D(\zeta_{12}^{5k})\big)/D(i^k)$ is $0, \frac{4}{3}, 0, 2, 0, \frac{4}{3}, 0, -\frac{4}{3}, 0, -2, 0, -\frac{4}{3}$ if $k \equiv 0, 1, \dots, 11$ modulo 12. Thus one can express $m^+\big(r(x)\big)$ as an explicit rational multiple of d_4.

3. Notice that the polynomials $F_8(x) = x^4 + 1$ and $F_{12}(x) = x^4 + x^2 + 1$ in Propositions 4 and 6 have Galois group V_4, the Klein 4-group. The significance of this will be apparent in Section 9 where we considerably extend Propositions 4 and 6.

8 Quadratic Fields II (Non-Monic Quadratics)

Next we consider the crossing polynomial $C(x) = ax^2 + bx + a$ with $a > 1$, $0 < b < 2a$ and root $\alpha = (-b + \sqrt{b^2 - 4a^2})/(2a)$ on the unit circle. Browkin [Br] has searched for *cyclotomic relations* for such α, *i.e.* functions in $\mathcal{R}$ with $|r(\alpha)| = 1$. This is less restrictive than requiring $r \in \mathcal{R}(C)$ since such $r(x)$ may have other crossings of the unit circle. By a result of Schinzel, [Sc], there is a finite set of independent relations for each such α. Browkin determined 23 such α for which a cyclotomic relation holds but did not always determine a complete set of relations because of his restriction to F_n of degree at most 5. Even if one knows a basis of relations for a given $C(x)$, it is not entirely trivial to determine the set $\mathcal{R}(C)$ as we illustrate below. On the other hand, our situation is somewhat simpler in that we know which F_n may appear as admissible numerators and denominators and hence can restrict consideration to relations involving these F_n.

In our computations, we decided to start afresh so the results here are independent of [Br] and provide an independent confirmation of those results. It is worthwhile

https://doi.org/10.4153/CJM-2002-016-9 Published online by Cambridge University Press

noting that we did not find any further quadratic α of this form which satisfy a cyclotomic relation, so perhaps the list of 23 found by Browkin is complete. However, we did find one cyclotomic relation that does not appear in [Br]: the relation $F_3^4 F_5^{-5} F_{13}$ for $\alpha = (-3+\sqrt{-7})/4$ can be added to the 7 relations determined there for this α. (Note that Browkin considers the equivalent $-\bar{\alpha} = (3+\sqrt{-7})/4$ so one must apply the permutation (24).)

We considered all $C(x)$ as above, with $1 < a \leq 30$. The first step was to identify those which satisfy *some* cyclotomic relation. For this we applied the LLL algorithm to find linear relations of the form $\sum c_n \log |F_n(\alpha)| = 0$, where the sum is over $n \leq 30$, and the c_n are *small* integers. We ignored any $C(x)$ for which the LLL algorithm did not produce at least one such relation and thus may have missed some possible relations, but this seems rather unlikely since we used 100 decimal place accuracy in this step. Each relation found can be verified by exact arithmetic (so there are no false positives).

In the next stage, for each of the $C(x)$ identified in the first stage, we determined the admissible numerators and denominators. Then we again used the LLL algorithm to find linear relations of the form $\sum c_n \log |F_n(\alpha)| = 0$, where now the sum is over only admissible n, identifying those that satisfy such a relation and determining a plausible set of basis elements for such relations.

From these relations, we formed linear combinations if necessary to form relations in which the signs of the c_n are correct, *i.e.* so that admissible numerators appear with a positive c_n and denominators with a negative c_n. At this stage, we only know that we have $r(x) = \prod_n F_n(x)^{c_n}$ of the correct shape with $|r(\alpha)| = 1$, but $r(x)$ may cross the circle at other points besides α and $\bar{\alpha}$. (Recall that we refer to such $r(x)$ as *admissible*.)

Plotting $|r(e^{it})|$ for $t \in [0, \pi]$ for the various r that we had now identified, we attempted to form products of these r to obtain $|r(e^{it})| < 1$ in $t < t_0 = \arg(\alpha)$ and $|r(e^{it})| > 1$ in $t > t_0$. If we found more than one independent $r(x)$ we obtained a multi-parameter family in $\mathcal{R}(C)$. Finally, using the methods of Proposition 3 and 5, we were able to show that we had identified all of $\mathcal{R}(C)$.

The results are exhibited in Table 1 which exhibit a "basis" of each $\mathcal{R}(C)$ for those $C(x) = ax^2 + bx + a$ for which the above succeeded. More explicitly, if there is a basis of solutions (as for $C(x) = x^2 + 1$) then these are listed. If the situation is as for $C(x) = x^2 + x + 1$, in which there are inequality constraints, we list a set of independent elements that generate an infinite subset of $\mathcal{R}(C)$, as in Remark 1 following Proposition 5.

As an example, consider $C(x) = 4x^2 + 5x + 4$ with $\alpha = (-5+\sqrt{-39})/8$ with $t_0 = \arg(\alpha) = 2.245927\cdots$ so $\arg(\alpha)/\pi = .714901\cdots > \frac{5}{7}$. Here the admissible numerators are F_n with $n \in \{1, 6, 4, 10, 3, 14\}$ and F_2 is the only admissible denominator. Using LLL we found the cyclotomic relations $r_1 = F_3 F_6 / F_2^4$ and $r_2 = F_6^3 F_{14} / F_2^{12}$ (both appear in [Br]). These are both admissible and it can be shown they form a basis for the admissible functions by considering the norms of $F_n(\alpha)$ with $n \in \{1, 2, 3, 4, 6, 10, 14\}$.

Observe that $|r_1(e^{it})| < 1$ for $0 \leq t < t_0$ and $|r_1(e^{it})| > 1$ for $t_0 < t \leq \pi$, hence $r_1 \in \mathcal{R}(C)$. However, r_2 has two further crossings of the circle at arguments $t_1 = 1.886385\cdots$ and $t_2 = 2.241937\cdots$. We find that $|r_2(e^{it})| < 1$ for $0 < t < t_1$ and $t_2 < t < t_0$ and that the inequality is reversed in $t_1 < t < t_2$ and $t_0 < t < \pi$.

Since $|r_1(e^{it})|$ achieves a maximum strictly less than 1 in $[t_1, t_2]$, it is clear that there is a smallest constant c_0 such that $r_1^c(e^{it})r_2(e^{it})$ crosses the circle exactly at $t = t_0$ for all $c > c_0$. A calculation gives $c_0 = 11.3855380775153$ where the last decimal place has been rounded up. So $r = r_1^a r_2^b$ will be in $\mathcal{R}(C)$ provided a, b are non-negative integers with $a \geq c_0 b$. For such a, b, we have

$$m^+(r_1^a r_2^b) = \frac{1}{18}a + \frac{4}{21}b. \tag{49}$$

In Table 1, we have listed simply r_1 and $r_1^{12}r_2$.

9 Quadratic Fields III (4-Group Quartics)

Theorem 2 gives us a way of producing examples with $m^+(r)$ of the form $s_f d_f$ from quartics with Galois group V_4. Here we are interested in $C(x) = ax^4+bx^3+cx^2+bx+a$ with all four roots on the unit circle and with Galois group V_4. For fixed a, there are only a finite number of $C(x)$ with all roots on the unit circle and these are easily enumerated: one writes $C(x) = x^2A(x+1/x)$, where $A(y) = ay^2+by+(c-2a)$, and $A(y)$ has two real roots in the interval $(-2, 2)$.

We considered all such polynomials with $a \leq 30$. The first step was to select the polynomials with $\mathrm{Gal}(C) = V_4$ using Pari's function "polgalois". There are a surprisingly large number of these, but one must keep in mind that there are only 3 possibilities for $\mathrm{Gal}(C)$, *i.e.* V_4, C_4 and $B_2 = D_4$ as we observed earlier. Next, the LLL algorithm was used, as in Section 8, to determine cyclotomic relations involving F_n with $n \leq 18$. Finally, these were combined as already described to yield suitable rational $r(x)$.

The largest a for which a cyclotomic relation was found is $a = 17$. If $C(x) = 17x^4 + 12x^3 + 17x^2 + 12x + 17$, then C has roots

$$\alpha = (-6 + 5\sqrt{13} + \sqrt{-15} + 2\sqrt{-195})/17$$

and its conjugates, all of absolute value 1. Let

$$\beta = (-6 - 5\sqrt{13} - \sqrt{-15} + 2\sqrt{-195})/17$$

be the other conjugate on the upper half circle. We find the single cyclotomic relation

$$\begin{aligned} r &= F_1^{-6}F_2^2F_3^3F_5^3F_6^{-3}F_8^{-2} \\ &= (x-1)^{-17}(x^2-1)^5(x^3-1)^6(x^4-1)^2(x^5-1)^3(x^6-1)^{-3}(x^8-1)^{-2}. \end{aligned} \tag{50}$$

If we let

$$R(x) = 17D(x) - \frac{5}{2}D(x^2) - 2D(x^3) - \frac{1}{2}D(x^4) - \frac{3}{5}D(x^5) + \frac{1}{2}D(x^6) + \frac{1}{4}D(x^8),$$

as in Proposition 1, then from the proof of Theorem 2, $R(\alpha) \pm R(\beta)$ will be rational multiples of πd_{15} and πd_{195}. We find numerically that

$$\begin{aligned} \frac{R(\alpha) - R(\beta)}{\pi} &\overset{?}{=} \frac{2}{5}d_{15}, \\ \frac{R(\alpha) + R(\beta)}{\pi} &\overset{?}{=} \frac{1}{20}d_{195}. \end{aligned} \tag{51}$$

Unfortunately, $r \notin \mathcal{R}(C)$ (there is an extra pair of crossings) so (51) does not give us a value for $m^+(r)$.

The five polynomials of this form to which Theorem 2 can be applied are given in Table 1. The polynomial $C(x) = 2x^4 + 2x^3 + x^2 + 2x + 2$ is the most fruitful. This has roots $\alpha = (-1 + \sqrt{-1} + \sqrt{7} + \sqrt{-7})/4$ and its conjugates, $\beta = \sqrt{-1}\alpha$, $\bar{\alpha}$ and $\bar{\beta}$. We found that there are two independent relations $r_1 = F_1F_2/F_3$ and $r_2 = F_2^2F_6/F_5$, giving a basis of $\mathcal{R}(C)$ so there is a complete two parameter family $r_1^a r_2^b$ with

$$m^+(r_1^a r_2^b) = \left(a + \frac{14}{15}b\right) d_4,$$

for non-negative a, b not both zero.

As an unexpected byproduct of this computation, we found one further example $r = F_2^2F_4/F_{12} = (x-1)^2(x^4-1)^{-2}(x^6-1)^{-1}(x^{12}-1)$ with crossing polynomial the reducible sextic $C(x) = (2x^4 + 2x^3 + x^2 + 2x + 2)(2x^2 + x + 2)$ which has all its roots on the unit circle. The first factor has the roots α, β of the previous paragraph on the upper half circle and the second factor has the root $\gamma = (-3 + \sqrt{-7})/2$. Hence if $R(x) = 2D(x) - \frac{1}{2}D(x^4) - \frac{1}{6}D(x^6) + \frac{1}{12}D(x^{12})$, then

$$m^+(r) = \big(R(\alpha) - R(\gamma) + R(\beta)\big)/\pi \stackrel{?}{=} \frac{1}{2}d_7, \tag{52}$$

according to Theorem 1 and the remark following Theorem 2. (As usual $\frac{1}{2}$ here represents a number known to be rational and equal to $\frac{1}{2}$ to 50 decimal places.)

10 Quadratic Fields IV (d_{19})

The smallest discriminant that does not appear in Table 1 is -19. For example, taking α as a root of $5x^2 + 9x + 5$ the LLL algorithm finds two independent cyclotomic relations, but there are no elements of $\mathcal{R}(P)$ that one can construct from these. One of these is $r(x) = F_2^4F_3F_4/F_{12}^2 = q(x)/p(x)$. We find that

$$p(x) - q(x) = -x(5x^2 + 9x + 5)(x^4 + x^3 + x^2 + x + 1).$$

This has $\alpha = (-9 + \sqrt{-19})/10$ and its inverse, and the 5-th roots of unity as its roots on the unit circle. On the other hand $p(x) + q(x)$ is irreducible and has no roots on the unit circle. If we write

$$R(x) = 5D(x) - \frac{1}{2}D(x^2) - \frac{1}{3}D(x^3) - \frac{3}{4}D(x^4) - \frac{1}{3}D(x^6) + \frac{1}{6}D(x^{12}),$$

then Proposition 1 gives

$$m\big(yp(x) + q(x)\big) = \frac{1}{\pi}\big(R(\zeta_5) - R(\zeta_5^2) + R(\alpha)\big),$$

where $\zeta_5 = \exp(2\pi i/5)$. Numerically, $R(\alpha)/\pi \stackrel{?}{=} \frac{1}{6}d_{19}$ (to 50 d.p.). For the conductor $f = 5$, we take $\chi_5(2) = i$ and write $L'(-1, \chi_5) = d_5' + d_5'' i$. Then can easily check that $\big(R(\omega) - R(\omega^2)\big)/\pi = \frac{13}{6}(d_5' - d_5'')$. Hence

$$m_1 := m\big(yp(x) + q(x)\big) \stackrel{?}{=} \frac{1}{6}(d_{19} + 13d_5' - 13d_5'').$$

https://doi.org/10.4153/CJM-2002-016-9 Published online by Cambridge University Press

We can eliminate d_5' and d_5'' by finding further polynomials in which these quantities appear. For example,

$$m_2 := m\big(y(x^2+x+1)+(x+1)\big) = \frac{1}{3}(d_5' + d_5''),$$

and

$$m_3 := m\big(y(x^4+1)^2+(x^2-x+1)\big) = d_5' - \frac{1}{3}d_5''.$$

Combining the last three equations, we have

$$4m_1 + 13(m_2 - m_3) \stackrel{?}{=} \frac{2}{3}d_{19},$$

which gives d_{19} as a rational multiple (presumably $\frac{3}{2}$) of the measure of the *rational function*

$$\big(yp(x)+q(x)\big)^4\big(y(x^2+x+1)+(x+1)\big)^{13}\big(y(x^4+1)^2+(x^2-x+1)\big)^{-13}.$$

As mentioned earlier, Chinburg [Ch] has shown that one may obtain *every* d_f as a rational multiple of $m(r_f)$ for a rational function r_f. In general his method produces rather more complicated r_f than in this example.

We should remark that the authors have recently constructed a polynomial $P(x, y)$ of degree 2 in y such that $m(P) \stackrel{?}{=} \frac{2}{5}d_{19}$. This will appear in a future article.

11 Quartic Fields

Our final set of examples, summarized in Table 2, are of rational functions with crossing polynomial quartics $C(x) = x^4 + bx^3 + cx^2 + bx + 1$ that have two real and two complex roots, the pair of complex roots lying on the unit circle. Then Theorem 1 applies and gives a relation $m^+\big(r(x)\big) = s_F Z_F$, where Z_F is as in (25) and $s_F \in \mathbb{Q}^*$. The values of s_F given in Table 2 are those surmised from 50 decimal place approximations. The techniques are identical with those used in the quadratic case.

1. Notice that the discriminant $-507 = -3 \cdot 13^2$ appears 6 times in Table 2, associated to two different crossing polynomials. If α is the root on the upper half circle of $C(x) = x^4 + x^3 - x^2 + x + 1$ then $-\bar{\alpha}^2$ is the corresponding root of $Q(x) = x^4 + 3x^3 + x^2 + 3x + 1$.

For $C(x) = x^4 + x^3 - x^2 + x + 1$, and $\alpha = (-1 + \sqrt{13} + \sqrt{-2 - 2\sqrt{13}})/4$, one obtains the four admissible solutions $r_1 = F_3F_6/F_1^2$, $r_2 = F_2^2F_6$, $r_3 = F_3F_7/F_1^2$ and $r_4 = F_2^2F_7$. (Here it is convenient to use the normalization $|r(1)| > 1$, in defining $\mathcal{R}(C)$.) If one lets $f_n = |F_n(\alpha)|^2$, then one finds that $f_n \in \mathbb{Z}[\sqrt{13}]$ with $\mathrm{Nm}(f_n) = 3, -1, 3^2, 3^2, 2^4, 1, 1$ for $n = 1, \ldots, 7$. Using this, one can prove that all admissible solutions are of the form $\prod r_k^{a_k}$ for non-negative integers $a_1, \ldots, a_4$. However, r_1 is the only one of the r_k with crossing polynomial P. As in the discussion of the example $4x^2 + 5x + 4$, above, we find that $r_1r_2 \in \mathcal{R}(C)$ and that there are constants

https://doi.org/10.4153/CJM-2002-016-9 Published online by Cambridge University Press

$c_3 = .7945464942\cdots$ and $c_4 = 2.0678582845\cdots$ so that $r_1^{c_3}r_3$ and $r_1^{c_4}r_4$ have a single crossing of the half circle $0 \le t \le \pi$ at $t_0 = \arg(\alpha)$. Then $r_1^{a_1}r_2^{a_2}r_3^{a_3}r_4^{a_4} \in \mathcal{R}(C)$ if $a_1 \ge a_2 + c_3a_3 + c_4a_4$, and one has

$$m^+(r_1^{a_1}r_2^{a_2}r_3^{a_3}r_4^{a_4}) = \Big(2a_1 + \frac{2}{3}a_2 + \frac{64}{21}a_3 + \frac{12}{7}a_4\Big)Z_F. \tag{53}$$

The examples given in Table 2 are r_1, r_1r_2, r_1r_3 and $r_1r_3r_4$.

2. The construction of the unique example for $C(x) = x^4+x^3-2x^2+x+1$ with $\Delta_F = -1156$ is an interesting illustration of a point mentioned earlier—that knowing a basis for the admissible relations does not necessarily make it easy to determine $\mathcal{R}(C)$. In this case if $t_0 = \arg(\alpha)$, then $t_0/\pi = .241823\cdots$ so the admissible numerators are F_n for $n \in \{2,\ldots,9\}$ and the admissible denominator F_1 (again adopting the convention of requiring $|r(1)| > 1$). LLL finds four independent relations involving $F_1,\ldots,F_9$, namely $v_1 = F_4F_8^{-2}F_9$, $v_2 = F_2^2F_8^{-1}F_9$, $v_3 = F_1^{-2}F_8$, and $v_4 = F_3^{-1}F_6^2F_9^{-1}$. Of these only v_3 is admissible (since only F_1 may appear as a denominator) but v_3 is not in $\mathcal{R}(C)$ (because of extra crossings of the circle).

Let $r_1 = v_3 = F_1^{-2}F_8$, $r_2 = v_2v_3 = F_1^{-2}F_2^2F_9$ and $r_3 = v_1v_3^2 = F_1^{-4}F_4F_9$. Then r_1, r_2, r_3 can be proved to form a basis of the admissible solutions but none is in $\mathcal{R}(C)$. Examining the graphs of $|r_k(e^{it})|$ we find that all are > 1 in $0 < t < t_0$, but each has other intervals in $t > t_0$ in which they are > 1. Furthermore, r_1 and r_2 share such an interval so that no product $r_1^{a_1}r_2^{a_2}$ can be in $\mathcal{R}(C)$. However, noticing that $\min_{t\ge t_0}(|r_1|,|r_3|) \le 1$ with equality at $t = t_0$ and $t = \pi/3$ it is conceivable that $r_1^nr_3 \in \mathcal{R}(C)$ for some n. In fact that this does hold for the unique choice $n = 3$, giving the solution $r = r_1^3r_3 = F_1^{-10}F_4F_8^3F_9 \in \mathcal{R}(C)$ given in Table 2. Note that $r(x)$ touches the unit circle at $x = e^{i\pi/3}$. If instead $|r(e^{it})| < 1$ had held for $t > t_0$, then combinations of the form r^nr_k would have been in $\mathcal{R}(C)$ for sufficiently large n. However, one can prove that the powers of r exhaust $\mathcal{R}(C)$.

3. Notice the interesting trio of examples: $\alpha = (1+\beta)/(1-\beta)$, with $\beta = (-2)^{1/4}$, $(-3)^{1/4}$ and $(-5)^{1/4}$, with discriminants -2048, -6912 and -2000 respectively.

4. To generate Table 2, we tested all irreducible $C(x) = x^4 + bx^3 + cx^2 + bx + 1$ with $0 \le b \le 30$ with 2 real roots and two complex roots on $|x| = 1$. The only $C(x)$ found for which $\mathcal{R}(C)$ is non-empty are those listed in Table 2. For each $C(x)$ listed in Table 2, we have determined the set $\mathcal{R}(C)$ by the method illustrated above for $C(x) = x^4 + x^3 - x^2 + x + 1$. It seems likely that there are no other P of this form for which $\mathcal{R}(C)$ is non-empty.

For $b > 12$, we found only one example of an admissible relation, namely $r(x) = F_1^{24}F_2^{-36}F_3^5$ for $x^4+20x^3+22x^2+20x+1$ which generates a field of discriminant -1600. Unfortunately, $r(x)$ has other crossings of the circle so $r \notin \mathcal{R}(C)$.

We also considered more general quartics $C(x) = ax^4 + bx^3 + cx^2 + bx + a$ with a pair of roots on the circle and with $a > 1$. Although many cyclotomic relations were found, especially for small a, none of these were admissible. This is perhaps somewhat surprising in view of the examples discussed in Section 8. We also considered monic sextic polynomials with 4 real roots and a pair of complex roots on the unit circle. Although some cyclotomic relations were found, none of these was admissible.

12 Verification of an Identity Relating Two Mahler Measures

The tables in Section 14 imply several relations between Mahler measures; for example, taking the first two corresponding to the quadratic field $K = \mathbb{Q}(\sqrt{-7})$ we find, to high accuracy, that

$$m\big((x+1)^2y + x^2 + x + 1\big) \stackrel{?}{=} \frac{9}{4}m\big((x+1)^4y + x^4 + 1\big). \tag{54}$$

The purpose of this section is to sketch a rigorous proof of this identity. In principle, all the implied relations between Mahler measures in our tables can similarly be proven in a finite amount of time. However, there is no know general algorithm for the calculations outlined below and the process is rather long.

Let $\alpha = (-3 + \sqrt{-7})/4$ and

$$\xi_1 = [\alpha] - [\alpha^2] + \frac{1}{3}[\alpha^3], \quad \xi_2 = 4[\alpha] - 2[\alpha^2] - \frac{1}{4}[\alpha^4] + \frac{1}{8}[\alpha^8].$$

Using Proposition 1 we see that (54) is equivalent to proving that $D(\xi') = 0$, where

$$\xi' = 18\xi_1 - 8\xi_2 = -14[\alpha] - 2[\alpha^2] + 6[\alpha^3] + 2[\alpha^4] - [\alpha^8].$$

The calculations that follow are entirely due to Don Zagier. First, by using the duplication formula

$$D(x) + D(-x) = \frac{1}{2}D(-x^2)$$

with $x = \alpha^4$ we reduce the question to showing that $D(\xi) = 0$, where

$$\xi = 7[\alpha] + [\alpha^2] - 3[\alpha^3] + [-\alpha^4].$$

We will verify that $D(\xi) = 0$ is a consequence of the 5-term functional equation satisfied by the dilogarithm, which we now recall (how to actually find which combination of such relations to consider is of course another matter).

$$D\big(V(x,y)\big) = 0, \quad x, y \in \mathbb{C} \setminus \{0,1\}, \tag{55}$$

where

$$V(x,y) = [x] + [y] + \left[\frac{1-x}{1-xy}\right] + [1-xy] + \left[\frac{1-y}{1-xy}\right] \in \mathbb{Z}[\mathbb{C}].$$

Limiting cases of this identity yield the following

$$D(x) = -D\Big(\frac{1}{x}\Big) = -D(1-x) = D\Big(\frac{1}{1-x}\Big) = D\Big(1-\frac{1}{x}\Big) = -D\Big(\frac{x}{1-x}\Big) \tag{56}$$

https://doi.org/10.4153/CJM-2002-016-9 Published online by Cambridge University Press

for any $x \in \mathbb{C} \setminus \{0, 1\}$; recall that also

$$D(\bar{x}) = -D(x) \tag{57}$$

We let $a = (1 + \sqrt{-7})/2$ and $b = \bar{a}$ and consider

$$\eta = V(a^{-1}, b) - V(2, b) - V(b, b^{-4}) + 2V(-a^{-1}, -b) - 2V(-b, -b^2).$$

Using (56) and (57) repeatedly one may check that $D(\xi) = D(\eta)$ and hence by (55) $D(\xi) = 0$ as claimed.

13 Concluding Remarks

As we have pointed out, because the formula (8) of Proposition 1 involves only α on the unit circle, the fields that appear in Theorem 1 are necessarily of even degree and with Galois group a subgroup of the hyperoctahedral group $C_2 \wr S_n$. It would be interesting to obtain formulas relating $m(P)$ to $\zeta_F(2)$ for other fields. Some such examples have been obtained in the course of the investigation described in [Bo5], where the measures $m\big(P(x, y)\big)$ of the "A-polynomials" of certain hyperbolic 3-manifolds have been evaluated as sums of dilogarithms of algebraic numbers obtained from representations of the fundamental group of the manifold into $\mathrm{SL}(2, \mathbb{C})$. The terms in these sums are interpreted as "pseudovolumes" of the manifold in question. A simple example, related to the complement of the knot 5_2 is the polynomial

$$P(x, y) = y^3 + y^2(x^2 - x^3 + 2x^5 + 2x^6 - x^7) + y(-1 + 2x + 2x^2 - y^4 + y^5) + 1. \tag{58}$$

It can be shown by the methods of [Bo5] that

$$\pi m(P) = \mathrm{vol}\,(5_2) = 3D(\alpha) = 2.8281220883\cdots,$$

is the volume of the manifold in question. Here α is the complex root of $f(x) = x^3 - 3x^2 + 2x - 1$ with $\mathrm{Im}(\alpha) > 0$. Notice that $|\alpha| \neq 1$. If $F = \mathbb{Q}(\alpha)$ then $\Delta_F = -23$ and if

$$Z_F = \frac{3(23)^{3/2}\zeta_F(2)}{2^3\pi^5} = 0.1500365366\cdots,$$

then

$$m(P) \stackrel{?}{=} 6Z_F, \tag{59}$$

where here again the multiplier is known to be rational and is numerically equal to 6 to 50 decimal place accuracy.

Another example is constructed from the manifold $m019$ of the survey [HW]. From the A-polynomial of this manifold we derive

$$\begin{aligned} P(x, y) = {} & x^4y^4 + x^3(-y^8 - y^7 + 2y^6 + 2y^5 - 5y^4 - y^3) \\ & + x^2(y^7 + 2y^6 - y^5 + 2y^4 - y^3 + 2y^2 + y) \\ & + x(-y^5 - 5y^4 + 2y^3 + 2y^2 - y - 1) + y^4. \end{aligned} \tag{60}$$

https://doi.org/10.4153/CJM-2002-016-9 Published online by Cambridge University Press

p	q	C(x)	Δ_F	s
F_2	1	x^2+x+1	-3	1
F_2	F_4	x^2+x+1	-3	$3/2$
F_2	F_{10}	x^2+x+1	-3	$11/5$
F_2^2	F_5	x^4-x^2+1	-3	$8/5$
F_2^4	$F_1^2F_6$	$7x^2+2x+7$	-3	$20/3$
$F_2^2F_5F_6$	$F_3^2F_4^2$	$3x^4+3x^3+4x^2+3x+3$	-3	$21/5$
$F_1^2F_5^2F_6$	$F_3^2F_4^4$	$3x^4+3x^3+4x^2+3x+3$	-3	$32/5$
F_2	F_1	x^2+1	-4	1
F_3	1	x^2+1	-4	$2/3$
F_3	F_6	x^2+1	-4	$4/3$
F_2^2	F_4F_6	x^4+1	-4	$4/3$
F_2^4	$F_1^2F_3$	$5x^2+6x+5$	-4	$8/3$
F_3	F_1F_2	$2x^4+2x^3+x^2+2x+2$	-4	1
F_5	$F_2^2F_6$	$2x^4+2x^3+x^2+2x+2$	-4	$14/15$
F_2^2	F_3	$2x^2+3x+2$	-7	$1/3$
F_2^4	F_8	$2x^2+3x+2$	-7	$3/4$
F_3^2	$F_1^2F_4$	$4x^2+x+4$	-7	$5/6$
$F_2^2F_6F_{12}$	$F_4^2F_5$	$4x^4+3x^3+2x^2+3x+4$	-7	$4/5$
$F_2^2F_4$	F_{12}	$(2x^4+2x^3+x^2+2x+2)(2x^2+3x+2)$	-7	$1/2$
F_2^4	$F_1^2F_4$	$3x^2+2x+3$	-8	1
$F_2^2F_3$	F_4^2	$3x^2+2x+3$	-8	$2/3$
F_2^{12}	$F_1^8F_{10}$	$3x^2+2x+3$	-8	$16/5$
$F_2^2F_3$	F_6^2	$5x^2+x+5$	-11	$2/3$
$F_1^2F_2^2F_6^3$	$F_3^2F_5^2$	$5x^4+6x^3+3x^2+6x+5$	-11	$14/15$
F_2^2	F_6	$2x^2+x+2$	-15	$1/6$
F_3	F_4	$2x^2+x+2$	-15	$1/12$
$F_1^6F_2^4F_6$	F_5^3	$6x^4+6x^3+x^2+6x+6$	-15	$7/15$
F_2^4	F_3F_4	$3x^2+4x+3$	-20	$1/6$
$F_2^2F_3$	$F_1^2F_4$	$5x^2+2x+5$	-24	$1/6$
F_3^2	F_4F_6	$3x^2+x+3$	-35	$1/12$
F_2^4	F_3F_6	$4x^2+5x+4$	-39	$1/18$
F_2^{60}	$F_3^{12}F_6^{15}F_{14}$	$4x^2+5x+4$	-39	$6/7$
$F_2^2F_3$	F_{10}	$4x^2+3x+4$	-55	$1/30$
F_2^4	F_4F_6	$5x^2+4x+5$	-84	$1/36$

Table 1: (Quadratic Fields)

https://doi.org/10.4153/CJM-2002-016-9 Published online by Cambridge University Press

p	q	C(x)	Δ_F	s
$F_2^2F_3$	1	$x^4+3x^3+3x^2+3x+1$	-275	$20/3$
F_3^2	F_4	$x^4+3x^3+3x^2+3x+1$	-275	$35/6$
$F_2^2F_6$	F_1^2	$x^4+2x^3-2x^2+2x+1$	-400	$10/3$
$F_4^3F_6$	F_1^6	$x^4+2x^3-2x^2+2x+1$	-400	$25/3$
$F_2^4F_4$	F_1^5	$x^4+8x^3-2x^2+8x+1$	-400	10
$F_2^2F_4$	1	$x^4+2x^3+x^2+2x+1$	-448	2
F_4F_3	F_6	$x^4+2x^3+x^2+2x+1$	-448	$8/3$
F_2^{12}	$F_3F_4^4$	$x^4+8x^3+10x^2+8x+1$	-448	$40/3$
F_2^4	F_3	$x^4+5x^3+7x^2+5x+1$	-475	$10/3$
F_3F_6	F_1^2	$x^4+x^3-x^2+x+1$	-507	2
$F_2^2F_3F_6^2$	F_1^2	$x^4+x^3-x^2+x+1$	-507	$8/3$
$F_3^2F_6F_7$	F_1^4	$x^4+x^3-x^2+x+1$	-507	$106/21$
$F_2^2F_3^2F_6F_7^2$	F_1^4	$x^4+x^3-x^2+x+1$	-507	$142/21$
F_3	F_1	$x^4+3x^3+x^2+3x+1$	-507	$5/3$
$F_2^2F_4$	F_6	$x^4+3x^3+x^2+3x+1$	-507	$5/2$
$F_2^2F_4^3$	F_1^6	$x^4+4x^3-2x^2+4x+1$	-1024	2
F_4^6	$F_1^8F_6$	$x^4+4x^3-2x^2+4x+1$	-1024	$8/3$
$F_4F_8^3F_9$	F_1^{10}	$x^4+x^3-2x^2+x+1$	-1156	$139/72$
F_2^2	F_1	$x^4+5x^3+4x^2+5x+1$	-1156	$1/2$
$F_2^2F_3^3$	$F_4^2F_6$	$x^4+5x^3+4x^2+5x+1$	-1156	$4/3$
$F_3^3F_4$	F_6^3	$x^4+4x^3+x^2+4x+1$	-1375	$5/4$
F_4F_3	F_1^2	x^4+2x^3+2x+1	-1728	$1/3$
F_5	F_6	x^4+2x^3+2x+1	-1728	$1/5$
$F_2^2F_3$	F_4	$x^4+4x^3+4x^2+4x+1$	-1792	$1/3$
$F_2^6F_3$	F_4^3	$x^4+6x^3+6x^2+6x+1$	-2000	$5/6$
F_2^{16}	$F_1^{12}F_4$	$x^4+12x^3+6x^2+12x+1$	-2048	2
$F_2^2F_4$	F_1^2	x^4+3x^3+3x+1	-2312	$1/8$
$F_2^2F_3$	F_6	$x^4+4x^3+3x^2+4x+1$	-5616	$1/12$
$F_2^8F_3^2$	$F_1^4F_4^3$	$x^4+8x^3+6x^2+8x+1$	-6912	$1/6$
F_3^6	$F_4F_6^4$	$x^4+6x^3+3x^2+6x+1$	-7616	$1/6$
F_2^4	F_6	$x^4+5x^3+5x^2+5x+1$	-8619	$1/18$
$F_2^4F_4F_6^2$	F_1^{10}	$x^4+6x^3-6x^2+6x+1$	-9248	$1/12$
F_2^6	F_8F_{10}	$x^4+2x^3-4x^2+2x+1$	-9408	$3/80$
$F_2^4F_3^2$	$F_1^2F_4F_6$	$x^4+6x^3+4x^2+6x+1$	-9408	$1/18$
$F_2^6F_3F_4^2$	$F_1^4F_6^3$	x^4+7x^3+7x+1	-9747	$1/14$
$F_2^2F_3^2F_4^3$	$F_1^8F_6$	$x^4+6x^3-2x^2+6x+1$	-24336	$1/36$
$F_2^2F_3^2F_4$	$F_1^4F_6$	x^4+5x^3+5x+1	-26136	$1/72$
$F_3F_4F_5^2$	$F_1^4F_6^3$	$x^4+5x^3-2x^2+5x+1$	-168100	$1/960$

Table 2: (Quartic Fields)

https://doi.org/10.4153/CJM-2002-016-9 Published online by Cambridge University Press

Again, $\pi m(P)$ is the volume of $m019$ and this leads to

$$\pi m(P) = \text{vol}\,(m019) = 3D(\alpha) = 2.4622842205\cdots. \tag{61}$$

Here α is the complex root of $f(x) = x^4 - x - 1$ with $\text{Im}(\alpha) > 0$. If $F = \mathbb{Q}(\alpha)$ then $\Delta_F = -283$ and the Galois group is S_4, so this field does not appear in Table 1. Once again one finds that $m(P) \overset{?}{=} 6Z_F$. It would be interesting to prove this and (59) by computations in the Bloch group as in [Za2].

14 Tables

The construction of Tables 1 and 2 have been described in the preceding sections. Recall that the quantity s is the ratio

$$s = \frac{m\big(p(x)y - q(x)\big) \cdot 2^{2n-3} \cdot \pi^{2n-1}}{3 \cdot |\Delta_F|^{3/2}\zeta_F(2)}.$$

Except for the quadratic fields of discriminants -3 and -4, where the methods of Section 7 apply, we know that $s \in \mathbb{Q}$ but we do not have an *a priori* bound for its height, so the tabulated values are the most likely value inferred from the numerical calculations.

References

[Bo1] D. W. Boyd, *Speculations concerning the range of Mahler's measure*. Canad. Math. Bull. **24**(1981), 453–469.

[Bo2] ———, *Two sharp inequalities for the norm of a factor of a polynomial*. Mathematika **39**(1992), 341–349.

[Bo3] ———, *Mahler's Measure and Special Values of L-functions*. Experiment. Math. **37**(1998), 37–82.

[Bo4] ———, *Mahler's measure and special Values of L-functions—some conjectures*. In: Number Theory in Progress **1**, (eds., K. Győry, H. Iwaniec and J. Urbanowicz), de Gruyter, Berlin, 1999, 27–34.

[Bo5] ———, *Mahler's measure and invariants of hyperbolic manifolds*. In: Number Theory for the Millennium, (ed., B. C. Berndt et al.), A. K. Peters, Boston, 2002.

[Br] J. Browkin, *Conjectures on the Dilogarithm*. K-Theory **3**(1989), 29–56.

[Ch] T. Chinburg, *Mahler measures and derivatives of L-functions at non-positive integers*. 1984, preprint.

[HW] M. Hildebrand and J. Weeks, *A computer generated census of cusped hyperbolic 3-manifolds*. In: Computers and Mathematics, (eds., E. Kaltofen and S. Watts), Springer-Verlag, New York, 1989, 53–59.

[La] F. Lalande, *Corps de nombres engendrés par un nombre de Salem*. Acta Arith. **88**(1999), 191–200.

[Le] L. Lewin, *Polylogarithms and associated functions*. North Holland, 1981.

[MPV] M. J. Mossinghoff, C. G. Pinner and J. D. Vaaler, *Perturbing polynomials with all their roots on the unit circle*. Math. Comp. **67**(1998), 1707–1726.

[Ra] G. A. Ray, *Relations between Mahler's measure and values of L-series*. Canad. J. Math. **39**(1987), 694–732.

[RV] F. Rodriguez Villegas, *Modular Mahler measures I*. Topics in Number Theory, (eds., S. D. Ahlgren, G. E. Andrews and K. Ono), Kluwer, Dordrecht, 1999, 17–48.

[Sc] A. Schinzel, *Primitive divisors of the expression $A^n - B^n$ in algebraic number fields*. J. Reine Angew. Math. (9) **268**(1974), 27–33.

[Sm] C. J. Smyth, *On measures of polynomials in several variables*. Bull. Austral. Math. Soc. **23**(1981), 49–63.

[Za1] D. Zagier, *The Bloch-Wigner-Ramakrishnan polylogarithm function*. Math. Ann. **286**(1990), 613–624.
[Za2] ———, *Special Values and Functional Equations of Polylogarithms*. In: Structural Properties of Polylogarithms, (ed., L. Lewin), Amer. Math. Soc., Providence, 1991, 377–400.

Department of Mathematics
University of British Columbia
Vancouver, British Columbia
V6T 1Z2
e-mail: boyd@math.ubc.ca

Department of Mathematics
University of Texas at Austin
Austin, Texas 78712
USA
e-mail: villegas@math.utexas.edu

https://doi.org/10.4153/CJM-2002-016-9 Published online by Cambridge University Press