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VOLUME COMPARISON OF BISHOP-GROMOV TYPE

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Bishop-Gromov type comparison theorems for the volume of a tube about a submanifold of a complete Riemannian manifold whose Ricci curvature is bounded from below are proved. The Kähler analogue is also proved.

1. Introduction

In Riemannian geometry, it is a fundamental question to ask how the geometric invariants of Riemannian manifolds are influenced by curvature restrictions. The volume of a geodesic ball is one of the basic invariants, for which the Bishop-Gromov comparison theorem is well-known (see [**1**, **6**]).

In this article we prove Bishop-Gromov type comparison theorems for the volume of a tube about a submanifold of a complete Riemannian manifold whose Ricci curvature is bounded from below. The Kähler analogue is also proved.

To be more specific let M be a complete Riemannian manifold of dimension n and let $P \subset M$ be a topologically embedded connected submanifold of dimension q with compact closure. For $r \geqslant 0$ let $V_P^M(r)$ denote the n-dimensional volume of a tube of radius r about P and let $A_P^M(r)$ denote the $(n-1)$-dimensional volume of its boundary. Let λ be a constant which may be positive, negative, or zero and let $K^n(\lambda)$ denote the n-dimensional space of constant curvature λ.

Theorem 1. *Let $P \subset M$ and suppose that the Ricci curvature ρ^M of M satisfies $\rho^M \geqslant (n-1)\lambda = \rho^{K^n(\lambda)}$. Let $\overline{P}$ denote a q-dimensional totally geodesic submanifold of $K^n(\lambda)$ such that volume $(\overline{P}) = \mathrm{volume}(P)$.*

(i) *If $q = \dim P = 0$, then $V_m^M(t) \leqslant V_{\overline{m}}^{K^n(\lambda)}(t)$ and $V_m^M(t)/V_{\overline{m}}^{K^n(\lambda)}(t)$ is a nonincreasing function of t for $0 < t \leqslant e_c(m)$. Here P is a point $m \in M$ and $e_c(m)$ is the minimal distance to the cut locus of m.*

(ii) *If $1 \leqslant q \leqslant n-2$, then for any $\varepsilon > 0$ there is small $t_0 > 0$ depending on ε and $P \subset M$ such that $V_P^M(t, t_0) \leqslant (1+\varepsilon)C(t_0)V_{\overline{m}}^{K^n(\lambda)}(t, t_0)$ for $t_0 \leqslant t \leqslant e_c(P)$, where $V_P^M(t, t_0) = V_P^M(t) - V_P^M(t_0)$ and $C(t_0) =$*

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$A_{\overline{P}}^{K^n(\lambda)}(t_0)/A_{\overline{m}}^{K^n(\lambda)}(t_0)$. Moreover $V_P^M(t,t_0)/V_{\overline{m}}^{K^n(\lambda)}(t,t_0)$ is a nonincreasing function of t for $t_0 \leqslant t \leqslant e_c(P)$. Here $e_c(P)$ is the minimal focal distance of P in M.

(iii) *If $q = n-1$ and P is minimal, then $V_P^M(t) \leqslant V_{\overline{P}}^{K^n(\lambda)}(t)$ and $V_P^M(t)/V_{\overline{P}}^{K^n(\lambda)}(t)$ is a nonincreasing function of t for $0 \leqslant t \leqslant e_c(P)$.*

REMARK. Observe that the right-hand sides of inequalities in (i), (ii), (iii) do not depend on the embedding of P into M. (i) is just the Bishop-Gromov comparison theorem for geodesic balls. If $\dim P \geqslant 1$, then for sufficiently small $r > 0$, $V_P^M(r)$ depends strongly on P as the power series expansion in [5] shows. This fact forces us to consider $V_P^M(t,t_0)$ in (ii). Even though $t_0 > 0$ depends on ε and $P \subset M$ in (ii), sufficiently small t_0 always satisfies the inequality. Also observe that $t \mapsto V_P^M(t)/V_{\overline{m}}^{K^n(\lambda)}(t)$ is nonincreasing for $0 < t \leqslant e_c(P)$ in (ii). When P is a hypersurface of M, an additional assumption on P (that is, P is minimal) is needed to get the global volume comparison in (iii), since the principal curvatures of P strongly affects $V_P^M(r)$ for relatively large r.

Next, to state the Kähler analogue of Theorem 1, let M be a complete Kähler manifold of real dimension $2n$. Let K_h^M (K_{ah}^M) denote the holomorphic (antiholomorphic) sectional curvature of M. The antiholomorphic Ricci curvature ρ_{ah}^M of M is the sum of antiholomorphic sectional curvatures (see for example [2]). Let $P \subset M$ be a topologically embedded connected complex submanifold of real dimension $2q$ with compact closure.

THEOREM 2. *Let $P \subset M$ and suppose that $K_h^M \geqslant 4\lambda$ and $\rho_{ah}^M \geqslant (2n-2)\lambda$. Let $\overline{P}$ denote a totally geodesic complex submanifold of real dimension $2q$ of $K_h^n(\lambda)$ such that volume$(\overline{P})$ = volume(P), where $K_h^n(\lambda)$ is a Kähler manifold of complex dimension n with constant holomorphic sectional curvature 4λ.*

(i) *If $q = 0$ (that is, P is a point $m \in M$), then $V_m^M(t) \leqslant V_{\overline{m}}^{K_h^n(\lambda)}(t)$ and $V_m^M(t)/V_{\overline{m}}^{K_h^n(\lambda)}(t)$ is a nonincreasing function of t for $0 < t \leqslant e_c(m)$.*

(ii) *If $1 \leqslant q \leqslant n-2$, then for any $\varepsilon > 0$ there is small $t_0 > 0$ depending on ε and $P \subset M$ such that $V_P^M(t,t_0) \leqslant (1+\varepsilon)C(t_0)V_{\overline{m}}^{K_h^n(\lambda)}(t,t_0)$ for $t_0 \leqslant t \leqslant e_c(P)$, where $C(t_0) = A_{\overline{P}}^{K_h^n(\lambda)}(t_0)/A_{\overline{m}}^{K_h^n(\lambda)}(t_0)$. Moreover $V_P^M(t,t_0)/V_{\overline{m}}^{K_h^n(\lambda)}(t,t_0)$ is a nonincreasing function of t for $t_0 \leqslant t \leqslant e_c(P)$.*

(iii) *If $q = n-1$ (that is, P is a complex hypersurface of M), then $V_P^M(t) \leqslant V_{\overline{P}}^{K_h^n(\lambda)}(t)$ and $V_P^M(t)/V_{\overline{P}}^{K_h^n(\lambda)}(t)$ is a nonincreasing function of t for $0 \leqslant t \leqslant e_c(P)$.*

REMARK. (i) is the result essentially due to Nayatani [7] (see also [4, page 194]). In (ii) $V_P^M(t)/V_{\overline{m}}^{K_h^n(\lambda)}(t)$ is also a nonincreasing function of t for $0 < t \leqslant e_c(P)$.

We shall prove these theorems following the ideas in [4]. In Section 2 we review some preliminary results. The proofs of theorems are given in Section 3.

2. PRELIMINARIES [3, 4]

Let M be a complete Riemannian manifold of dimension n and let $P \subset M$ be a topologically embedded submanifold of dimension q which is relatively compact.

Let $t \mapsto \gamma(t)$ be a unit speed geodesic in M normal to P with $\gamma(0) = p \in P$. Assume that $t \geqslant 0$ is less than or equal to the distance between P and its nearest focal point. Denote by $S(t)$ the second fundamental form at the point $\gamma(t)$ of the tubular hypersurface at a distance t from P. Also let $R(t) : M_{\gamma(t)} \to M_{\gamma(t)}$ be the symmetric linear transformation defined by $\langle R(t)x, y\rangle = R^M_{\gamma'(t)x\gamma'(t)y}$, where $\langle\ ,\ \rangle$ and R^M are the metric and the Riemannian curvature tensor field of M respectively, $M_{\gamma(t)}$ denotes the tangent space to M at $\gamma(t)$, and x, $y \in M_{\gamma(t)}$. Then $S(t)$ satisfies the differential equation

$$S'(t) = S(t)^2 + R(t). \tag{1}$$

Let ω be the Riemannian volume form of M, and let $(x_1, \dots, x_n)$ be a system of Fermi coordinates such that

$$\omega\left(\frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_n}\right)(\gamma(t)) > 0.$$

For $u \in P_p^\perp$ with $\|u\| = 1$, put

$$\theta_u(t) = \omega\left(\frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_n}\right)(\gamma(t)).$$

Then $\theta_u(0) = 1$ and

$$\frac{\theta_u'(t)}{\theta_u(t)} = -\left(\frac{n-q-1}{t} + trS(t)\right). \tag{2}$$

Let $V_P^M(r) = n$-dimensional volume of $\{m \in M \mid d(m,P) \leqslant r\}$ and $A_P^M(r) = (n-1)$-dimensional volume of $\{m \in M \mid d(m,P) = r\}$. Then

$$A_P^M(t) = \int_P \int_{S^{n-q-1}(1)} t^{n-q-1}\theta_u(t)\, du\, dP \tag{3}$$

and

$$V_P^M(r) = \int_0^r A_P^M(t)\, dt, \tag{4}$$

where $S^{n-q-1}(1)$ is the unit sphere in $P_p^\perp$. If $\overline{P} \subset K^n(\lambda)$ is a totally geodesic submanifold,

$$(5) \qquad A_{\overline{P}}^{K^n(\lambda)}(t) = \frac{2\pi^{(n-q)/2}}{\Gamma\left(\frac{n-q}{2}\right)} \left(\frac{\sin t\sqrt{\lambda}}{\sqrt{\lambda}}\right)^{n-q-1} \left(\cos t\sqrt{\lambda}\right)^q \operatorname{volume}\left(\overline{P}\right),$$

and if $\overline{P} \subset K_h^n(\lambda)$ is a totally geodesic complex submanifold,

$$(6) \qquad A_{\overline{P}}^{K_h^n(\lambda)}(t) = \frac{2\pi^{n-q}}{(n-q-1)!} \left(\frac{\sin t\sqrt{\lambda}}{\sqrt{\lambda}}\right)^{2n-2q-1} \left(\cos t\sqrt{\lambda}\right)^{2q+1} \operatorname{volume}\left(\overline{P}\right).$$

In (5) and (6) we interpret $\left(\sin t\sqrt{\lambda}\right)/\sqrt{\lambda}$ as $\left(\sinh t\sqrt{-\lambda}\right)/\sqrt{-\lambda}$ (respectively t) when $\lambda < 0$ (respectively $\lambda = 0$).

3. Proof of Theorems

We will use many results of [4]. The proof of Theorem 2 is similar to but a little more complicated than that of Theorem 1.

Proof of Theorem 1: (i) This is just the Bishop-Gromov volume comparison theorem for geodesic balls (see [**1**, **4**, **6**]). Observe that the proof of case (ii) gives us (i) easily.

(ii) Let $1 \leqslant q = \dim P \leqslant n-2$, and $0 < t \leqslant e_c(P)$.

Then $t \rightarrow \theta_u(t)/\left(\sin t\sqrt{\lambda}/t\sqrt{\lambda}\right)^{n-1} t^q$ is nonincreasing (see [4, p. 181]). For any $\varepsilon > 0$, there is small $t_0 > 0$ depending on ε and $P \subset M$ such that $\theta_u(t_0) \leqslant (1+\varepsilon)\left(\sin t_0\sqrt{\lambda}/t_0\sqrt{\lambda}\right)^{n-q-1}\left(\cos t_0\sqrt{\lambda}\right)^q$ since $\lim_{t_0\to 0}\theta_u(t_0) = 1$. Then we have for $0 < t_0 \leqslant s \leqslant t$

$$(7) \quad s^{n-q-1}\theta_u(s) \leqslant (1+\varepsilon)t_0^{n-q-1}\left(\frac{\sin t_0\sqrt{\lambda}}{t_0\sqrt{\lambda}}\right)^{n-q-1}\left(\cos t_0\sqrt{\lambda}\right)^q \Big/ \left(\frac{\sin t_0\sqrt{\lambda}}{\sqrt{\lambda}}\right)^{n-1}$$

and

$$(8) \qquad t^{n-q-1}\theta_u(t)\left(\frac{\sin s\sqrt{\lambda}}{\sqrt{\lambda}}\right)^{n-1} \leqslant s^{n-q-1}\theta_u(s)\left(\frac{\sin t\sqrt{\lambda}}{\sqrt{\lambda}}\right)^{n-1}.$$

Integrating (7) over the unit sphere $S^{n-q-1}(1)$ in $P_p^\perp$ and using (3), (5), we obtain

$$(9) \qquad A_P^M(s) \leqslant (1+\varepsilon)\frac{A_{\overline{P}}^{K^n(\lambda)}(t_0)}{A_{\overline{m}}^{K^n(\lambda)}(t_0)} A_{\overline{m}}^{K^n(\lambda)}(s).$$

Then integrating (9) with respect to s from t_0 to t and using (4) we get

$$V_P^M(t,t_0) \leqslant (1+\varepsilon)C(t_0)V_{\overline{m}}^{K^n(\lambda)}(t,t_0), \tag{10}$$

where $C(t_0) = A_{\overline{P}}^{K^n(\lambda)}(t_0)/A_{\overline{m}}^{K^n(\lambda)}(t_0)$. Similarly from (8) we obtain

$$V_P^M(t,t_0)A_{\overline{m}}^{K^n(\lambda)}(t) \geqslant V_{\overline{m}}^{K^n(\lambda)}(t,t_0)A_P^M(t). \tag{11}$$

But (11) implies that

$$\frac{d}{dt}\left\{\frac{V_P^M(t,t_0)}{V_{\overline{m}}^{K^n(\lambda)}(t,t_0)}\right\} = \frac{1}{\{V_{\overline{m}}^{K^n(\lambda)}(t,t_0)\}^2}\times$$
$$\left(V_{\overline{m}}^{K^n(\lambda)}(t,t_0)A_P^M(t) - V_P^M(t,t_0)A_{\overline{m}}^{K^n(\lambda)}(t)\right) \leqslant 0.$$

(iii) Let P be a minimal hypersurface of M and $0 \leqslant t \leqslant e_c(P)$. Then Lemma 8.28 in [4, page 181] shows that

$$\theta_u(t) \leqslant \left(\cos t\sqrt{\lambda}\right)^{n-1} \tag{12}$$

and

$$\theta_u(t)\left(\cos s\sqrt{\lambda}\right)^{n-1} \leqslant \theta_u(s)\left(\cos t\sqrt{\lambda}\right)^{n-1}. \tag{13}$$

Integrating (12) and (13) we get

$$V_P^M(t) \leqslant V_{\overline{P}}^{K^n(\lambda)}(t) \tag{14}$$

and

$$V_P^M(t)A_{\overline{P}}^{K^n(\lambda)}(t) \geqslant A_P^M(t)V_{\overline{P}}^{K_n(\lambda)}(t). \tag{15}$$

But (15) implies that $d\left(V_P^M(t)/V_{\overline{P}}^{K^n(\lambda)}(t)\right)/dt \leqslant 0$. This completes the proof of Theorem 1. $\square$

Proof of Theorem 2: Let $\{e_1, e_{1^*}, \ldots, e_n, e_{n^*}\}$ be an orthonormal basis of the tangent space M_p such that $e_{i^*} = Je_i$ and $e_1, e_{1^*}, \ldots, e_q, e_{q^*}$ are tangent to P. Let $\gamma(t)$ be a unit speed geodesic with $\gamma(0) = p$ and $\gamma'(0) = u = e_n$. Extend $e_1, e_{1^*}, \ldots, e_n, e_{n^*}$ to orthonormal vector fields $E_1(t), E_{1^*}(t), \ldots, E_n(t), E_{n^*}(t)$ along γ so that $E_n(t) = \gamma'(t)$ and the other $E_i(t)$ diagonalise $S(t)$. Consider the functions

$$f_i(t) = \langle S(t)E_i(t), E_i(t)\rangle, \qquad i \neq n. \tag{16}$$

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Taking the derivative of both sides of (16) and using (1) and the Cauchy-Schwarz inequality, we get

$$
(17)\qquad \begin{aligned} f_i'(t) &= \langle S'E_i, E_i\rangle = \langle S^2E_i + RE_i, E_i\rangle = \|SE_i\|^2 + \langle RE_i, E_i\rangle \\ &\geqslant \langle SE_i, E_i\rangle^2 + \langle RE_i, E_i\rangle = f_i^2(t) + \langle R(t)E_i(t), E_i(t)\rangle. \end{aligned}
$$

Let

$$
(18)\qquad f(t) = \frac{1}{2n-2}\sum_{i=1}^{(n-1)^*} f_i(t).
$$

Since $\rho_{ah}^M = (2n-2)f(t) \geqslant (2n-2)\lambda$ and $K_h^M \geqslant 4\lambda$, we have

$$
(19)\qquad f'(t) \geqslant f^2(t) + \lambda
$$

and

$$
(20)\qquad f_{n^*}'(t) \geqslant f_{n^*}^2(t) + 4\lambda.
$$

The differential inequalities (19) and (20) can be solved explicitly (see for example [4, pp.174–175]).

(i) This is the result essentially due to Nayatani [7]. Observe that the proof of case (ii) gives us (i) easily.

(ii) Let $1 \leqslant q = \dim P/2 \leqslant n-2$ and $0 < t \leqslant e_c(P)$. Since $f(0) = f_{n^*}(0) = -\infty$, we have from (19) and (20)

$$
f(t) \geqslant -\sqrt{\lambda}\cot t\sqrt{\lambda} \quad \text{and} \quad f_{n^*}(t) \geqslant -2\sqrt{\lambda}\cot 2t\sqrt{\lambda}.
$$

Summing the functions $f_i(t)$, we find that

$$
(21)\qquad tr\, S(t) \geqslant -(2n-2)\sqrt{\lambda}\cot t\sqrt{\lambda} - 2\sqrt{\lambda}\cot 2t\sqrt{\lambda}.
$$

From (2) and (21) we obtain

$$
\frac{d}{dt}\ln\theta_u(t) = -\frac{2n-2q-1}{t} - tr\, S(t) \leqslant \frac{d}{dt}\ln\alpha(t),
$$

where

$$
\alpha(t) = \left(\frac{\sin t\sqrt{\lambda}}{t\sqrt{\lambda}}\right)^{2n-2q-1}\left(\sin t\sqrt{\lambda}\right)^{2q}\cos t\sqrt{\lambda}.
$$

Then $d(\ln(\theta_u(t)/\alpha(t)))/dt \leqslant 0$, and $\theta_u(t)/\alpha(t)$ is a nonincreasing function of t. For any $\varepsilon > 0$, there is $t_0 > 0$ depending on ε and $P \subset M$ such that $\theta_u(t_0) \leqslant$

$(1+\varepsilon)\left(\sin\left(t_0\sqrt{\lambda}\right)\Big/\left(t_0\sqrt{\lambda}\right)\right)^{2n-2q-1}\left(\cos t_0\sqrt{\lambda}\right)^{2q+1}$ since $\lim_{t_0\to 0}\theta_u(t_0)=1$. Then we have for $0<t_0\leqslant s\leqslant t$

$$(22)\quad s^{2n-2q-1}\theta_u(s)\leqslant(1+\varepsilon)\frac{A_{\overline{m}}^{K_h^n(\lambda)}(s)}{A_{\overline{m}}^{K_h^n(\lambda)}(t_0)}t_0^{2n-2q-1}\left(\frac{\sin t_0\sqrt{\lambda}}{t_0\sqrt{\lambda}}\right)^{2n-2q-1}\left(\cos t_0\sqrt{\lambda}\right)^{2q+1}$$

and

$$(23)\quad t^{2n-2q-1}\theta_u(t)\left(\frac{\sin s\sqrt{\lambda}}{\sqrt{\lambda}}\right)^{2n-2q-1}\left(\cos s\sqrt{\lambda}\right) \leqslant s^{2n-2q-1}\theta_u(s)\left(\frac{\sin t\sqrt{\lambda}}{\sqrt{\lambda}}\right)^{2n-2q-1}\left(\cos t\sqrt{\lambda}\right).$$

Integrating (22) using (3) we obtain

$$(24)\quad A_P^M(t)\leqslant(1+\varepsilon)\frac{A_{\overline{P}}^{K_h^n(\lambda)}(t_0)}{A_{\overline{m}}^{K_h^n(\lambda)}(t_0)}A_{\overline{m}}^{K_h^n(\lambda)}(t).$$

Integrating (24) with respect to s from t_0 to t we get

$$(25)\quad V_P^M(t,t_0)\leqslant(1+\varepsilon)C(t_0)V_{\overline{m}}^{K_h^n(\lambda)}(t,t_0),$$

where $C(t_0)=A_{\overline{P}}^{K_h^n(\lambda)}(t_0)/A_{\overline{m}}^{K_h^n(\lambda)}(t_0)$. Similarly, from (23) we obtain

$$(26)\quad V_P^M(t,t_0)A_{\overline{m}}^{K_h^n(\lambda)}(t)\geqslant A_P^M(t)V_{\overline{m}}^{K_h^n(\lambda)}(t,t_0).$$

But (26) implies that $d\left(V_P^M(t,t_0)/V_{\overline{m}}^{K_h^n(\lambda)}(t,t_0)\right)/dt\leqslant 0$.

(iii) Let P be a complex hypersurface of M and $0\leqslant t\leqslant e_c(P)$. Then $f_i(0)$, $1\leqslant i\leqslant(n-1)^*$, are finite and $f(0)=0$ since the mean curvature vector fields of a Kähler submanifold of M vanishes. Hence we obtain from (19) and (20)

$$f(t)\geqslant\sqrt{\lambda}\cot t\sqrt{\lambda}\quad\text{and}\quad f_{n^*}(t)\geqslant-2\sqrt{\lambda}\cot 2t\sqrt{\lambda}.$$

Summing the functions $f_i(t)$, we find that

$$(27)\quad tr\,S(t)\geqslant 2(n-2)\sqrt{\lambda}\cot t\sqrt{\lambda}-2\sqrt{\lambda}\cot 2t\sqrt{\lambda}.$$

From (2) and (27) it follows that

$$\frac{d}{dt}\ln\theta_u(t)=-\frac{1}{t}-tr\,S(t)\leqslant\frac{d}{dt}\ln\beta(t),$$

where $\beta(t) = \left(\sin t\sqrt{\lambda} \big/ t\sqrt{\lambda}\right)\left(\cos t\sqrt{\lambda}\right)^{2n-1}$. Then $d\ln\left(\theta_u(t)/\beta(t)\right)/dt \leqslant 0$, and $\theta_u(t)/\beta(t)$ is a nonincreasing function of t, whose value for $t = 0$ is 1, whence

$$\theta_u(t) \leqslant \beta(t) \tag{28}$$

and

$$t\theta_u(t) A_{\overline{P}}^{K_h^n(\lambda)}(s) \leqslant s\theta_u(s) A_{\overline{P}}^{K_h^n(\lambda)}(t) \tag{29}$$

for $0 \leqslant s \leqslant t$, follows. Integrating (28) using (3), (4), we get

$$V_P^M(t) \leqslant V_{\overline{P}}^{K_h^n(\lambda)}(t). \tag{30}$$

Similarly integrating (29), we obtain

$$V_P^M(t) A_{\overline{P}}^{K_h^n(\lambda)}(t) \geqslant A_P^M(t) V_{\overline{P}}^{K_h^n(\lambda)}(t). \tag{31}$$

But (31) implies that $V_P^M(t)/V_{\overline{P}}^{K_h^n(\lambda)}(t)$ is a nonincreasing function of t. This completes the proof of Theorem 2. □

References

[1] R.L. Bishop and R.J. Crittenden, *Geometry of manifolds* (Academic Press, New York, 1964).

[2] F. Giménez and V. Miquel, 'Volume estimates for real hypersurfaces of a Kaehler manifold with strictly positive holomorphic sectional and antiholomorphic Ricci curvatures', *Pacific J. Math.* **142** (1990), 23–39.

[3] A. Gray, 'Comparison theorems for the volumes of tubes as generalizations of the Weyl tube formula', *Topology* **21** (1982), 201–228.

[4] A. Gray, *Tubes* (Addison-Wesley, New York, 1990).

[5] A. Gray and L. Vanhecke, 'The volume of tubes in a Riemannian manifold', *Rend. Sem. Math. Univ. Politec. Torino* **39** (1983), 1–50.

[6] M. Gromov, *Structures Métriques pour les Variétés riemanniennes* (Cédic Nathan, Paris, 1981).

[7] S. Nayatani, 'On the volume of positively curved Kähler manifolds', *Osaka J. Math.* **25** (1988), 223–231.

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